On Locally Multiplication Modules

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Abstract

The main aim of this paper is to study locally multiplication modules and to extend some properties of multiplication modules to locally multiplication modules. Some conditions are obtained under which locally multiplication modules satisfy ascending and descending chain conditions on certain types of submodules. Also, those locally multiplication modules which are finitely generated are classified. In addition to the above, some conditions are given each of which makes a proper submodule of an $R$–module as a prime submodule.

Keywords: multiplication modules, locally multiplication modules, prime submodules, primal submodules and faithful modules

1 Introduction

Throughout this paper $R$ is a commutative ring with identity and $M$ is an $R$–module unless otherwise stated. A non-empty subset $S$ of $R$ is called a
multiplicatively closed set in $R$ if $0 \notin S$ and $a, b \in S$ implies $ab \in S$ [14].

If $S$ is a multiplicatively closed set in $R$, then one can easily make $M_S$ as an $R_S$–module under the module operations $\frac{z}{s} + \frac{y}{t} = \frac{tx + yS}{st}$ and $\frac{r}{s} \frac{z}{t} = \frac{rz}{st}$, for $\frac{z}{s} \in R_S$ and $\frac{y}{t} \in M_S$ [15], so that when we say $M_S$ is a module we mean $M_S$ is an $R_S$–module. If $P$ is a prime ideal of $R$, then $S = R \setminus P$ is a multiplicatively closed set in $R$, in this case we denote $R_{R \setminus P}$ and $M_{R \setminus P}$ by $R_P$ and $M_P$ respectively. $R$ is called a Noetherian (Artinian) ring if it satisfies the ascending (descending) chain condition for ideals [21] and $R$ is called Locally Noetherian if $R_P$ is a Noetherian ring for each prime ideal $P$ of $R$ [11].

The Jacobson radical of $R$, denoted by $rad(R)$ [4] (or $J(R)$), is defined as $J(R) = rad(R) = \cap P, P$ is a maximal ideal of $R$. $M$ is called a Noetherian module if it satisfies the ascending chain condition for submodules [21]. Let $I$ be an ideal of $R$ and $N$ be a submodule of $M$. An element $r \in R$ is said to be prime to $I$ [5] (prime to $N$ [2]) if $x \in R(x \in M)$ such that $rx \in I(rx \in N)$, then $x \in I(x \in N)$, equivalently $r$ is not prime to $I$ (not prime to $N$) if there exists $m \in R \setminus I(m \in M \setminus N)$ such that $rm \in I(rm \in N)$.

The set of all elements of $R$ that are not prime to $I$ (not prime to $N$) is denoted by $S_R(I)$ [5] ($S_M(N)$ [2]), that is $S_R(I) = \{r \in R : rx \in I, \text{for some } x \notin I\}$ and $S_M(N) = \{r \in R : rx \in N, \text{for some } x \notin N\}$, especially, we have $S_R(0) = \{r \in R : rx = 0, \text{for some } 0 \neq x \in R\}$, $S_M(0) = \{r \in R : rx = 0, \text{for some } 0 \neq x \in M\}$ and $S_R(R) = \phi = S_M(M)$. If $I$ is an ideal of $R$ and $N, L$ are submodules of $M$, then $(N : I) = \{m \in M : Im \subseteq N\}$ and $(N : M) = \{r \in R : rM \subseteq N\}$, which is an ideal of $R$ [17], in case that $N = 0$, then $(0 : M)$ is called the annihilator of $M$ and denoted by $Ann(M)$, that is $Ann(M) = (0 : M) = \{r \in R : rM = 0\}$. $M$ is called a faithful module if $Ann(M) = (0 : M) = 0$ [16]. A proper submodule $K$ of $M$ is called a prime submodule if $r \in R$ and $x \in M$ such that $rx \in K$, then $x \in K$ or $r \in (K : M)$ [3] (equivalently $rM \subseteq K$) and $K$ is called a primal submodule of $M$ if $S_M(K)$ forms an ideal of $R$ [2]. The prime and primal spectrum of $M$ are defined as $Spec(M) = \{L : L$ is a prime submodule of $M\}$ [18] and $pSpec(M) = \{L : L$ is a primal submodule of $M\}$ [10]. $M$ is called a finitely generated $R$–module if there exist $x_1, x_2, ..., x_n \in M$ such that $M = Rx_1 + Rx_2 + ... + Rx_n$, and it is called a cyclic module if it is generated by a single element, that is $M = Rx$ for some $x \in M$ [21]. For the $R$–module $M$ we define, $C_M = \{IM : IM \neq 0, I$ is an ideal of $R\} and $C = \cap C_M = \bigcap_{IM \neq 0} IM$, where $I$ is an ideal of $R$ [9]. A submodule $N$ is said to have a presentation ideal if $N = IM$, for some ideal $I$ of $R$ and $I$ is called a presentation ideal of $N$ [3]. If $a, b \in M$, then by $ab$ is meant the product $(Ra)(Rb)$ and if $N, K$ are submodules of $M$ such that $N = IM$ and $K = JM$, for some ideals $I, J$ of $R$, then the product $NK$, of $N$
and $K$, is defined as $NK = IJM$ and it is a submodule of $M$ [3]. $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IJM$ [3] or equivalently, if for any submodule $N$ of $M$, we have $N = (N : M)M$ [19]. If $L, N$ are submodules of $M$, then $(L :_M N)$ is defined as $(L :_M N) = \{ m \in M : (Rm)(Rn) \subseteq N, \text{ for all } n \in N \}$ [13]. $M$ is called a locally multiplication module if $M_P$ is a multiplication $R_P$-module for each maximal ideal $P$ of $R$ [7] and it is called a prime module if the zero submodule of $M$ is prime [3].

2 Locally Multiplication Modules Which Satisfy Chain Conditions

We introduce the following definition.

**Definition 2.1.** We say that an $R$-module $M$ is locally Noetherian if $M_P$ is a Noetherian $R_P$-module for each prime ideal $P$ of $R$.

