On Fuzzy JB-semigroups

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Abstract

In this paper, we introduce the notion of fuzzy JB-semigroups and we investigate some of its properties.

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1 Introduction and Preliminaries

In [4], J. Neggers and H.S. Kim introduced the notion of B-algebra. A B-algebra is an algebra $(X; *, 0)$ of type $(2, 0)$ (that is, a nonempty set $X$ with a binary operation $*$ and a constant $0$) satisfying the following axioms: (I) $x * x = 0$, (II) $x * 0 = x$, and (III) $(x * y) * z = x * (z * (0 * y))$. In [1], J.C. Endam and J.P. Vilela introduced the notion of JB-semigroup. A JB-semigroup is a nonempty set $X$ together with two binary operations $*$ and $\cdot$ and a constant 0 satisfying the following: $(X; *, 0)$ is a B-algebra, $(X, \cdot)$ is a semigroup, and the operation $\cdot$ is left and right distributive over the operation $*$, that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

In [5], a nonempty subset $N$ of a B-algebra $(X; *, 0)$ is called a subalgebra of $X$ if $x * y \in N$ for any $x, y \in N$. It is called normal in $X$ if for any $x * y, a * b \in N$ implies $(x * a) * (y * b) \in N$. A normal subset of $X$ is a subalgebra of $X$. 
Throughout this paper, $X$ means a JB-semigroup $(X; *, \cdot, 0)$. In [1], a nonempty subset $S$ of $X$ is called a sub JB-semigroup of $X$ if $x * y, x \cdot y \in S$ for all $x, y \in S$. If $a \cdot b = b \cdot a$ for all $a, b \in X$, then $X$ is called commutative.

Example 1.1 [1] Let $X = \{0, a, b, c\}$ be a set with the following tables of operations:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then $(X; *, \cdot, 0)$ is a JB-semigroup.

Example 1.2 [1] Let $X = \{0, a, b, c\}$ be a set with the following tables:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then $(X; *, \cdot, 0)$ is a JB-semigroup.

Example 1.3 [1] The algebras $(\mathbb{Z}; *, \cdot, 0)$, $(\mathbb{Q}; *, \cdot, 0)$, $(\mathbb{R}; *, \cdot, 0)$, and $(\mathbb{C}; *, \cdot, 0)$ are JB-semigroups, where $x * y = x - y$ and $\cdot$ is the usual multiplication.

Definition 1.4 [1] A nonempty subset $I$ of $X$ is called a JB-ideal of $X$ if the following hold: i. $(x * a) * (y * b) \in I$ for any $x * y, a * b \in I$, and ii. $a \cdot x, x \cdot a \in I$ for any $a \in I, x \in X$.

Let $X$ be the JB-semigroup in Example 1.2, the set $I = \{0, b\}$ is a JB-ideal of $X$, while the set $J = \{0, a\}$ is not.

Remark 1.5 [1] Let $I$ be a JB-ideal of $X$. Then $I$ is a sub JB-semigroup of $X$ and $I$ is a JB-ideal for every sub JB-semigroup of $X$ containing $I$.

Theorem 1.6 [1] Let $\{I_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty collection of JB-ideals of $X$. Then $\bigcap_{\alpha \in \mathcal{A}} I_\alpha$ is a JB-ideal of $X$.

Theorem 1.7 [1] Let $I$ be a JB-ideal of $X$. Then $(X/I; *, \cdot, [0]_I)$ is a JB-semigroup, where $*$ and $\cdot$ defined as $[x]_I * [y]_I = [x * y]_I$ and $[x]_I \cdot [y]_I = [x \cdot y]_I$. If $X$ is commutative or has a unity, then the same is true of $X/I$. 
The JB-semigroup $X/I$ in Theorem 1.7 is called quotient JB-semigroup of $X$ by $I$.

We now introduce the concept of fuzzy sets.

