Idempotent and Inverse Elements in
Strong Ternary Semirings

Rabah Kellil

College of Science at Al-Zulfi, Majmaah University, Saudi Arabia
&
Faculty of Sciences of Monastir Tunisia

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Abstract

In continuation of a previous works on semirings [5], in the present paper we introduce the notions of a strong ternary semiring (ST-semiring) (that is a ternary semirings with an additional condition called the left invertive law). We prove that many results obtained in [5] for semirings are still valid in the present case. We establish some relationships between the idempotents for both the addition and the multiplication. We prove in the case of ST-semiring, that the set of multiplicative idempotent; $E^*(S)$ is closed under the multiplication and so $(S, +, .)$ is an orthodox ST-semiring.

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1 Introduction and Preliminaries

The ring of integers $\mathbb{Z}$ has a great role in the theory of rings and as we can remark the two subsets $\mathbb{Z}^+$ and $\mathbb{Z}^-$ have distinct nature. They are both semigroups but the first is closed under the ring product and the second is not. If
we define the triple product; we then make \( \mathbb{Z}^- \) closed under this product. So ternary semirings arise naturally.

The ternary operations are used to study the static hazards in combinatorial switching circuits by means of a suitable ternary switching algebra. The ternary operations appear also in the study of Quark model to explain the non-observability of isolated quarks as a phenomenon of algebraic confinement [8].

In this paper and in continuation of our previous works on semirings [5], we introduce the notions of a strong ternary semiring (ST-semiring) (that is a ternary semirings with an additional condition called the left invertive law). We prove that many results obtained in [5] for semirings are still valid in the present case. We establish some relationships between the idempotents for both the addition and the multiplication. We prove in the case of ST-semiring, that the set of multiplicative idempotent; \( E^*(S) \) is closed under the multiplication and so \( (S, +, \cdot) \) is an orthodox ST-semiring.

As pointed in [5]; the references [9], [3], [4], [11] and [12] can be used as a background for the subject.

**Definition 1.1.** Let \( S \) be a non-empty set endowed with a binary operation "+" and a ternary operation "\(*\)". \( S \) is called an ST-semiring (i.e. strong ternary semiring) if:

1. \( (S, +) \) is a semigroup,
2. \( \forall x, y, z, t, v \in S; \quad (x \cdot y \cdot z) \cdot t \cdot v = x \cdot (y \cdot z \cdot t) \cdot v = x \cdot y \cdot (z \cdot t \cdot v), \)
3. \( \forall x, y, z, t \in S; \quad x \cdot (y + z) \cdot t = x \cdot y \cdot t + x \cdot z \cdot t, \)
4. \( \forall x, y, z, t \in S; \quad x \cdot y \cdot (z + t) = x \cdot y \cdot z + x \cdot y \cdot t \)
5. \( \forall x, y, z, t \in S; \quad (z + t) \cdot x \cdot y = z \cdot x \cdot y + t \cdot x \cdot y, \)
6. \( x \cdot y \cdot z = z \cdot y \cdot x \) for all \( x, y, z \in S \); called the left invertive law.

**Definition 1.2.**

1. A non-empty subset \( U \) of a ST-semiring \( S \) is said to be a ST-subsemiring \( S \) if \( (U, +) \) is subsemigroup and \( U \cdot U \cdot U \subset U \).
2. A left (resp. right) additive ideal \( I \) of a ST-semiring \( S \) is a non-empty subset \( I \) of \( S \) such that \( I + S \subset I \) (resp. \( S + I \subset I \)).
3. A left (resp. right, lateral) multiplicative ideal \( I \) of a ST-semiring \( S \) is a non-empty subset \( I \) of \( S \) such that \( I \cdot S \cdot S \subset I \) (resp. \( S \cdot I \cdot S \subset I \)).
4. A left (resp. right, lateral) ideal $I$ of an ST-semiring $S$ is a non-empty subset $I$ of $S$ such that $I + I \subset I$ and $I \ast S \ast S \subset I$ (resp. $S \ast S \ast I \subset I$, $S \ast I \ast S \subset I$).

5. If $I$ is both a left and a right ideal (resp. additive ideal, multiplicative ideal) of an ST-semiring $S$, then we say that $I$ is an ideal (resp. additive ideal, multiplicative ideal) of $S$.

6. If $I$ is such that $S + I \subset I, I + S \subset I, S \ast S \ast I \subset I, I \ast S \ast S \subset I, S \ast I \ast S \subset I$, we say that $I$ is a strong two sided ideal of $S$.