It is known that, if $M$ is a Noetherian $R$-module, then $M_S$ is a Noetherian $R_S$-module for every multiplicatively closed set $S$ in $R$ [14], so that if $P$ is any prime ideal of $R$, then $S = R \setminus P$ is a multiplicatively closed set in $R$, so that $M_P$ is a Noetherian $R_P$-module and this proves that every Noetherian module is locally Noetherian. However, the converse is not true in general. An example of a locally Noetherian ring which is not Noetherian is that given in [11, Example 3.4] and if we consider this ring $R$ as an $R$-module, then clearly it is the requested example of a locally Noetherian module which is not Noetherian.

Now, we give a condition that makes a locally Noetherian module as a Noetherian module.

**Theorem 2.2.** Let $M$ be a locally Noetherian $R$-module. If $R$ is a semi local ring, then $M$ is a Noetherian module.

Proof. Let $P_1, P_2, ..., P_k$ be the maximal ideals of $R$. Since $M$ is locally Noetherian, so $M_{P_i}$ is Noetherian for each $i$. Now, let $L_1 \subseteq L_2 \subseteq ... \subseteq L_n \subseteq ...$ be any ascending chain of submodules of $M$, then for each $i (1 \leq i \leq k)$, we have $(L_1)_{P_i} \subseteq (L_2)_{P_i} \subseteq ... \subseteq (L_n)_{P_i} \subseteq ...$, so that, for each $i$, there exists $m_i \in \mathbb{Z}^+$ such that $(L_{m_i})_{P_i} = (L_{m_i+1})_{P_i} = ...$. Let $m = max\{m_1, m_2, ..., m_k\}$, then it is obvious that, for each $i$, we have $(L_m)_{P_i} = (L_{m+1})_{P_i} = ...$, that means $(L_m)_P = (L_{m+1})_P = ...$ for each maximal ideal $P$ of $R$, so that by [8, Corollary 2.2], we get $L_m = L_{m+1} = ...$. Hence, $M$ is a Noetherian module.

The same argument is true for Artinian modules, that means every Artinian module is locally Artinian [6, Proposition 2.13], but the converse is not true.
in general and the same condition that given in Theorem 2.2, makes a locally Artinian module as an Artinian one, so that we can state.

**Theorem 2.3.** Let $M$ be a locally Artinian $R$–module. If $R$ is semi local, then $M$ is Artinian.

Proof. By using the same technique as in Theorem 2.2, one can easily get the result.

Now, we introduce the following definition.

**Definition 2.4.** Let $M$ be an $R$–module and $P$ be a maximal ideal of $R$, we define $H_P = \{ N : N \in pSpec(M) \text{ and } S_M(N) \subseteq P \}$.

**Theorem 2.5.** Let $M$ be a locally multiplication $R$–module and $P$ be a maximal ideal of $R$. If $R$ is a Noetherian ring, then $M$ satisfies ascending chain condition on $H_P$.

Proof. Since $M$ is a locally multiplication module, so $M_P$ is a multiplication $R_P$–module and as $R$ is Noetherian, we have $R_P$ is also Noetherian. Hence, by [20, Lemma 2], we get $M_P$ is a Noetherian $R_P$–module. Now, let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ be any ascending chain of submodules of $M$ in $H_P$. By [8, Proposition 2.1], we have $(N_1)_P \subseteq (N_2)_P \subseteq \ldots \subseteq (N_n)_P \subseteq \ldots$ and since $M_P$ is Noetherian, there exists $k \in \mathbb{Z}^+$ such that $(N_k)_P = (N_{k+1})_P = \ldots$. Now, let $x \in N_k$, then $\frac{x}{1} \in (N_{k+1})_P$ and as $N_{k+1} \in H_P$, we have $S_M(N_{k+1}) \subseteq P$, so by [6, Lemma 2.1], we get $x \in N_{k+1}$, so that $N_k \subseteq N_{k+1}$. Similarly, we can show that $N_{k+1} \subseteq N_k$. Hence, $N_k = N_{k+1}$. Since, $N_i \in H_P$, for all $i$, so by applying the same argument as in above we get that $N_k = N_{k+1} = N_{k+2} = \ldots$. Hence, $M$ satisfies ascending chain condition on $H_P$.

**Corollary 2.6.** Let $M$ be a locally multiplication $R$–module. If $R$ is a semi local Noetherian ring, then $M$ is Noetherian.

Proof. Let $P$ be any maximal ideal of $R$. As $R$ is Noetherian, it is locally Noetherian, so that $R_P$ is Noetherian and since $M$ is locally multiplication, so $M_P$ is multiplication, then by [20, Lemma 2], $M_P$ is Noetherian, that means, $M$ is locally Noetherian. Hence, by Theorem 2.2, $M$ is a Noetherian $R$–module.

In the following result a condition is given which makes locally multiplication modules cyclic.

**Theorem 2.7.** Let $M$ be a locally multiplication $R$–module. If $P$ is a maximal ideal of $R$ such that $S_M(Rx) \subseteq P$, for every $x \in M$ and $M$ is Artinian, then $M$ is cyclic.

Proof. Since $M$ is Artinian, so by [6, Proposition 2.13], we have $M$ is locally Artinian, that means $M_P$ is Artinian and as $M$ is locally multiplication, we have $M_P$ is a multiplication $R_P$– module, then by [20, Theorem 9], $M_P$ is cyclic, so $M_P = R_P \frac{m}{m}$, for some $x \in M$ and $p \notin P$, then by the hypothesis we have $S_M(Rx) \subseteq P$. Now, let $m \in M$, then $\frac{m}{1} \in M_P = R_P \frac{m}{m}$, so that
Let $m = \frac{r}{q} = \frac{r}{q}$, for some $r \in R$, $q \notin P$, then $uqpm = urx \in Rx$, for some $u \notin P$. If $m \notin Rx$, then $uqpm \in S_M(Rx) \subseteq P$, that is a contradiction. Hence, $m \in Rx$, so that $M \subseteq Rx$ and as $Rx \subseteq M$, we get $M = Rx$. Hence $M$ is cyclic.