**Definition 1.8** [6] Let $X$ be any set. A mapping $\mu : X \to [0, 1]$ is called a fuzzy set of $X$.

**Definition 1.9** [3] Let $X$ and $Y$ be two nonempty sets, $\mu$ a fuzzy set of $Y$, and $f : X \to Y$ a mapping. The preimage of $\mu$ under $f$, denoted by $\mu^f$, is the fuzzy set of $X$ defined by $\mu^f(x) = \mu(f(x))$ for all $x \in X$, that is, $\mu^f = \mu \circ f$.

**Definition 1.10** [3] Let $\mu$ be a fuzzy set of $X$ and $f : X \to Y$ a mapping. The mapping $f(\mu) : Y \to [0, 1]$ defined by

$$f(\mu)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \{\mu(x)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\
0, & \text{if } f^{-1}(y) = \emptyset,
\end{cases}$$

is called the image of $\mu$ under $f$, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

**Definition 1.11** [3] Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy sets of $X$, where $\mathcal{A}$ is an arbitrary index set. The intersection of $\mu_\alpha$, denoted by $\bigwedge_{\alpha \in \mathcal{A}} \mu_\alpha$, is defined by $\bigwedge_{\alpha \in \mathcal{A}} \mu_\alpha(x) = \inf_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}$ for all $x \in X$.

**Definition 1.12** [3] Let $\{\mu_\alpha : \alpha \in \mathcal{A}\}$ be a nonempty family of fuzzy sets of $X$, where $\mathcal{A}$ is an arbitrary index set. The union of $\mu_\alpha$, denoted by $\bigvee_{\alpha \in \mathcal{A}} \mu_\alpha$, is defined by $\bigvee_{\alpha \in \mathcal{A}} \mu_\alpha(x) = \sup_{\alpha \in \mathcal{A}} \{\mu_\alpha(x)\}$ for all $x \in X$.

## 2 Fuzzy sub JB-semigroups

This section introduces the notion of fuzzy sub JB-semigroups and provides some related properties.

**Definition 2.1** A fuzzy JB-semigroup $\mu$ of $X$ is called a fuzzy sub JB-semigroup of $X$ if it satisfies the following conditions: For all $x, y \in X$, i. $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$, and ii. $\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}$.

**Example 2.2** Let $X = \{0, a, b, c\}$ be the JB-semigroup in Example 1.1. Define a fuzzy subset $\mu : X \to [0, 1]$ by $\mu(0) = 0.8$, $\mu(a) = 0.3$ for all $x \neq 0$. Then by routine calculations, $\mu$ is a fuzzy sub JB-semigroup of $X$. 

Lemma 2.3 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \). Then \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Moreover, if \( \mu \) is onto, then \( \mu(0) = 1 \).

Theorem 2.4 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \). Then there exists a sequence \( \langle x_n \rangle \) in \( X \) such that \( \lim_{n \to \infty} \mu(x_n) = 1 \) if and only if \( \mu(0) = 1 \).

Theorem 2.5 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \). Then the set \( X_\mu = \{ x \in X : \mu(x) = \mu(0) \} \) is a sub JB-semigroup of \( X \).

Proof: Clearly, \( 0 \in X_\mu \) and so \( X_\mu \neq \emptyset \). If \( x, y \in X_\mu \), then \( \mu(x) = \mu(y) = \mu(0) \) and so \( \mu(x * y) \geq \min\{\mu(x), \mu(y)\} = \mu(0) \). By Lemma 2.3, \( \mu(x * y) = \mu(0) \). Hence, \( x * y \in X_\mu \). Similarly, \( x \cdot y \in X_\mu \). Therefore, \( X_\mu \) is a sub JB-semigroup of \( X \). \( \square \)

Theorem 2.6 The intersection of any nonempty family of fuzzy sub JB-semigroups of \( X \) is also a fuzzy sub JB-semigroup of \( X \).

