A Geometric Treatment of Mixed/Multiple Hexachordal Combinatoriality

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Abstract

A direct geometric treatment of mixed/multiple hexachordal combinatoriality is provided without the need to first generate all instances of hexachordal combinatoriality.

1 Introduction

A decidedly geometric approach was taken in [6] to the systematic exploration of all four types of hexachordal combinatoriality: prime (P), inversional (I), retrograde (R) and retrograde inversional (RI). (The reader is directed to [8, pp. 222-230] for requisite definitions and musical context.)

Now, some hexachords possess mixed/multiple combinatorial properties: either of more than one of the types P, I, R and RI or of a single type at more than one transpositional/inversional level (or both). Of course, such mixed/multiple hexachordal combinatorialities may be gleaned directly from the charts of each type of combinatoriality constructed in [6].

Instead, it is the express purpose of the present paper to construct ab initio all mixed/multiple combinatorial hexachords using elementary geometric arguments thereby obviating the need to generate a priori all hexachords possessing any mode of combinatoriality. Specifically, the geometric properties [4] of rotational symmetry, rotational antisymmetry, reflectional symmetry and reflectional antisymmetry will be employed to systematically generate all instances of hexachords possessing mixed/multiple combinatoriality.

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In essence, the paper before you forms a natural sequel to that of [6]. As such, the current reader is presumed to have first studied this prior work. Of particular utility to the present development is the following consequence of the geometric theory of symmetry groups therein derived. The order, \( N \), of the collection of symmetric/antisymmetric transformations associated with each type of hexachordal combinatoriality must be a divisor of 6. Thus, \( N = 1, 2, 3, 6 \).

The following fundamental results will also serve us well in the forthcoming investigation.

**Lemma 1 (Dihedral Group [9])** The product of two transpositions is a transposition and the product of two inversions is a transposition while the product of an inversion with a transposition is an inversion.

**Proof:**

\[
T_m \circ T_n = T_{m+n}, \\
T_m I \circ T_n I = T_{m-n}, \\
T_m \circ T_n I = T_{m+n} I, \\
T_m I \circ T_n = T_{m-n} I.
\]

\[\blacksquare\]

**Theorem 1 (Hexachordal Prime Forms)** All hexachords in prime form must open with one of the following five triadic patterns:

- \([0, 1, 2, \cdot, \cdot, \cdot]\)
- \([0, 1, 3, \cdot, \cdot, \cdot]\)
- \([0, 1, 4, \cdot, \cdot, \cdot]\)
- \([0, 2, 3, \cdot, \cdot, \cdot]\)
- \([0, 2, 4, \cdot, \cdot, \cdot]\)

**Proof:** See [5] for a straightforward geometric demonstration. \[\blacksquare\]

Before embarking upon our geometric exploration of mixed/multiple hexachordal combinatoriality, it will prove to be most propitious to first introduce some relevant terminology. The **order of combinatoriality**, \( N \), of a hexachord, \( H \), is the number of transpositional/inversional levels at which \( H \) is combinatorial [7, p. 118]. If \( N = 1 \) then the combinatoriality is **simple (unary)** whereas if \( N > 1 \) then the combinatoriality is **multiple**. In particular, \( N = 2, 3, 6 \) are referred to as **binary, ternary, senary** combinatoriality, respectively.
The degree of combinatoriality, \( M \), of a hexachord, \( H \), is the number of types of combinatoriality (P, I, R, RI) which \( H \) possesses. If \( M = 1 \) then the combinatoriality is pure (unitary) whereas if \( M > 1 \) then the combinatoriality is mixed. In particular, \( M = 2, 3 \) are referred to as bipartite, tripartite combinatoriality, respectively, whereas a hexachord with \( M = 4 \) is said to be all-combinatorial (abbreviated AC) [1].

It should be noted that the ensuing collection of theorems in large measure constitutes the codification of results scattered throughout the works of Milton Babbitt, though originally presented without proof [2].