It is known that, if $R$ is an integral domain and $P$ is a prime ideal of $R$, then $R_P$ is also an integral domain, since if $\frac{a}{p} = 0$, for $a, b \in R$ and $p, q \notin P$, then we have $\frac{a}{pq} = 0$, that means $uab = 0$, for some $u \notin P$, and thus $u \neq 0$, so we get $a = 0$ or $b = 0$, from which we get that $\frac{a}{p} = 0$ or $\frac{b}{q} = 0$. However, the converse is not true, in general, as we see in the following example.

Take $Z_6$ and $P = \{0, 2, 4\}$ as a prime ideal of $Z_6$. By simple calculations one can easily get that $(Z_6)_{\{0, 2, 4\}} = \left\{\frac{0}{1}, \frac{1}{1}\right\}$, which is clearly a field and hence an integral domain but clearly $Z_6$ is not an integral domain.

In the following result, we give a condition which makes the converse is also true.

**Proposition 2.8.** Let $R$ be a commutative ring with identity and $P$ be a prime ideal of $R$ such that $R_P$ is an integral domain. If $S_R(0) \subseteq P$, then $R$ is also an integral domain.

Proof. Let for $a, b \in R$ we have $ab = 0$, then $\frac{a}{1} \frac{b}{1} = a \frac{b}{1} = 0$. As $R_P$ is an integral domain, we have $\frac{a}{1} = 0$ or $\frac{b}{1} = 0$, then by [6, Lemma 2.1], we get $a = 0$ or $b = 0$. Hence, $R$ is an integral domain.

By combining all the above we get the following corollary.

**Corollary 2.9.** Let $R$ be a commutative ring with identity and $P$ be a prime ideal of $R$ such that $S_R(0) \subseteq P$, then $R$ is an integral domain if and only if $R_P$ is an integral domain.

**Proposition 2.10.** Let $M$ be a faithful $R$–module and $P$ be a prime ideal of $R$. If $S_M(0) \subseteq P$, then $M_P$ is a faithful $R_P$–module.

Proof. We have $(0 : M) = 0$. To show $(0 : M_P) = 0$. Let $\frac{x}{p} \in (0_P : M_P)$, where $r \in R$ and $p \notin P$. Then, $\frac{r}{p}M_P = 0$. Let $x \in M$ be any element, then we have $\frac{rx}{p} = \frac{rx}{p} \frac{1}{1} = 0$. As $S_M(0) \subseteq P$, by [6, Lemma 2.1], we get $rx = 0$, this gives that $rM = 0$, so that $r \in (0 : M) = 0$, thus $r = 0$ and that $\frac{r}{p} = 0$. Therefore, $(0_P : M_P) = 0$. Hence, $M_P$ is a faithful $R_P$–module.

**Theorem 2.11.** Let $R$ be an integral domain and $M$ be a locally multiplication $R$–module. Let $P$ be a maximal ideal of $R$ such that $S_M(0) \subseteq P$. If $M$ is an Artinian faithful $R$–module, then $M$ is a prime module.

Proof. As $M$ is Artinian, by [6, Proposition 2.13], we have $M$ is locally Artinian, that is, $M_P$ is Artinian and as $M$ is locally multiplication, $M_P$ is a multiplication $R_P$–module. Since, $M$ is faithful, so by Proposition 2.10, we have $M_P$ is a faithful $R_P$–module. Next, since $R$ is an integral domain, so we have $R_P$ is an integral domain. Hence, by [20, Theorem 10], we get that
$M_P$ is a prime module, that is $0_P$ is a prime submodule of $M_P$. We claim that the zero submodule of $M$ is prime. Let for $r \in R$ and $x \in M$, we have $rx = 0$, but $x \neq 0$. To show $rM = 0$, we have $\frac{rx}{1} = 0$. If $\frac{x}{1} = 0$, then as $S_M(0) \subseteq P$, by [6, Lemma 2.1], we get $x = 0$, which is a contradiction, so that $\frac{x}{1} \neq 0$ and as $0_P$ is a prime submodule of $M_P$, we must have $\frac{x}{1}M_P = 0$, then by [8, Proposition 2.8], we have $(rM)_P = \frac{x}{1}M_P = 0$, so again by [6, Lemma 2.1], we get $rM = 0$, so that the zero submodule of $M$ is prime. Hence, $M$ is a prime $R$-module.

3 Locally Multiplication Modules Which are Finitely Generated

In this section we study faithful locally multiplication modules and we determine some of their properties. We start with the following lemma.

Lemma 3.1. Let $M$ be an $R$-module and $P$ be a prime ideal of $R$. Let $N$ be a submodule of $M$ and $A$ an ideal of $R$ such that $S_M(N), S_R(A) \subseteq P$.

1. If $x \in M$ such that $qx \in N$, for some $q \notin P$, then $x \in N$.
2. If $r \in R$ such that $pr \in A$, for some $p \notin P$, then $r \in A$.

Proof. (1) If $x \notin N$, then as $qx \in N$, we get $q \in S_M(N) \subseteq P$, which is a contradiction. Hence, $x \in N$.

(2) By applying the same technique as in (1), the result follows.

Theorem 3.2. If $M$ is a locally multiplication module and $N$ is a proper submodule of $M$ such that $S_M(N) \subseteq J(R)$, then $N = (N : M)M$.

Proof. Let $P$ be any maximal ideal of $R$, so that $S_M(N) \subseteq P$. As $M$ is locally multiplication, we have $M_P$ is a multiplication $R_P$-module, then by [20, Lemma 1], we have $N_P = (N_P : M_P)M_P$. As $N$ is a proper submodule of $M$, by [8, proposition 2.17], we have $N_P$ is a proper submodule of $M_P$ and since, $S_M(N) \subseteq P$, so by [8, Theorem 2.21], we get $(N : M)_P = (N_P : M_P)$. Hence, we get $N_P = (N_P : M_P)M_P = (N : M)_PM_P = ((N : M)M)_P$, then by [8, Corollary 2.2], we get $N = (N : M)M$.

Theorem 3.3. Let $M$ be an $R$-module and $N$ be a submodule of $M$. If $A, P$ are ideals of $R$ with $P$ prime such that $S_M(N) \subseteq P$, then $(N : I)_P = (N_P : I_P)$.