Proposition 2.7 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \). Then for all \( x, y \in X \),

i. \( \mu(0 * x) = \mu(x) \),

ii. \( \mu(x * (0 * y)) \geq \min\{\mu(x), \mu(y)\} \),

iii. \( \mu(x * y) = \mu(y * x) \).

Proposition 2.8 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \) and \( x, y \in X \). If \( \mu(x * y) = \mu(0) \), then \( \mu(x) = \mu(y) \).

The converse of Proposition 2.8 need not be true. Consider the fuzzy sub JB-semigroup \( \mu \) in Example 2.2. Now, \( \mu(a) = 0.3 = \mu(b) \). But \( \mu(a * b) = \mu(c) = 0.3 \neq 0.8 = \mu(0) \).

Let \( \mu \) be a fuzzy set in \( X \) and \( 0 \leq t \leq 1 \). The upper level set \( U(\mu; t) \) is the set \( U(\mu; t) = \{ x \in X : \mu(x) \geq t \} \).

Theorem 2.9 Let \( \mu \) be a fuzzy set in \( X \), where \( U(\mu; t) \neq \emptyset \). Then \( \mu \) is a fuzzy sub JB-semigroup of \( X \) if and only if \( U(\mu; t) \) is a sub JB-semigroup of \( X \) for every \( 0 \leq t \leq 1 \).

Proof: Suppose that \( \mu \) is a fuzzy sub JB-semigroup of \( X \) and \( U(\mu; t) \neq \emptyset \). Let \( x, y \in U(\mu; t) \). Then \( \mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t \). Thus, \( x * y \in U(\mu; t) \). Also, \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \geq t \). Thus, \( x \cdot y \in U(\mu; t) \). Hence, \( U(\mu; t) \) is a sub JB-semigroup of \( X \). Conversely, suppose \( U(\mu; t) \) is a sub JB-semigroup of \( X \). If there are \( x_0, y_0 \in X \) such that \( \mu(x_0 * y_0) < \min\{\mu(x_0), \mu(y_0)\} \), then taking \( t_0 = \frac{1}{2}(\mu(x_0 * y_0) + \min\{\mu(x_0), \mu(y_0)\}) \), \( \mu(x_0 * y_0) < t_0 < \min\{\mu(x_0), \mu(y_0)\} \). Hence, \( x_0, y_0 \in U(\mu; t_0) \), but \( x_0 * y_0 \notin U(\mu; t_0) \), a contradiction. Thus, for all \( x, y \in X \), \( \mu(x * y) \geq \min\{\mu(x), \mu(y)\} \). Similarly, \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \in X \). Therefore, \( \mu \) is a fuzzy sub JB-semigroup of \( X \). \( \square \)
Theorem 2.10 Two upper level sets \( U(\mu; s) \) and \( U(\mu; t) \) (\( s < t \)) of a fuzzy sub JB-semigroup \( \mu \) of \( X \) are equal if and only if there is no \( x \in X \) such that \( s \leq \mu(x) < t \).

Proof: Suppose \( s < t \) and \( U(\mu; s) = U(\mu; t) \). If there exists \( x \in X \) such that \( s \leq \mu(x) < t \), then \( U(\mu; t) \) is a proper subset of \( U(\mu; s) \), a contradiction. Conversely, suppose there is no \( x \in X \) such that \( s \leq \mu(x) < t \). From \( s < t \), \( U(\mu; t) \subseteq U(\mu; s) \). If \( x \in U(\mu; s) \), then \( \mu(x) \geq s \) and so \( \mu(x) \geq t \). Hence, \( x \in U(\mu; t) \). Therefore, \( U(\mu; s) = U(\mu; t) \).

