Some Identities on the Generalized Twisted $q$-Euler Polynomials with Weak Weight

Cheon Seoung Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Copyright © 2016 Cheon Seoung Ryoo. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we study the symmetry for generalized twisted $q$-Euler numbers $E^{(\alpha)}_{n,\chi,q,\zeta}$ and polynomials $E^{(\alpha)}_{p,\chi,q,\zeta}(x)$ with weak weight $\alpha$. We obtain some interesting identities of the power sums and generalized twisted $q$-Euler polynomials $E^{(\alpha)}_{n,\chi,q,\zeta}(x)$ with weak weight $\alpha$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Euler numbers and polynomials, generalized twisted $q$-Euler numbers and polynomials with weak weight, symmetric properties, power sums

1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\pi_1}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$(see [1-9]). Throughout this paper we use
the notation:

\[ [x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case.

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( g \in UD(\mathbb{Z}_p) \), the fermionic \( p \)-adic invariant \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \quad \text{see [1, 2]}. \tag{1.1}
\]

If we take \( g_n(x) = g(x + n) \) in (1.1), then we see that

\[
q^n I_q(g_n) + (-1)^{n-1} I_q(g) = \left[ 2 \right]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{1.2}
\]

Let \( T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), where \( C_{p^N} = \{ \zeta | \zeta^{p^N} = 1 \} \) is the cyclic group of order \( p^N \). For \( \zeta \in T_p \), we denote by \( \phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \zeta^x \) (see [7, 8]). In [7], we introduced generalized twisted \( q \)-Euler numbers \( E_{n,\chi,q,\zeta}^{(\alpha)} \) and polynomials \( E_{n,\chi,q,\zeta}^{(\alpha)}(x) \) attached to \( \chi \). Let \( \chi \) be the primitive Dirichlet character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod } 2) \) and \( \zeta \in T_p \). We assume that \( \alpha \in \mathbb{Z} \) and \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \). Let \( g(y) = \chi(y)\phi_\zeta(y)e^{(y+x)t} \). By (1.1), we derive

\[
\int_X \chi(y)\phi_\zeta(y)e^{(y+x)t}d\mu_{-q^\alpha}(y) = \left[ 2 \right]_q \sum_{\alpha=0}^{d-1} \chi(a)(-1)^a \zeta^a q^\alpha e^{at} \frac{\zeta^d q^{\alpha d}e^{dt} + 1}{e^{xt}}
= \sum_{n=0}^{\infty} E_{n,\chi,q,\zeta}^{(\alpha)}(x)\frac{t^n}{n!}.
\tag{1.3}
\]

By using Taylor series of \( e^{(y+x)t} \) in the above equation (1.3), we obtain

\[
\sum_{n=0}^{\infty} \left( \int_X \chi(y)\phi_\zeta(y)(y + x)^n d\mu_{-q^\alpha}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,\chi,q,\zeta}^{(\alpha)}(x)\frac{t^n}{n!}.
\]

By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we have the Witt formula for the generalized twisted \( q \)-Euler polynomials attached to \( \chi \) as follows:

\[
\textbf{Theorem 1.1} \quad \text{For positive integers } n \text{ and } \zeta \in T_p, \text{ we have}
E_{n,\chi,q,\zeta}^{(\alpha)}(x) = \int_X \chi(y)\phi_\zeta(y)(y + x)^n d\mu_{-q^\alpha}(y). \tag{1.4}
\]
Observe that for $x = 0$, the equation (1.4) reduces to (1.5).

**Corollary 1.2** For positive integers $n$ and $\zeta \in T_p$, we have

$$E^{(\alpha)}_{n,\chi,q,\zeta} = \int_X \chi(y) x^n \phi_{\zeta}(x) d\mu_a(x).$$  \hspace{1cm} (1.5)

By (1.4) and (1.5), we have the following theorem.