Proof. Let $\frac{m}{t} \in (N : I)_P$, where $m \in M$ and $t \notin P$, then $qm \in (N : I)$, for some $q \notin P$. This means that $Iqm \subseteq N$. Now, let $\frac{i}{p} \in I_P$ be any element where $i \in R$ and $p \notin P$, then $\frac{i}{p} = \frac{j}{q}$, for some $j \in I, q \notin P$, then we get that $uqi = upj \in I$, for some $u \notin P$, so that $uqim \in N$, then $\frac{im}{t} = \frac{uqi}{t} = \frac{upj}{t} = \frac{ujq}{t} \in N_P$, so we get $I_PM_T \subseteq N_P$, that means $\frac{m}{t} \in (N_P : I_P)$. Hence,
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$(N : I)_P \subseteq (N_P : I_P)$. Conversely, let $\frac{m}{t} \in (N_P : I_P)$, where $m \in M$ and $t \notin P$, then $I_P \frac{m}{t} \subseteq N_P$. Now, for any $r \in I$, we have $\frac{r}{t} \in I_P$ and thus $\frac{rm}{t} = \frac{r}{t} \frac{m}{1} \in N_P$, this gives that $l(rm) \in N$ for some $l \notin P$. Then by Lemma 3.1, we get $rm \in N$. Hence, we get that $Im \subseteq N$, that is, $m \in (N : I)$, so we get $\frac{m}{t} \in (N : I)_P$ and thus $(N_P : I_P) \subseteq (N : I)_P$. Hence, $(N : I)_P = (N_P : I_P)$.

Next, we introduce the following definition.

**Definition 3.4.** Let $M$ be an $R$–module. We define $F_M = \{ P : P \in Spec(R) \text{ and } PM \text{ is a submodule of } M \text{ such that } S_M(PM) = P \}$.

It is clear that for any $R$–module $M$ we have always $F_M \subseteq Spec(R)$. Now, we give some examples to illustrate the above definition.

**Examples 3.5.** (1) Consider the $Z$–module $Z_6$. The ideal $< 2 >$ is a prime ideal of $Z$. We have $< 2 >^2 Z_6 = \{ 0, 2, 4 \}$. Then, $S_{Z_6}(< 2 > Z_6) = S_{Z_6}(\{ 0, 2, 4 \}) = \{ m \in Z : mx \in \{ 0, 2, 4 \}, \text{for some } x \notin \{ 0, 2, 4 \} = \{ m \in Z : mx = 0 \text{ or } 2 \text{ or } 4, \text{for some } x \in \{ 1, 3, 5 \} \} =< 2 >$. Similarly, we can get for the prime ideal $< 3 >$ of $Z$ that, $S_{Z_6}(< 3 > Z_6) =< 3 >$. But if we take the prime ideal $< 5 >$ of $Z$, we have $< 5 > Z_6 = Z_6$, so that $S_{Z_6}(< 5 > Z_6) = S_{Z_6}(Z_6) = \phi \neq< 5 >$, and for the prime ideal $< 7 >$ of $Z$, one can get $S_{Z_6}(< 7 > Z_6) = S_{Z_6}(Z_6) = \phi \neq< 7 >$. Hence, $F_{Z_6} \neq Spec(Z)$.

(2) Let us consider $Z_6$ as a $Z_6$–module. The only two prime ideals of $Z_6$ are $\{ 0, 2, 4 \}$ and $\{ 0, 3 \}$, for which $S_{Z_6}(\{ 0, 2, 4 \}) = \{ 0, 2, 4 \}$ and $S_{Z_6}(\{ 0, 3 \}) = \{ 0, 3 \}$. Now, we have $\{ 0, 2, 4 \} Z_6 = \{ 0, 2, 4 \}$ and $\{ 0, 3 \} Z_6 = \{ 0, 3 \}$, so that $S_{Z_6}(\{ 0, 2, 4 \} Z_6) = S_{Z_6}(\{ 0, 2, 4 \}) = \{ 0, 2, 4 \}$ and $S_{Z_6}(\{ 0, 3 \} Z_6) = S_{Z_6}(\{ 0, 3 \}) = \{ 0, 3 \}$, this means that $S_{Z_6}(PZ_6) = P$, for all prime ideals $P$ of $Z_6$. Hence, we get $F_{Z_6} = Spec(Z_6)$.

(3) For the $Z$–module $Z$, we have $PZ = P$, for every $P \in Spec(Z)$, so that $S_Z(PZ) = S_Z(P) = P$, for every $P \in Spec(Z)$. Hence, $F_Z = Spec(Z)$.

Now, we prove a property for faithful locally multiplication modules.

**Theorem 3.6.** Let $M$ be a faithful locally multiplication $R$–module. If $S_M(0) \subseteq J(R)$ and $F_M = Spec(R)$, then the following statements are equivalent.

(i) $M_P$ is finitely generated for each maximal ideal $P$ of $R$.

(ii) If $A$ and $B$ are ideals of $R$ such that $AM \subseteq BM$, then $A \subseteq B$.

(iii) $M \neq AM$ for any proper ideal $A$ of $R$.

(iv) $M \neq PM$ for any maximal ideal $P$ of $R$.

Proof. Let $P$ be any maximal ideal of $R$, then we have $S_M(0) \subseteq J(R) \subseteq P$ and as $M$ is faithful, by Proposition 2.10, we have $M_P$ is a faithful $R_P$–module and since $M$ is locally multiplication, so $M_P$ is a multiplication $R_P$–module, that means, $M_P$ is a faithful multiplication $R_P$–module. $(i) \rightarrow (ii)$. Assume that $(i)$ holds and suppose that $A$ and $B$ are ideals of $R$ such that $AM \subseteq
BM. We have to show that $A \subseteq B$. Now, $A_P$ and $B_P$ are ideals of $R_P$ and $AM \subseteq BM$ gives $A_PM_P \subseteq B_PM_P$ and as $M_P$ is finitely generated, so by [20, Theorem 4], we have $A_P \subseteq B_P$ and this holds for every maximal ideal $P$ of $R$, so by [14, Proposition 3.13], we get $A \subseteq B$.