**Definition 1.3**.

An element $a$ of a ST-semiring $S$ is multiplicative regular if there exists $a^*$ in $S$ such that

$$a \ast a^* \ast a = a; \quad \text{and} \quad a^* \ast a \ast a^* = a^*.$$

The element $a^*$ is called a multiplicative inverse of $a$.

**Definition 1.4**.

An element $a$ of a semigroup $(S, +)$ is called regular if there exists $a'$ in $S$ such that

$$a + a' + a = a; \quad \text{and} \quad a' + a + a' = a'.$$

The element $a'$ is called an additive inverse or simply an inverse of $a$ in the semigroup $S$.

**Definition 1.5**.

An additive (resp. multiplicative) inverse ST-semiring $S$ is a ST-semiring such that any element of $S$ has a unique additive (resp. multiplicative) inverse.

**Definition 1.6**.

Let $S$ be an additive inverse ST-semiring and $x'$ denotes the unique inverse of the element $x$. We say that $S$ satisfies the conditions (A), (B) or (C) if for all $a, b \in S$

1. (A) $a \ast (a + a') \ast a = a + a'$,
2. (B) $a \ast a \ast (b + b') = (b + b') \ast a \ast a = a \ast (b + b') \ast a$,
3. (C) $a + a \ast (b + b') \ast a = a$.

**Definition 1.7**.

Let $S$ be a ST-semiring, we denote by $E^+(S) = \{a \in S \mid a + a = a\}$ the set of additive idempotents and by $E^*(S) = \{e \in S \mid e \ast e \ast e = e\}$ the set of multiplicative idempotents.

Note that $E^+(S)$ is a multiplicative ideal of $S$.

In the sequel the binary operation $\ast$ will be simply denoted by ",".
2 Inverses in a ST-semiring

Lemma 2.1 Any ST-semiring S satisfies the medial law: for all \(a_i, b_i, c_i \in S\);
\[(a_1.a_2.a_3).(b_1.b_2.b_3).(c_1.b_1.c_3) = (a_1.b_1.a_3).(a_2.b_2.c_2).(c_1.b_3.c_3).\]

Proof. For all \(a_i, b_i, c_i \in S\) one has
\[(a_1.a_2.a_3).(b_1.b_2.b_3).(c_1.c_2.c_3) = a_1.[a_2.a_3(b_1.b_2.b_3)].(c_1.c_2.c_3) =\]
\[(a_1.a_2.a_3).(b_1.a_3.b_2.b_3).(c_1.c_2.c_3) = a_1.[(b_1.a_3.a_2).b_2.b_3].(c_1.c_2.c_3) =\]
\[(a_1.b_1.a_3).(a_2.b_2.b_3).(c_1.c_2.c_3) = (a_1.b_1.a_3).[(a_2.b_2.b_3).c_1.c_2].c_3 =\]
and the result follows. ■

Proposition 2.2 Let S be an additive inverse ST-semiring.

1. If \(e \in E^*(S)\) then \(e' \in E^*(S)\) and \(e + e' \in E^+(S)\).

2. \(E^*(S).E^*(S).E^*(S) \subset E^*(S)\). in this case S is called a ST-orthodox semiring.

Proof.

1. \(e'.e'.e' = e'.e'.(e' + e + e')' = e'.e'.e' + e'.e'.e + e'.e'.e' + e'.e'.e = e'.e'(e + e' + e) = e'.e'e\) so \(e'.e'.e\) is an additive inverse of \(e'.e'.e'\) and since S is an additive inverse ST-semiring; \((e'.e'.e')' = e'.e'.e\). In another hand \(e'.e'.e = e'.(e' + e + e')e = e'.e'.e + e'.e.e + e'.e.e\) and then \((e'.e'.e)' = e'.e.e\). Using the unicity; \((e'.e'.e)' = e'.e'.e = (e'.e.e)'\). Finally using the same expansion we get \((e.e.e)' = e'.e'.e'\) but as \(e \in E^*(S)\);
\((e + e') + (e + e') = (e + e' + e) + e' = e + e'.\)

2. Let \(e, f, g \in E^*(S)\). From the lemma 2.1;
\[(e.f.g).(e.f.g).(e.f.g) = (e.e.g).(f.f.f).(e.g.g) = (e.e.g).(f.f.f).(e.g.g) =\]
\[(e.e.g).f.(e.g.g) = e.e[(g.f.e).g.g] = e.e[(e.f.g).g.g] = (e.e.e).f.(g.g.g) = e.f.g;\]
and then \(e.f.g \in E^*(S)\). ■
Proposition 2.3 Let $S$ be a ST-semiring. If $a', b', c' \in S$ denote additive inverses of $a, b$ and $c$ then $a'b'c'$ is an additive inverse of $a'b'c'$.