2 Multiple Combinatoriality

![Multiple R-Combinatorial Hexachords](image)

Figure 1: Multiple R-Combinatorial Hexachords

Recall the following definitions. A hexachord \( H \) is said to be
• R-combinatorial if and only if it maps onto itself under some \( T_n \), i.e. if and only if it is rotationally symmetric. \((T_n : \ H \rightarrow H)\)

• P-combinatorial if and only if it maps onto its complement under some \( T_n \), i.e. if and only if it is rotationally antisymmetric. \((T_n : \ H \rightarrow \overline{H})\)

• RI-combinatorial if and only if it maps onto itself under some \( T_nI \), i.e. if and only if it is reflectionally symmetric. \((T_nI : \ H \rightarrow H)\)

• I-combinatorial if and only if it maps onto its complement under some \( T_nI \), i.e. if and only if it is reflectionally antisymmetric. \((T_nI : \ H \rightarrow \overline{H})\)

2.1 Multiple R-Combinatoriality

A hexachord \( H \) is R-combinatorial if and only if it maps onto itself under some \( T_n \), i.e. if and only if it is rotationally symmetric. If \( T_n \) is the identity
transformation, this R-combinatoriality is trivial as it is possessed by all hexachords. Thus, the consideration of multiple R-combinatoriality concerns the identification of such nontrivial $T_n$ where $n \neq 0$.

The geometric analysis of [6] reveals that instances of such nontrivial R-combinatoriality are exceedingly rare. The resultant four prime forms are on display in Figure 1. The first three are the all-combinatorial hexachords at multiple transpositional/inversional levels (6, 3 and 2, respectively) and they are collected for future reference in Figure 2 ($H_{114}, H_{13}, H_2$). The fourth is the only hexachord in prime form possessing only R- and I-combinatoriality at precisely two transpositional/inversional levels (in fact, it turns out to be the only hexachord with binary bipartite combinatoriality) and is isolated in Figure 3 ($H_{123}$) for future consideration. Note that multiple R-combinatoriality implies multiple I-combinatoriality. As will eventually become evident, the paucity of such multiple R-combinatoriality places severe restrictions upon the remaining instances of multiple combinatoriality.

### 2.2 Multiple P-Combinatoriality

**Theorem 2** Multiple P-combinatoriality implies multiple all-combinatoriality.

**Proof:** Suppose that $H$ is multiply P-combinatorial. Then,

\[ T_k : H \rightarrow \overline{H} \& T_l : H \rightarrow \overline{H} \; (k \neq l) \Rightarrow \]

$H$ maps onto itself under $T_k \circ T_k = T_{2k}$, $T_l \circ T_l = T_{2l}$ and $T_k \circ T_l = T_{k+l}$, which cannot all coincide. Therefore, $H$ is multiply R-combinatorial and, consequently, $H \in \{H_{114}, H_{13}, H_2\}$, since $H_{123}$ is not P-combinatorial. Thus, $H$ is multiply all-combinatorial. ■

### 2.3 Multiple I-Combinatoriality

**Theorem 3** Multiple I-combinatoriality implies multiple R-combinatoriality.

**Proof:** Suppose that $H$ is multiply I-combinatorial. Then,

\[ T_kI : H \rightarrow \overline{H} \& T_lI : H \rightarrow \overline{H} \; (k \neq l) \Rightarrow T_kI \circ T_lI = T_{k-l} : H \rightarrow H \Rightarrow \]

$H$ is R-combinatorial at level $k - l \neq 0 \Rightarrow H \in \{H_{114}, H_{13}, H_2, H_{123}\}$. Thus, $H$ is multiply R-combinatorial. ■
2.4 Multiple RI-Combinatoriality

Theorem 4 Multiple RI-combinatoriality implies multiple all-combinatoriality.

Proof: Suppose that \( H \) is multiply RI-combinatorial. Then,

\[
T_k I : H \rightarrow H \ & T_l I : H \rightarrow H \ (k \neq l) \Rightarrow T_k I \circ T_l I = T_{k-l} : H \rightarrow H \Rightarrow
\]

\( H \) is R-combinatorial at level \( k - l \neq 0 \Rightarrow H \in \{H_{123}, H_{13}, H_2\} \), since \( H_{123} \) is not RI-combinatorial. Thus, \( H \) is multiply all-combinatorial. ■

2.5 Summary of Multiple Combinatoriality

Corollary 1 The only hexachords possessing binary combinatoriality are \{\( H_{114}, H_{123} \)\}, the former being 2nd order all-combinatorial while the latter possesses only binary R- and I-combinatoriality. The nontrivial R-combinatoriality of both hexachords occurs under the half-turn \( T_6 \).

In summary:

- Multiple P-combinatoriality implies multiple all-combinatoriality.
- Multiple I-combinatoriality is equivalent to multiple R-combinatoriality.
- Multiple RI-combinatoriality implies multiple all-combinatoriality.
- There is only one prime form, \( H_{123} \), that is both multiply R- and I-combinatorial yet is not all-combinatorial.
- Multiple combinatoriality is equivalent to multiple R-combinatoriality (is equivalent to multiple I-combinatoriality) (Figure 1).