(ii) $\rightarrow$ (iii). Suppose that (ii) holds and $M = AM$, for some proper ideal $A$ of $R$. As $R$ has the identity, we have $M = RM$ so we get that $RM = AM$, especially $RM \subseteq AM$, so by the condition (ii), we get $R \subseteq A$ and that $A = R$, that is a contradiction. Hence, $M \neq AM$, for any proper ideal $A$ of $R$.

(iii) $\rightarrow$ (iv). Suppose that (iii) holds. Since every maximal ideal is proper, so $M \neq PM$, for any maximal ideal $P$ of $R$.

(iv) $\rightarrow$ (i). Suppose that (iv) holds, so that we have $M \neq PM$. Now, if \possible suppose that $M_P = P_PM_P = (PM)_P$. Let $x \in M$, then $\frac{x}{1} \in (PM)_P$, so that $qx \in PM$, for some $q \notin P$ and as $P \in F$, we have $S_M(PM) = P \subseteq P$, so by Lemma 3.1, we get $x \in PM$, that gives $M \subseteq PM$. Hence, we have $M = PM$, that is a contradiction, so that $M_P \neq P_PM_P$. Since, $P_P$ is the unique maximal ideal of $R_P$, so by [20, Theorem 4], we get $M_P$ is finitely generated and the proof is complete.

It is known that, if $R$ is a local commutative ring with identity and $P$ is its unique maximal ideal, then an element $x \in R$ is a unit if and only if $x \notin P$. Now, we use this fact to prove the following result.

**Proposition 3.7.** Let $M$ be an $R$–module. If $R$ is a local ring with the unique maximal ideal $P$, then $M_P \cong M$.

Proof. Let $\frac{m}{p} \in M_P$, for $m \in M, p \notin P$, then by what we have mentioned above, we get $p$ is a unit in $R$, so that $p^{-1} \in R$. Now, define $f : M_P \to M$ by $f(\frac{m}{p}) = p^{-1}m$. One can easily prove that $f$ is an isomorphism, so that $M_P \cong M$.

Combining Theorem 3.6 and Proposition 3.7, we get the following corollary.

**Corollary 3.8.** Let $R$ be a local ring with $P$ as its unique maximal ideal and $M$ be a faithful locally multiplication $R$–module. If $S_M(0) \subseteq P$ and $F = Spec(R)$, then the following statements are equivalent.

(i) $M$ is finitely generated.

(ii) $M_P$ is finitely generated for each maximal ideal $P$ of $R$.

(iii) If $A$ and $B$ are ideals of $R$ such that $AM \subseteq BM$, then $A \subseteq B$.

(iv) $M \neq AM$ for any proper ideal $A$ of $R$.

(v) $M \neq PM$.

Proof. Since, $P$ is the unique maximal ideal of $R$, so that $J(R) = P$, thus we have $S_M(0) \subseteq P = J(R)$. By Proposition 3.7, we have $M_P \cong M$, so that $M$ is finitely generated if and only if $M_P$ is finitely generated and since $P$ is the only maximal ideal of $R$, so we can say that $M$ is finitely generated if and only
if $M_P$ is finitely generated for each maximal ideal $P$ of $R$, then by Theorem 3.6, the result follows at once.

4 Submodules of Locally Multiplication Modules Which are Prime

In this section we classify the proper submodules of locally multiplication modules that are prime and we give several conditions each of which makes a proper submodule of a locally multiplication $R$–module as a prime submodule.

Now, we introduce the following definition.

**Definition 4.1.** Let $M$ be an $R$–module and $P$ be a maximal ideal of $R$. We define $E_P = \{K : K$ is a proper submodule of $M$, with $K \subseteq C = \bigcap C_M$ and $S_M(K) \subseteq P\}$.

**Proposition 4.2.** Let $M$ be a locally multiplication $R$–module and $P$ be a maximal ideal of $R$. If $N, K \in E_P$, then $(NK)_P = N_PK_P$.

Proof. As $N, K \in E_P$, we have $N, K$ are proper submodules of $M$ with $N, K \subseteq C$ and $S_M(N), S_M(K) \subseteq P$, so that by [9, Theorem 2.7], $N$ and $K$ have presentation ideals, so that $I$ be a presentation ideal of $N$ and $J$ be a presentation ideal of $K$, that is $N = IM$ and $K = JM$, so that $NK = IJM$ and then $(NK)_P = (IJM)_P = I_PK_P$. Next, we have $N_P = (IM)_P = I_PM_P$ and $K_P = (JM)_P = J_PM_P$, that means $I_P$ is a presentation ideal of $N_P$ and $J_P$ is a presentation ideal of $K_P$, so that $N_PK_P = I_PK_PM_P = (NK)_P$.

**Lemma 4.3.** Let $R$ be a commutative ring with identity. If $\overline{A}$ is a prime ideal of $R_P$, then $A = \{a \in R : \frac{a}{1} \in \overline{A}\}$ is a prime ideal of $R$ and $\overline{A} = A_P$.

Proof. By [8, Proposition 2.16], $A = \{a \in R : \frac{a}{1} \in \overline{A}\}$ is an ideal of $R$ and $\overline{A} = A_P$. It remains to show that $A$ is prime. Since $\overline{A} = A_P$ is a proper ideal of $R_P$, so we have $A$ is a proper ideal of $R$. Now, let for $a, b \in R$ we have $ab \in A$. Then, $\frac{a}{1} \cdot \frac{b}{1} = \frac{ab}{1} \in A_P = \overline{A}$, then as $\overline{A}$ is prime, we get $\frac{a}{1} \in \overline{A} \text{ or } \frac{b}{1} \in \overline{A}$, which gives that $a \in A$ or $b \in A$. Hence, $A = \{a \in R : \frac{a}{1} \in \overline{A}\}$ is a prime ideal of $R$ and $\overline{A} = A_P$.

**Lemma 4.4.** Let $M$ be an $R$–module and $Q$ be a prime ideal of $R$. If $QM$ is a proper submodule of $M$, then $S_R(Q) \subseteq S_M(QM)$.