**Theorem 1.3** For positive integers $n$ and $\zeta \in T_p$, we have

$$E^{(\alpha)}_{n,\chi,q,\zeta}(x) = \sum_{l=0}^n \binom{n}{l} E^{(\alpha)}_{l,\chi,q,\zeta} x^{n-l}.$$  

2 Identities for generalized twisted $q$-Euler polynomials with weak weight

In this section, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We obtain some interesting identities of the power sums and generalized twisted $q$-Euler polynomials $E^{(\alpha)}_{n,\chi,q,\zeta}(x)$ with weak weight $\alpha$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$. If $n$ is odd from (1.2), we obtain

$$q^n I_q(g_n) + I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l g(l).$$  \hspace{1cm} (2.1)

Substituting $g(x) = \chi(x) \zeta^x e^{xt}$ into the above, we obtain

$$q^{\alpha nd} \int_X \chi(x + nd) \zeta^{x+nt} e^{(x+nt) t} d\mu_{-q^a}(x) + \int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^a}(x)$$

$$= [2]_q \sum_{j=0}^{nd-1} (-1)^j \chi(j) \zeta^j q^{\alpha j} e^{jt}. \hspace{1cm} (2.2)$$

For $k \in \mathbb{Z}_+$, let us define the $p$-adic functional $T^{(\alpha)}_{k,\chi,q,\zeta}(n)$ as follows:

$$T^{(\alpha)}_{k,\chi,q,\zeta}(n) = \sum_{l=0}^n (-1)^l \chi(l) q^{\alpha l} \zeta^l k.$$  \hspace{1cm} (2.3)

After some calculations, we have

$$q^{\alpha nd} \int_X \chi(x) \zeta^{x+nt} e^{(x+nt)t} d\mu_{-q^a}(x) + \int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^a}(x)$$

$$= (1 + \zeta^{nd} q^{\alpha nd} e^{nt}) \frac{[2]_q \sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^{\alpha a} e^{at}}{\zeta^d q^{\alpha d} e^{dt} + 1}.$$
From the above, we get
\[
q^{\text{odd}} \int_X \chi(x) \zeta^{x+nd} e^{(x+nd)t} d\mu_{-q^a}(x) + \int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^a}(x) = \frac{[2]_q \int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^a}(x)}{\int_{Z_p} \zeta^{ndx} q((nd-1)x) e^{ndtx} d\mu_q(x)}.
\]
(2.4)

By (2.2), (2.3), and (2.4), we arrive at the following theorem:

**Theorem 2.1** Let \( n \) be odd positive integer. Then we obtain
\[
\int_X \chi(x) \zeta^x e^{xt} d\mu_{-q^a}(x) = \sum_{m=0}^{\infty} \left( \frac{[2]_q}{[2]_q^m} T_{m,\chi,q,\zeta}(nd-1) \right) \frac{t^m}{m!}.
\]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By Theorem 2.1, and after some calculations, we have the following theorem.

**Theorem 2.2** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have
\[
\int_X \chi(x) \zeta^{w_1x_1 + w_2x_2} e^{w_2x_2} d\mu_{-q^a}(x) = \frac{[2]_q^{w_2}}{[2]_q} \sum_{m=0}^{\infty} T_{m,\chi,q,\zeta}^{(w_2)} (w_1 d - 1) w_2^m \frac{t^m}{m!}.
\]

Then we set
\[
S(w_1, w_2) = \int_X \int_X \chi(x_1) \chi(x_2) \zeta^{(w_1x_1 + w_2x_2)} e^{(w_1x_1 + w_2x_2)} d\mu_{-q^a}(x_1) d\mu_{-q^a}(x_2).
\]

By \( S(w_1, w_2) \) and Theorem 2.2, after calculations, we obtain
\[
S(w_1, w_2) = \left( \int_X \chi(x_1) \zeta^{w_1x_1} e^{w_1x_1} d\mu_{-q^a}(x_1) \right) \left( \int_X \chi(x_2) \zeta^{w_2x_2} e^{w_2x_2} d\mu_{-q^a}(x_2) \right)
\[
= \left( \sum_{m=0}^{\infty} E_{m,\chi,q,\zeta}^{(w_1)} (w_2 x_1) \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} E_{m,\chi,q,\zeta}^{(w_2)} (w_2 x_2) \frac{t^m}{m!} \right).
\]

By using Cauchy product in the above, we obtain
\[
S(w_1, w_2) = \sum_{m=0}^{\infty} \left( \frac{[2]_q^{w_2}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} E_{j,\chi,q,\zeta}^{(w_1)} (w_2 x_1) w_1^j T_{m-j,\chi,q,\zeta}^{(w_2)} (w_2 d - 1) w_2^m \frac{t^m}{m!} \right).
\]