3 Simple Combinatoriality

Theorem 5 There are no hexachords with tripartite combinatoriality:

1. P-combinatoriality + I-combinatoriality ⇒ RI-combinatoriality.
2. I-combinatoriality + RI-combinatoriality ⇒ P-combinatoriality.
3. RI-combinatoriality + P-combinatoriality ⇒ I-combinatoriality.

Proof:
Mixed/multiple hexachordal combinatoriality

1. $T_k : \overline{H} \rightarrow H \& T_l : H \rightarrow \overline{H} \Rightarrow T_k \circ T_l I = T_{k+l} I : H \rightarrow H$.

2. $T_k I : H \rightarrow \overline{H} \& T_l I : H \rightarrow H \ (l \neq k) \Rightarrow T_k I \circ T_l I = T_{k-l} I : H \rightarrow \overline{H}$.

3. $T_k I : H \rightarrow H \& T_l : H \rightarrow \overline{H} \Rightarrow T_l \circ T_k I = T_{k+l} I : H \rightarrow \overline{H}$.

As a consequence of this theorem, any simple combinatoriality must be all-, bipartite or pure.

### 3.1 Simple All-Combinatoriality

![Figure 4: Simple All-Combinatorial Hexachords](image)

**Theorem 6** If $H$ is simple P-combinatorial at transpositional level $k$ ($T_k : H \leftrightarrow \overline{H}$), simple I-combinatorial at inversional level $l$ ($T_l I : H \leftrightarrow \overline{H}$) and simple RI-combinatorial at inversional level $m$ ($T_m I : \overline{H} \rightarrow \overline{H}$), then $k \equiv_{12} 6$, $l$ is odd and $m \equiv_{12} l + 6$ (so that $m$ is also odd).

**Proof:**

$$T_l I \circ T_k = T_{l-k} I : H \rightarrow H \Rightarrow m \equiv_{12} l - k,$$

since $H$ is simple RI-combinatorial.

$$T_m I \circ T_k = T_{m-k} I : H \rightarrow \overline{H} \Rightarrow l \equiv_{12} m - k,$$

since $H$ is simple I-combinatorial.

$$T_m I \circ T_l I = T_{m-l} I : H \rightarrow \overline{H} \Rightarrow k \equiv_{12} m - l,$$

since $H$ is simple P-combinatorial.

$$\therefore \ l - k \equiv_{12} m \equiv_{12} l + k \Rightarrow 2m \equiv_{12} 2l \Rightarrow m \equiv_{12} l + 6 \Rightarrow k \equiv_{12} 6.$$
Figure 5: All-Combinatorial Hexachords

Since \( T_l I : H \leftrightarrow \bar{H} \) can have no fixed points, \( l \) must be odd. Otherwise, the two pitch-classes at the endpoints of the axis of reflection would be fixed points of the associated inversion.

This theorem may be used to find all hexachords possessing simple all-combinatoriality as follows. For each of the five possible opening triadic patterns of a hexachord in prime form (Theorem 1):

- Exclude the half-turn images of this opening triad. (P-combinatoriality under \( T_6 \))
- Attempt to fill in the remainder of the hexachord \( H \) so that it is in prime form and possesses reflectional symmetry under \( T_m I \) (\( m = 1, 3, 5, 7, 9, 11 \)). (RI-combinatoriality under \( T_m I \))
- By Theorem 6, each such \( H \) will automatically possess I-combinatoriality under \( T_l I, l \equiv_{12} m - 6 \).

The resulting 3 simple all-combinatorial hexachords are shown in Figure 4. Combining Figures 2 and 4, we obtain the complete collection of all-combinatorial hexachords appearing in Figure 5.
3.2 Simple Bipartite Combinatoriality

Figure 6: P-Combinatorial Hexachords

Figure 6 portrays the 7 hexachords possessing P-combinatoriality which were constructed geometrically in [6]. By removing the 6 all-combinatorial hexachords we arrive at the 1 hexachord with simple bipartite P-combinatoriality of Figure 7.

Figure 8 portrays the 19 hexachords possessing I-combinatoriality which were constructed geometrically in [6]. By removing the 4 multiple R-combinatorial hexachords and the 3 simple all-combinatorial hexachords we arrive at the 12 hexachords possessing simple bipartite I-combinatoriality of Figure 9.