Proof. Let $r \in S_R(Q)$, then $rq \in Q$, for some $q \notin Q$ and as $Q$ is prime, we have $r \in Q$. Since, $QM \neq M$, so there exists $m \in M$ and $m \notin QM$, but then $rm \in QM$, then we get $r \in S_M(QM)$. Hence, we get that $S_R(Q) \subseteq S_M(QM)$.

**Theorem 4.5.** Let $M$ be a locally multiplication $R$–module and $N$ be a proper submodule of $M$ and let $P$ be a maximal ideal of $R$ such that
$S_M(N), S_M(0) \subseteq P$. If $Rm \in E_P$, for all $m \in M$ and $S_M(K) \subseteq P$, for all $K \in C_M$, then the following statements are equivalent:

(i) $N$ is a prime submodule of $M$.

(ii) If $L, K \in E_P$ such that $LK \subseteq N$, then $L \subseteq N$ or $K \subseteq N$.

(iii) If $m, n \in M$ such that $(Rm)(Rn) \subseteq N$, then $m \in N$ or $n \in N$.

(iv) $AM = N$ for some prime ideal $A$ of $R$ with $Ann(M) \subseteq A$.

Proof. (i) $\rightarrow$ (ii) Let $N$ be a prime submodule of $M$ and $L, K \in E_P$ such that $LK \subseteq N$. To show $L \subseteq N$ or $K \subseteq N$. By Proposition 4.2, we get $(LK)_P = L_PK_P$. As $N$ is a prime submodule of $M$ and $S_M(N) \subseteq P$, by [12, Lemma 4.10], we have $N_P$ is a prime submodule of $M_P$. As $LK \subseteq N$, we get $L_PK_P = (LK)_P \subseteq N_P$ and as $M$ is locally multiplication, we have $M_P$ is a multiplication module, thus by [17, Theorem 2.5] we get $L_P \subseteq N_P$ or $K_P \subseteq N_P$ and since $S_M(N) \subseteq P$, by [12, Lemma 4.8], we get $L \subseteq N$ or $K \subseteq N$.

(ii) $\rightarrow$ (iii) Suppose that the condition (ii) holds. Let $m, n \in M$ such that $(Rm)(Rn) \subseteq N$. As $Rm, Rn$ are submodules of $M$ and $Rm, Rn \in E_P$, by the condition (ii), we get $Rm \subseteq N$ or $Rn \subseteq N$ and as $1 \in R$, the former case gives $m \in N$ and the latter case gives $n \in N$.

(iii) $\rightarrow$ (iv) Suppose that the condition (iii) holds. As $N$ is a proper submodule of $M$, by [8, Proposition 2.17], $N_P$ is a proper submodule of $M_P$. Let for $\frac{m}{1}, \frac{n}{1} \in M_P$, where $m, n \in M$ and $t, l \notin P$, we have $(R_P \frac{m}{1})(R_P \frac{n}{1}) \subseteq N_P$. We will show that $\frac{m}{1} \in N_P$ or $\frac{n}{1} \in N_P$. Since, $Rm, Rn \in E_P$, so by Proposition 4.2, we get $((Rm)(Rn))_P = (Rm)_P(Rn)_P = (R_P \frac{m}{1})(R_P \frac{n}{1}) \subseteq N_P$ and as $S_M(N) \subseteq P$, we have $(Rm)(Rn) \subseteq N$. Hence, by the condition (iii), we get $m \in N$ or $n \in N$, which gives that $\frac{m}{1} \in N_P$ or $\frac{n}{1} \in N_P$. As $M$ is locally multiplication, we have $M_P$ is a multiplication $R_P$−module, so by [17, Theorem 2.5], we get $N_P = \overline{A}M_P$, for some prime ideal $\overline{A}$ of $R_P$ and $Ann(M_P) \subseteq \overline{A}$. By Lemma 4.3, $\overline{A} = A_P$, where $A = \{a \in R : \frac{a}{1} \in \overline{A}\}$ is a prime ideal of $R$. That is, $N_P = \overline{A}M_P = A_PM_P = (AM)_P$. Since $AM \subseteq C_M$, so we have $S_M(AM) \subseteq P$ and as $S_M(N) \subseteq P$, by [12, Lemma 4.8], we get $N = AM$. Next, let $x \in Ann(M)$, then $xM = 0$. Now, for any $\frac{m}{1} \in M_P$, we have $\frac{x}{1}M_P = \frac{xm}{1} = 0$, so that $\frac{x}{1}P = 0$, that is, $\frac{x}{1} \in Ann(M_P) \subseteq \overline{A}$, this gives $x \in A$. Hence, $Ann(M) \subseteq A$.

(iv) $\rightarrow$ (i) Suppose that $AM = N$, where $A$ is a prime ideal of $R$ with $Ann(M) \subseteq A$. Then, we get $N_P = (AM)_P = A_PM_P$ and $S_M(AM) = S_M(N) \subseteq P$. Now, we have $N$ is a proper submodule of $M$, that means $AM$ is a proper submodule of $M$, so by Lemma 4.4, we get $S_R(A) \subseteq S_M(AM) \subseteq P$. Hence, by [12, Proposition 3.4], we get $A_P$ is a prime ideal of $R_P$. Next, we will show $Ann(M_P) \subseteq A_P$. Let $\frac{a}{1} \in Ann(M_P)$, where $a \in R, p \notin P$. Then, $\frac{a}{1}P = 0$, so that $(aM)_P = \frac{a}{1}M_P = 0$, then by [6, Lemma 2.1], we get $aM = 0$,
so that \(a \in Ann(M) \subseteq A\) and that \(\frac{m}{p} \in A\). Hence, \(Ann(M_p) \subseteq A_p\). Next, as \(M\) is locally multiplication, we have \(M_p\) is a multiplication \(R_p\)–module, so by [17, Theorem 2.5], we have \(N_p\) is a prime submodule of \(N_p\) and as \(S_M(N) \subseteq P\), by [6, Proposition 2.21], we have \(N\) is a prime submodule of \(M\).

**Lemma 4.6.** Let \(M\) be an \(R\)–module and \(L, N\) be submodules of \(M\). Let \(P\) be a maximal ideal of \(R\) with \(S_M(L) \subseteq P\). If \(Rm \in E_P\), for all \(m \in M\), then \((L :_M N)_P = (L_P :_{M_P} N_P)\).