From the symmetry of \( S(w_1, w_2) \) in \( w_1 \) and \( w_2 \), we also see that
\[
S(w_1, w_2) = \left( \int_X \chi(x_2) \zeta^{w_2x_2} e^{w_2x_2} d\mu_{-q^a}(x_2) \right) \left( \int_X \chi(x_1) \zeta^{w_1x_1} e^{w_1x_1} d\mu_{-q^a}(x_1) \right)
\[
= \left( \sum_{m=0}^{\infty} E_{m,\chi,q,\zeta}^{(w_2)} (w_1 x_1) \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} E_{m,\chi,q,\zeta}^{(w_1)} (w_2 x_2) \frac{t^m}{m!} \right).
\]
Thus we obtain
\[ S(w_1, w_2) = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^{j} E_{j, \chi, q, \zeta}^{(w_1)}(w_1 x) T_{m-j, \chi, q, \zeta}^{(w_1)}(w_2 d - 1) w_1^{m-j} \right) \frac{t^m}{m!}. \]

Thus we arrive at the following theorem:

**Theorem 2.3** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have
\[
[2]_{q^{w_1}} \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^{j} E_{j, \chi, q, \zeta}^{(w_1)}(w_1 x) T_{m-j, \chi, q, \zeta}^{(w_1)}(w_2 d - 1) = [2]_{q^{w_2}} \sum_{j=0}^{m} \binom{m}{j} w_1^{j} w_2^{m-j} E_{j, \chi, q, \zeta}^{(w_1)}(w_2 x) T_{m-j, \chi, q, \zeta}^{(w_2)}(w_1 d - 1),
\]

where \( E_{j, \chi, q, \zeta}^{(w_1)}(x) \) and \( T_{m-j, \chi, q, \zeta}^{(w_2)}(k) \) denote generalized twisted \( q \)-Euler polynomials with weak weight \( w_1 \) and \( p \)-adic functional, respectively.

By Theorem 2.3, we have the following corollary.

**Corollary 2.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
[2]_{q^{w_1}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^{j-k} E_{k, \chi, q, \zeta}^{(w_2)}(w_2 x) T_{m-j, \chi, q, \zeta}^{(w_1)}(w_1 d - 1) = [2]_{q^{w_2}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{j} w_2^{m-k} E_{k, \chi, q, \zeta}^{(w_1)}(w_1 x) T_{m-j, \chi, q, \zeta}^{(w_2)}(w_1 d - 1).
\]

Now we will obtain another interesting identities for generalized twisted \( q \)-Euler polynomials with weak weight using the symmetric property of \( S(w_1, w_2) \).

\[
S(w_1, w_2) = [2]_{q^{w_2}} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 j} \int_{X} \chi(x_1) \zeta^{w_1 x_1} e \left( \frac{x_1 + w_2 x + j w_2}{w_1} \right) \frac{w_2}{w_1} \, d\mu_{q^{w_1}}(x_1)
\]
\[
= \sum_{n=0}^{\infty} \left( [2]_{q^{w_2}} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 j} E_{n, \chi, q^{w_1}, \zeta}^{(w_1)} \left( w_2 x + j \frac{w_2}{w_1} \right) \frac{w_1^n}{n!} \right) \frac{t^n}{n!}.
\]

By using the symmetry property in the above equation, we also obtain
\[
S(w_1, w_2) = [2]_{q^{w_1}} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 j} \zeta^{w_2 x_2} e \left( \frac{x_2 + w_1 x + j w_1}{w_2} \right) \frac{w_1}{w_2} \, d\mu_{q^{w_2}}(x_2)
\]
\[
= \sum_{n=0}^{\infty} \left( [2]_{q^{w_1}} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 j} E_{n, \chi, q^{w_2}, \zeta}^{(w_2)} \left( w_1 x + j \frac{w_1}{w_2} \right) \frac{w_2^n}{n!} \right) \frac{t^n}{n!}.
\]
Thus we have the following theorem.

**Theorem 2.5** Let $w_1$ and $w_2$ be odd positive integers and $\zeta \in T_p$. Then we obtain

$$
[2]_{q^w_2} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} w_1^n E_n^{(w_1)}(w_2 x + j \frac{w_2}{w_1}) = [2]_{q^w_1} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} w_2^n E_n^{(w_2)}(w_1 x + j \frac{w_1}{w_2}).
$$

If we take $x = 0$ in Theorem 2.5, we also derive the interesting identity for generalized twisted $q$-Euler numbers with weak weight as follows:

**Corollary 2.6** Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

$$
[2]_{q^w_2} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} w_1^n E_n^{(w_1)}\left( \frac{j w_2}{w_1} \right) = [2]_{q^w_1} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} w_2^n E_n^{(w_2)}\left( \frac{j w_1}{w_2} \right).
$$

**References**


Received: January 19, 2016; Published: March 5, 2016