Proof. $a'b'.c' + a'b'.c' + a'b'.c = a'b.(c + c' + c) = a'b'.c$ and $a'b'.c' + a'b'.c' + a'b'.c' = a'b'(c' + c + c') = a'b'.c'$ and the conclusion follows. ■

Corollary 2.4 Let $S$ be an additive inverse ST-semiring. Then for any elements $a, b$ and $c$ in $S$;

1. $a'b'.c = a'b'.c' = a'b'.c$

2. $(a,b,c)' = a'b'.c = a'b'.c

Proof.

1. As made in the previous lemma we prove that $a'b'.c'$ and $a'b'.c'$ are also two additive inverses of $a'b'.c'$ and the conclusion follows from the uniqueness of additive inverse of any element of $S$.

2. The proof is trivial.

■

Corollary 2.5 Let $S$ be an additive inverse ST-semiring. For any elements $x$ in $S$, the following equalities hold

1. $(x')^*(x)^*$.

2. $(x + x' + x)^* = x^* + (x')^* + x^*$

Proof.

1. $(x')^*.x'.(x^*)' = [[[x^*.x^*]'']]' = [[[x^*]'']]' = [x^*]'$ by using the uniqueness.

   In the other hand $x'.[x^*]' = [[[x^*.x^*]'']]' = [[[x^*]'']]' = x'$ so the result follows.

2. $(x + x' + x)^* = x^* = x^* + (x')^* + x^*$ from the previous equality.

■

Proposition 2.6 1. If a ST-semiring $S$ satisfies the conditions (A) and (C), then for any additive inverse $a'$ of $a \in S$, we have $a + a + a' = a$.

2. If in addition $a \in E^*(S)$ and is has a multiplicative inverse then $3a = a$.

Proof.
1. From (C), one has \(a + a(a + a')a = a\). But from (A) as \(a(a + a').a = a + a'\), we deduce that \(a + a(a + a').a = a + (a + a') = a + a + a'\) and then \(a + a + a' = a\).

2. From (C) we also have \(a = a + a(a + a^*)a \iff a + a.a + a.a^*.a = a + a.a + a = a + a + a\) since \(a.a.a = a\) and then \(3a = a\).

**Definition 2.7**

Let \(S\) be a ST-semiring and \(A\) a subset of \(S\). We define,

\[
A^l = \{y \in S \mid yzt \in E^+(S), \forall z, t \in A\}
\]

and

\[
A^r = \{y \in S \mid zty \in E^+(S), \forall z, t \in A\}.
\]

**Remark 2.8** It is easy to show that \(E^+(S) \subset A^l \cap A^r\).

**Definition 2.9**

A ST-semiring \(S\) is said to be a cyclic ST-semiring if in the definition 1.1 the condition (6) is replaced by

\[(6')\quad a.b.c = b.c.a.\]

**Proposition 2.10** Let \(S\) be a cyclic ST-semiring. If \(I\) is a left ideal, then

\[S.S.I^l \subset I^r, \quad I^r.S.S \subset I^l.\]

**Proof.** Let \(a \in I^l, s, s' \in S\) and \(x, y \in I\).

\[x.y.(s.s'.a) = a.x.(y.s.s') = a.x.(s.s'.y).\]

But since \(I\) is a left ideal \(s.s'.y \in I\) and as \(a \in I^l, x \in I\) then \(a.x.(y.s.s') \in E^+(S)\) and so \(s.s'.a \in I^r\).

Now let \(a \in I^r, s, s' \in S\) and \(x, y \in I\).

\[(a.s.s').x.y = x.y.(a.s.s') = (x.y.a).s.s' = [x.y.a+x.y.a].s.s' = (a.s.s').x.y+(a.s.s').x.y\]

since \(a \in I^r\) and then \(x.y.a \in E^+(S)\). Finally \((a.s.s') \in I^l\).