Figure 10 portrays the 20 hexachords possessing RI-combinatoriality which were constructed geometrically in [6]. By removing the 6 all-combinatorial hexachords we arrive at the 14 hexachords possessing simple bipartite RI-combinatoriality of Figure 11.
3.3 Simple Pure (Trivial) Combinatoriality

The 16 hexachords possessing simple pure R-combinatoriality were derived geometrically in [6]. They appear here in Figure 12 for the sake of completeness.

4 Conclusion

<table>
<thead>
<tr>
<th>$M$</th>
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<th>$N = 2$</th>
<th>$N = 3$</th>
<th>$N = 6$</th>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Simple Bipartite P(1),I(12),RI(14)</td>
<td>Binary Bipartite I(1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>4</td>
<td>1st Order AC (3)</td>
<td>2nd Order AC (1)</td>
<td>3rd Order AC (1)</td>
<td>6th Order AC (1)</td>
</tr>
</tbody>
</table>

Table 1: Modes of Hexachordal Combinatoriality

The preceding geometric classification of the modes of hexachordal combinatoriality may be summarized as in Table 1. While the most notable feature of this table is surely the blank entries indicating impossible modes of combinatoriality, one is struck by the observation that the order of combinatoriality of each combinatorial type present in an instance of mixed combinatoriality is always the same.

This is a direct consequence of the fact that the collection of transformations of a hexachord $H$ associated with each type of P-, I- and RI-combinatoriality is entirely determined by the group of transformations associated with the R-combinatoriality of $H$ [6]. Specifically:

- The group of rotational symmetries of $H$ must be one of the following subgroups of the cyclic group $C_{12}$ [3, p. 283]:
  \[
  \{T_0\}
  \]
Figure 8: I-Combinatorial Hexachords

\{T_0, T_6\}
\{T_0, T_4, T_8\}
\{T_0, T_2, T_4, T_6, T_8, T_{10}\};

the order of the subgroup coinciding with the order of R-combinatoriality.

- If \(H\) is P-combinatorial then the corresponding collection of rotations carrying \(H\) onto its complement is the product of the respective subgroup of rotations with the square root of its group generator:

\{T_0\}
\{T_3, T_9\}
\{T_2, T_6, T_{10}\}
\{T_1, T_3, T_5, T_7, T_9, T_{11}\}. 
• If $H$ is RI-combinatorial then the group of rotational and reflectional symmetries of $H$ must be one of the following subgroups of the dihedral group $D_{24}$ [3, p. 446]:

\[
\{T_0, T_k I\} \\
\{T_0, T_6, T_k I, T_{k+6} I\} \\
\{T_0, T_4, T_8, T_k I, T_{k+4} I, T_{k+8} I\}
\]

\[
\{T_0, T_2, T_4, T_6, T_8, T_{10}, T_k I, T_{k+2} I, T_{k+4} I, T_{k+6} I, T_{k+8} I, T_{k+10} I\},
\]

with the permissible values of $k$ determined by the constraint that $H$ appear in prime form.

• If $H$ is I-combinatorial then the corresponding collection of reflections carrying $H$ onto its complement is the product of the respective subgroup of rotations with an inversion of odd suffix:

\[
\{T_{2k+1} I\}\]
Mixed/multiple hexachordal combinatoriality

Figure 10: RI-Combinatorial Hexachords

\[
\{T_{2k+1}I, T_{2k+7}I\} \\
\{T_{2k+1}I, T_{2k+5}I, T_{2k+9}I\} \\
\{T_{1}I, T_{3}I, T_{5}I, T_{7}I, T_{9}I, T_{11}I\},
\]

with the permissible values of $k$ determined by the constraint that $H$ appear in prime form.

In any event, the order of any type of combinatoriality of $H$ is always equal to the order of R-combinatoriality of $H$. Thus, the orders of combinatoriality of all combinatorial types present within $H$ are always identical.

Taken together with the geometric syntheses of [6], the preceding geometric cogitations provide a thorough geometric treatment of all salient aspects of hexachordal combinatoriality. The naturalness of this geometric perspective on hexachordal combinatoriality together with its attendant systematization stands in sharp contrast to the traditional ad hoc algebraic approach [7, 8]. This is not at all surprising since, although possessing both algebraic as well
as geometric facets, combinatoriality is primarily a geometric property of hexachords.
Figure 12: Simple Pure (Trivially) R-Combinatorial Hexachords

References


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