Proof. Let \(\frac{m}{p} \in (L :_M N)_P\), where \(m \in M, p \notin P\). Then, \(qm \in (L :_M N)\), for some \(q \notin P\) and so, \((Rqm)(Rn) \subseteq L\), for all \(n \in N\). We will show that \(\frac{m}{p} \in (L_P :_{M_P} N_P)\). Let \(\frac{x}{t} \in N_P\) be any element, where \(x \in M, t \notin P\), then \(sx \in N\), for some \(s \notin P\), so that we get \((Rqm)(Rsx) \subseteq L\). Now, \((R_P\frac{m}{p})(R_P\frac{x}{t}) = (R_P\frac{qm}{qP})(R_P\frac{sx}{sP}) = (Rqm)_P(Rs)_P = ((Rqm)(Rsx))_P \subseteq L_P\). Hence, \(\frac{m}{p} \in (L_P :_{M_P} N_P)\), so that \((L :_M N)_P \subseteq (L_P :_{M_P} N_P)\). Conversely, let for \(m \in M, p \notin P\), we have \(\frac{m}{p} \in (L_P :_{M_P} N_P)\). Let \(n \in N\) be any element, then \((R_P\frac{m}{p})(R_P\frac{n}{q}) \subseteq L_P\), that is \((Rm)_P(Rn)_P \subseteq L_P\). As, \(Rm, Rn \in E_P\), we have \(((Rm)(Rn))_P = (Rm)_P(Rn)_P \subseteq L_P\). Since, \(S_M(L) \subseteq P\), by [12, Lemma 4.8], we get \((Rm)(Rn) \subseteq L\). Hence, \(m \in (L :_M N)\), so that \(\frac{m}{p} \in (L :_M N)_P\), this gives that \((L_P :_{M_P} N_P) \subseteq (L :_M N)_P\) and thus \((L :_M N)_P = (L_P :_{M_P} N_P)\).

**Theorem 4.7.** Let \(M\) be a locally multiplication \(R\)–module and \(Rm \in E_P\), for all \(m \in M\). If \(P\) is a maximal ideal of \(R\) and \(L\) is a proper submodule of \(M\) such that \(S_M(L) \subseteq P\), then the following statements are equivalent:

(i) \(L\) is a prime submodule of \(M\).

(ii) If \(N\) is a submodule of \(M\) such that \(N \notin L\), then \((L :_M N) = L)\).

Proof. Since \(M\) is locally multiplication then \(M_P\) is a multiplication \(R_P\)–module.

(i) \(\rightarrow\) (ii) Let \(L\) be a prime submodule of \(M\) and \(N\) be a submodule of \(M\) such that \(N \notin L\). Since, \(S_M(L) \subseteq P\), so by [12, Lemma 4.10], we get \(L_P\) is a prime submodule of \(M_P\). Now, if \(N_P \subseteq L_P\), and as \(S_M(L) \subseteq P\), by [12, Lemma 4.8], we get \(N \subseteq L\), which is a contradiction. Hence, we must have \(N_P \notin L_P\) and as \(M_P\) is a multiplication \(R_P\)–module, by [13, Proposition 3.1], we get \((L_P :_{M_P} N_P) = L_P\) and since \(S_M(L) \subseteq P\), then by Lemma 4.6, we have \((L :_M N)_P = (L_P :_{M_P} N_P) = L_P\). We will show that \((L :_M N) = L\). As, \(S_M(L) \subseteq P\), by [12, Lemma 4.8], we have \((L :_M N) \subseteq L\). Now, let \(m \in L\), then \(\frac{m}{p} \in L_P = (L :_M N)_P\), so that \(qm \in (L :_M N)\), for some \(q \notin P\). Now, let \(n \in \frac{q}{t}\) be any element, then \(qn \in N\), so that \((Rqm)(Rqn) \subseteq L\). Next, we have \(((Rm)(Rn))_P = (Rm)_P(Rn)_P = (R_P\frac{m}{p})(R_P\frac{n}{q}) = (R_P\frac{qm}{qP})(R_P\frac{qn}{qP}) = (R_P\frac{qm}{q})(R_P\frac{qn}{q}) = ((Rqm)(Rqn))_P \subseteq L_P\) and as \(S_M(L) \subseteq P\), by [12, Lemma 4.8], we have \((Rm)(Rn) \subseteq L\). Hence, we get \(m \in (L :_M N)\), thus \(L \subseteq (L :_M N)\), therefore \((L :_M N) = L\).

(ii) \(\rightarrow\) (i) Let the condition (ii) holds. We will show that \(L_P\) is a prime
submodule of $M_P$. Let $\overline{N}$ be any submodule of $M_P$ such that $\overline{N} \not\subseteq L_P$. Then by [8, Proposition 2.16], $\overline{N} = N_P$, for some submodule $N$ of $M$. If $N \subseteq L$, then clearly $\overline{N} = N_P \subseteq L_P$, which is a contradiction, so that $N \not\subseteq L$, then by the condition $(ii)$, we have $(L :_M N) = L$, from which we get $(L :_M N)_P = L_P$. Since, $S_M(L) \subseteq P$, so by Lemma 4.6, we get $(L_P :_{M_P} \overline{N}) = (L :_M N)_P = L_P$ and since, $M_P$ is a multiplication $R_P$–module, so by [13, Proposition 3.1], we get $L_P$ is a prime submodule of $M_P$ and as $S_M(L) \subseteq P$, by [12, Lemma 4.10], we get $L$ is a prime submodule of $M$.

**Theorem 4.8.** Let $M$ be a locally multiplication $R$–module such that $\text{Ann}(M)$ is a prime ideal of $R$. If $P$ is a maximal ideal of $R$ such that $S_M(0) \subseteq P$, then $M$ is a prime module.