**Proposition 2.11** Let \(S\) be a cyclic ST-semiring. If \(+\) is commutative and \(I\) is an ideal then \(I^l \cap I^r\) is an ideal.
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**Proof.** Let $y \in I^l \cap I^r$, $s, s' \in S$ and $a, b \in I$.

$$(y.s.s').a.b + (y.s.s').a.b = a.b.(y.s.s') + a.b.(y.s.s') = (a.b.y).s.s' + (a.b.y).s.s' =$$

$$(a.b.y + a.b.y).s.s' = (a.b.y).s.s' = a.b.(y.s.s') = (y.s.s').a.b;$$

since $a.b.y \in E^+(S)$. Then $y.s.s' \in I^l$. \hspace{1cm} (A)

$$a.b.(y.s.s') = (y.s.s').a.b = (s.s'a).b.y = s.s'(a.b.y) = s.s'(a.b.y + a.b.y)$$

since $aby \in E^+(S)$. So $a.b.(y.s.s') \in E^+(S)$ and then $y.s.s' \in I^r$. \hspace{1cm} (B)

In the other hand, as for all $y \in I^l \cap I^r$, $s, s' \in S$ one has $s.s'y = y.s.s'$ and using the previous facts we get

$$S.S.(I^l \cap I^r) = (I^l \cap I^r).S.S' \subset I^l \cap I^r.$$

Now let $a, b \in I^l \cap I^r$, $x, y \in I$. Then $(a + b).x.y + (a + b).x.y = a.x.y + b.x.y + a.x.y + b.x.y = (a.x.y + a.x.y) + (b.x.y + b.x.y) = a.x.y + b.x.y$ since $a, b \in I^l$ and so $(a + b) \in I^l$.

With the same arguments we can easily prove that $(a + b) \in I^r$.

Finally

$$I^l \cap I^r + I^l \cap I^r \subset I^l \cap I^r.$$

\[\rule{1cm}{0.1cm}\]

**Remark 2.12** It is clear that if in addition; $S$ has a multiplicative identity say 1 then

$$S.S.(I^l \cap I^r) = (I^l \cap I^r).S.S' = I^l \cap I^r.$$

**Proposition 2.13** Let $S$ be a multiplicative inverse ST-semiring. If $a \in E^+(S)$ then $a^* = a$ and so $a^* \in E^+(S)$.

**Proof.** For every $a \in E^+(S)$, we have:

$$(a.a.a).(a^*.a.a).(a.a.a) = a.a[(a.a.a).a.(a.a.a)] = a.a[a.a.(a.a.a)] = a.a.[a.a.a] = a.a.a.$$ 

In the other hand;


so the multiplicative inverse of $a.a.a$ is $a^*.a.a$ but as $a.a.a = a$ and $a^*$ is the unique multiplicative inverse of $a$ then $a^* = a^*.a.a$.

With the same considerations we can easily prove that $a.a^*.a$ and $a.a.a^*$ are multiplicative inverses of $a.a.a$ and by the uniqueness of any multiplicative inverse we get $(a.a.a)^* = a.a^*.a = a.a.a^* = a^*.a.a$. But $a.a^*.a = a$ and then $a^* = a.a^*.a = a$. 

\[\rule{1cm}{0.1cm}\]
**Proposition 2.14** Let $S$ be a ST-semiring. Then for any permutation $\delta$ of the set $\{e, f, g\}$ and any $e, f, g \in E^*(S)$ such that $f.e.f = e.f.f$, we have $e.f.g = \delta(e)\delta(f)\delta(g)$

**Proof.** From the lemma 2.2 it is clear that $e.f.g \in E^*(S)$.

Now as $S_3$ can be generated by the transpositions $\tau_{1,2}$ and $\tau_{1,3}$ and since the condition 6) in the definition 1.1 gives $e.f.g = \tau_{1,3}(e)\tau_{1,3}(f)\tau_{1,3}(g)$ it suffices to prove that $e.f.g = \tau_{1,2}(e)\tau_{1,2}(f)\tau_{1,2}(g)$ that is $e.f.g = f.e.g$. Indeed

$$e.f.g = (e.f.g)(e.f.g)(e.f.g) = (f.e.f)(e.f.g)(e.f.g) = (f.e.e)(f.f.g)(e.f.g) = f.(f.f.g)(e.f.g)(e.f.g) = f.f.(f.f.g)(e.f.g) = f.f.(f.f.g)(e.f.g) = f.f.(f.f.g)(e.f.g) = f.f(f.f.e) = g.f.f$$

\[\blacksquare\]

**Corollary 2.15** Let $S$ be a ST-semiring with zero and $E^*(S)$ of characteristic 3 (that is $3x = 0 \quad \forall x \in E^*(S)$) and such that $e.f.g = \tau_{1,2}(e)\tau_{1,2}(f)\tau_{1,2}(g)$ for any elements $e, f$ and $g$ of $E^*(S)$, then $E^*(S)$ is a subsemiring.

**Proof.** Since $E^*(S)$ is closed with respect to the action of $\tau_{1,2}$ then it is a subsemiring.

\[\blacksquare\]

**Proposition 2.16** Let $S$ be a ST-semiring if each element has a multiplicative inverse then $\forall a, b, c \in S; c^*b^*a^*$ is a multiplicative inverse of $a \cdot b \cdot c$, where $a^*, b^*, c^*$ are respectively some multiplicative inverses of $a, b$ and $c$.

**Proof.** If $a, b, c \in S$ and $a^*, b^*, c^*$ are as required, then $(a \cdot b \cdot c \cdot a^* \cdot b^* \cdot c^*) = a \cdot b \cdot c \cdot (c^* \cdot b^* \cdot a^*) \cdot (b^* \cdot a^*) \cdot (a \cdot b \cdot c)$

\[\begin{align*}
    a \cdot b \cdot c &\cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] = a \cdot b \cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] \\
    a \cdot b &\cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] = a \cdot b \cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] \\
    a \cdot b &\cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] = a \cdot b \cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] \\
    a \cdot b &\cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)] = a \cdot b \cdot [c \cdot b \cdot (b^* \cdot a^* \cdot a)]
\end{align*}\]

In the other hand by replacing $a, b, c, a^*, b^*, c^*$ respectively by $c^*, b^*, a^*, c, b, a$ in the previous relation we get

$$(c^* \cdot b^* \cdot a^*) \cdot (a \cdot b \cdot c) = c^* \cdot b^* \cdot a^*$$

so $c^* \cdot b^* \cdot a^*$ is an inverse of $a \cdot b \cdot c$.

\[\blacksquare\]
**Definition 2.17** Let $x$ be an element of a ST-semiring $S$, we define the powers of $x$ as:
\[ x^3 = x.x.x, \quad x^5 = (x^3).x.x, \quad x^{2n+1} = (x^{2n-1}).x.x \quad \forall n \geq 1. \]

**Proposition 2.18** Let $S$ be a ST-semiring. If $x^*$ is an inverse of $x$ then
\[ x^*.x^*(x^3) = x.x.x^* \quad \text{and} \quad \forall n \geq 3; x^*.(x^{2n+1}) = x^{2n-1}. \]

**Proof.** For all $x \in S$ we have:
\[ x^*.x^*(x^3) = (x^*.x^*).x.x = (x.x^*).x.x = x.x^*.(x.x^*) = (x.x^*).x.x^* = x.x.x^*. \]

In the other hand:
\[ x^*.x^*(x^5) = (x^*.x^*(x^3)).x.x = (x.x.x^*).x.x = x.(x.x^*).x.x = x.x.x = x^3 \]

and so by induction for any $n \geq 2$:
\[ x^*.x^*(x^{2n+1}) = (x^*.x^*(x^{2n-1})).x.x = (x^{2n-3}).x.x = x^{2n-1}. \]

**Proposition 2.19** Let $S$ be a ST-semiring. If $x^*$ is an inverse of $x$ and $n$ is such $n \geq 1$; then
\[ x \in E^+(S) \implies x^3 \in E^+(S) \iff x^{2n+1} \in E^+(S). \]

**Proof.** The first implication is trivial.

For the equivalence; since $x^3 \in E^+(S)$ and $x^{2n+3} + x^{2n+3} = (x^{2n+1} + x^{2n-1}).x.x$; the direct implication can then be done by induction on $n \geq 1$.

The converse can be done by a decreasing induction:

In one hand we have $x^{2n+1} \in E^+(S)$ in the other hand, suppose that $x^{2n-p} \in E^+(S)$ \forall 1 \leq p \leq 2n - 5 then;
\[ x^{2n-p-2} + x^{2n-p-2} = x^*.x^*(x^{2n-p}) + x^*.x^*(x^{2n-p}) = x^*.x^*(x^{2n-p} + x^{2n-p}) = x^*.x^*(x^{2n-p-2}) \]

So by taking $p = 2n - 5$ we get $x^3 \in E^+(S)$.