**Proof.** First, we will show that $(\text{Ann}(M))_P$ is a prime ideal of $R_P$. As, $S_M(0) \subseteq P$, by [6, Proposition 2.5], we have $(\text{Ann}(M))_P = \text{Ann}(M_P)$. If $(\text{Ann}(M))_P = R_P$, then $\frac{1}{1} \in (\text{Ann}(M))_P$, this gives that $q \in \text{Ann}(M)$, for some $q \notin P$, that means $qM = 0$. As $M \neq 0$, there exists $0 \neq m \in M$, then $qm = 0$ and this implies that $q \in S_M(0) \subseteq P$, which is a contradiction, so that $(\text{Ann}(M))_P$ is a proper ideal of $R_P$. Next, let for $a, b \in R$ and $p, q \notin P$, we have $\frac{a}{b} \frac{b}{p} \in (\text{Ann}(M))_P$, then $sab \in \text{Ann}(M)$, for some $s \notin P$. As $\text{Ann}(M)$ is prime, we get $sa \in \text{Ann}(M)$ or $b \in \text{Ann}(M)$. The former case gives that $\frac{a}{p} \frac{b}{p} \in (\text{Ann}(M))_P$ and the latter case gives that $\frac{b}{p} \in (\text{Ann}(M))_P$, so that $(\text{Ann}(M))_P$ is a prime ideal of $R_P$ and as $(\text{Ann}(M))_P = \text{Ann}(M_P)$, we have $\text{Ann}(M_P)$ is a prime ideal of $R_P$, so by [1, Theorem 2.1], we get $M_P$ is a prime module, that means the zero submodule of $M_P$ is prime. To show $M$ is a prime module it is enough to show that the zero submodule of $M$ is prime. Let, $rx = 0$, but $x \neq 0$, where $r \in R, x \in M$, then we have $\frac{r}{x} \frac{x}{x} = r \frac{1}{1} = 0$. If $\frac{x}{1} = 0$, then as $S_M(0) \subseteq P$, by [6, Lemma 2.1], we get $x = 0$, which is a contradiction, so that $\frac{x}{1} \neq 0$ and as $M_P$ is a prime module, we get $\frac{x}{1}M_P = 0$, that is $(rM)_P = 0$, then by [6, Lemma 2.1], we get that $rM = 0$, that means the zero submodule of $M$ is a prime submodule. Hence, $M$ is a prime module.

In the following last two results we extend some other properties of multiplication modules to locally multiplication modules.

**Theorem 4.9.** Let $M$ be a locally multiplication $R$–module and $N$ be a prime submodule of $M$ and $P$ be a prime ideal of $R$ such that $S_M(N) \subseteq P$. If $N_1, N_2, ..., N_k$ are submodules of $M$ such that $N_i \in E_P$, for all $i(1 \leq i \leq k)$, then the following statements are equivalent.

(i) $N_j \subseteq N$ for some $j$ with $1 \leq j \leq k$.
(ii) $\bigcap_{i=1}^{k} N_i \subseteq N$.
(iii) $\prod_{i=1}^{k} N_i \subseteq N$.

**Proof.** Since $M$ is a locally multiplication $R$–module, so $M_P$ is a multipli-
cution \( R_P \)-module and since \( S_M(N) \subseteq P \), then by \([12, \text{Lemma 4.10}]\), we get \( N_P \) is prime.

(i) \( \rightarrow \) (ii) Suppose that \( N_j \subseteq N \), for some \( j(1 \leq j \leq k) \). Then, we get \((N_j)_P \subseteq N_P \) and so we have \((\bigcap_{i=1}^k N_i)_P = \bigcap_{i=1}^k (N_i)_P \subseteq (N_j)_P \subseteq N_P \), and since \( S_M(N) \subseteq P \), so we get \( \bigcap_{i=1}^k N_i \subseteq N \).

(ii) \( \rightarrow \) (iii) Let \( \bigcap_{i=1}^k N_i \subseteq N \). Then, \( \bigcap_{i=1}^k (N_i)_P = (\bigcap_{i=1}^k N_i)_P \subseteq N_P \), and as \( N_P \) is a prime submodule of the multiplication module \( M_P \), by \([20, \text{Theorem 11}]\), we get \( \prod_{i=1}^k (N_i)_P \subseteq N_P \) and since \( N_i \in E_P \), for all \( i(1 \leq i \leq k) \), so we have \((\prod_{i=1}^k N_i)_P = \prod_{i=1}^k (N_i)_P \subseteq N_P \), and as \( S_M(N) \subseteq P \), by \([12, \text{Lemma 4.8}]\), we get \( \prod_{i=1}^k N_i \subseteq N \).

(iii) \( \rightarrow \) (i) Suppose that \( \prod_{i=1}^k N_i \subseteq N \), then as \( N_i \in E_P \), for all \( i(1 \leq i \leq k) \), we have \( \bigcap_{i=1}^k (N_i)_P = (\bigcap_{i=1}^k N_i)_P \subseteq N_P \), so by \([20, \text{Theorem 11}]\), we get \((N_j)_P \subseteq N_P \) for some \( j \) with \( 1 \leq j \leq k \). As, \( S_M(N) \subseteq P \), by \([12, \text{Lemma 4.8}]\), we get \( N_j \subseteq N \).

**Theorem 4.10.** Let \( M \) be a locally multiplication \( R \)-module and \( N \) be a prime submodule of \( M \) and \( P \) be a maximal ideal of \( R \) such that \( S_M(N) \subseteq P \). If \( N_1, N_2, \ldots, N_k \) are submodules of \( M \) and \( N = \bigcap_{i=1}^n N_i \), then \( N = N_j \), for some \( j(1 \leq j \leq n) \).

Proof. Since \( M \) is a locally multiplication \( R \)-module then \( M_P \) is a multiplication \( R_P \)-module and since \( S_M(N) \subseteq P \), then by \([6, \text{Proposition 2.21}]\), we have \( N_P \) is a prime submodule of \( M_P \). Since, \( N = \bigcap_{i=1}^n N_i \), so \( N_P = (\bigcap_{i=1}^n N_i)_P = \bigcap_{i=1}^k (N_i)_P \), then by \([20, \text{Corollary 2}]\), we get \( N_P = (N_j)_P \) for some \( j(1 \leq j \leq n) \) and since \( S_M(N) \subseteq P \), then \( N_j \subseteq N \) and as \( N \subseteq N_j \), we get \( N = N_j \).

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