**Proposition 2.20** If $S$ is a cyclic ST-semiring then:

1. for all $n \geq 1, x, y$ and $z \in S$;
\[ (x^{2n+1}).(y^{2n+1}).(z^{2n+1}) = (x.y.z).(x.y.z)(x^{2n-1}.z^{2n-1}.y^{2n-1}). \quad (D) \]

2. If $x^*$ is a multiplicative inverse of $x$ then $(x^*)^{2n+1}$ is a multiplicative inverse of $x^{2n+1}$.

**Proof.**
1. Since $S; \ a^{2n+1} = a^{2n-1}a.a = a.a.a^{2n-1}$ then
\[ x^{2n+1}.y^{2n+1}.z^{2n+1} = \{x.x.x^{2n-1}\} (y.y.y^{2n-1}).(z.z.z^{2n-1}) = \]
\[ x.x.\{x^{2n-1}.(y.y.y^{2n-1}).(z.z.z^{2n-1})\} = x.x.\{x^{2n-1}.y.y^{2n-1}.(z.z.z^{2n-1})\} = \]
\[ x.x.\{y.x^{2n-1}.y.y^{2n-1}.(z.z.z^{2n-1})\} = (x.y.x).y^{2n-1}.x^{2n-1}.(z.z.z^{2n-1}) = (x.y.x).y.\{y^{2n-1}.x^{2n-1}.z\}.z.$z^{2n-1} = \]
\[ (x.y.x).y.\{z.z^{2n-1}.(y^{2n-1}.x^{2n-1}.z)\} = (x.y.x).y.z^{2n-1}.y^{2n-1}.x^{2n-1}.z = \]
\[ ((x.y.x).y.z).z^{2n-1}.(y^{2n-1}.x^{2n-1}) = [x.y.(z.x.y)].z^{2n-1}.(z.y^{2n-1}.x^{2n-1}) = \]
\[ [(x.y.z).x.y].z^{2n-1}.(z.y^{2n-1}.x^{2n-1}) = (x.y.z)\] \[x.y.z^{2n-1}.y^{2n-1}.x^{2n-1} = ((x.y.z).x.y).y^{2n-1}.x^{2n-1} = \]
\[ z.(x.y.z).x.y$.z^{2n-1}.y^{2n-1}.x^{2n-1} = z.((x.y.z).x.y).z^{2n-1}.y^{2n-1}.x^{2n-1} = \]
\[ (x.y.z).z.\{x.y.\{z^{2n-1}.y^{2n-1}.x^{2n-1}\}\} = (x.y.z).z.x.y.\{z^{2n-1}.y^{2n-1}.x^{2n-1}\} = \]
\[ (x.y.z).z.x.y.\{z^{2n-1}.y^{2n-1}.x^{2n-1}\} = (x.y.z).(x.y.z).z^{2n-1}.y^{2n-1}.x^{2n-1} = \]
\[ (x.y.z).(x.y.z).z^{2n-1}.y^{2n-1}.x^{2n-1} = (x.y.z).(x.y.z).(x^{2n-1}.z^{2n-1}.y^{2n-1}).\]

2. By induction on $n \geq 1$ and by replacing in the equality (D), y, z respectively by $x^*, x$ we have

For $n = 1; \ x^3(x^*)^3.x^3 = (x.x^*.x).\{x.x^*.x\}.x.x^*.x = x.x.\{x.x^*.x\} = x.x.x = x^3.$

Suppose that $x^{2n-1}.(x^*)^{2n-1}.x^{2n-1} = x^{2n-1}$, then
\[ (x^{2n+1}).\{(x^*)^{2n+1}\} = (x.x^*).\{x.x^*\}.(x^{2n-1}.x^{2n-1}).\{(x^*)^{2n-1}\} = \]
\[ x.x.\{x^{2n-1}.(x^*)^{2n-1}.x^{2n-1}\} = x.x.(x^{2n-1}) = x^{2n-1}.x.x = x^{2n+1}.\]

**Proposition 2.21** Any multiplicative inverse of an element of a ST-semiring is unique.

**Proof.** Let $a, b$ be two multiplicative inverses of an element $x \in S$; then
\[ a = a.x.a = (x.b.x).a = a.x.(b.x.a) = a.x.(a.x.b) = (a.x.a).x.b = a.x.b \]

and by inverting $a$ in $b$ and vice-versa we get $b = b.x.a$ and since $a.x.b = b.x.a$ we get the uniqueness of the multiplicative inverse. ■

**References**

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