Theorem 2.11 Let \( \emptyset \neq N \subseteq X \) and \( \mu_N \) be a fuzzy set in \( X \) defined by

\[
\mu_N(x) = \begin{cases} 
\alpha & \text{if } x \in N, \\
\beta & \text{otherwise}
\end{cases}
\]

for all \( x \in X \) and \( \alpha, \beta \in [0, 1] \) with \( \alpha > \beta \). Then \( \mu_N \) is a fuzzy sub JB-semigroup of \( X \) if and only if \( N \) is a sub JB-semigroup of \( X \). Moreover, in this case, \( X_{\mu_N} = N \).

Proof: Suppose that \( \mu_N \) is a fuzzy sub JB-semigroup of \( X \). Let \( x, y \in N \). Then \( \mu_N(x * y) \geq \min\{\mu_N(x), \mu_N(y)\} = \alpha \). Hence, \( \mu_N(x * y) = \alpha \) since \( \alpha > \beta \). Thus, \( x * y \in N \). Similarily, \( x \cdot y \in N \). Therefore, \( N \) is a sub JB-semigroup of \( X \). Conversely, suppose that \( N \) is a sub JB-semigroup of \( X \) and \( x, y \in X \). If \( x, y \in N \), then \( x \cdot y \in N \) and \( \mu_N(x \cdot y) = \alpha = \min\{\mu_N(x), \mu_N(y)\} \).

If \( x \notin N \) or \( y \notin N \), then \( \mu_N(x \cdot y) \geq \beta = \min\{\mu_N(x), \mu_N(y)\} \). Hence, \( \mu_N(x \cdot y) \geq \min\{\mu_N(x), \mu_N(y)\} \). Therefore, \( \mu_N(x \cdot y) \geq \min\{\mu_N(x), \mu_N(y)\} \). Similarly, \( \mu_N(x \cdot y) \geq \min\{\mu_N(x), \mu_N(y)\} \).

Moreover, \( X_{\mu_N} = \{x \in X : \mu_N(x) = \mu_N(0)\} = \{x \in X : \mu_N(x) = \alpha\} = N \).

Corollary 2.12 Let \( \chi_N \) be the characteristic function of a nonempty subset \( N \) of \( X \). Then \( \chi_N \) is a fuzzy sub JB-semigroup of \( X \) if and only if \( N \) is a sub JB-semigroup of \( X \).

Any sub JB-semigroup of \( X \) can be realized as an upper level set of some fuzzy sub JB-semigroup.

Corollary 2.13 Let \( N \) be a sub JB-semigroup of \( X \). Then there exists a fuzzy sub JB-semigroup \( \mu \) of \( X \) such that \( U(\mu; t) = N \) for any \( 0 < t < 1 \).

Proof: Consider the fuzzy sub JB-semigroup \( \mu \) of \( X \) defined by

\[
\mu(x) = \begin{cases} 
t & \text{if } x \in N, \\
0 & \text{otherwise}
\end{cases}
\]

where \( 0 < t < 1 \). Clearly, \( U(\mu; t) = N \).
Theorem 2.14 Let \( \mu \) be a fuzzy set in \( X \) with \( \text{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \) where \( \alpha_i < \alpha_j \) whenever \( i > j \). Let \( \{N_n : n = 0, 1, \ldots, k\} \) be a family of sub JB-semigroups of \( X \) such that

(i) \( N_0 \subset N_1 \subset \cdots \subset N_k = X \),
(ii) \( \mu(N'_n) = \alpha_n \), where \( N'_n = N_n \setminus N_{n-1} \) for \( n = 0, 1, \ldots, k \) and \( N_{-1} = \emptyset \).

Then \( \mu \) is a fuzzy sub JB-semigroup of \( X \).

Proof: Let \( x, y \in X \). Consider the following cases.

Case 1. If \( x, y \in N'_n \), then \( \mu(x) = \alpha_n = \mu(y) \). Since \( x, y \in N_n \) and \( N_n \) is a sub JB-semigroup, \( x \ast y \in N_n \). By WOP, there exists a smallest \( m \) such that \( x \ast y \in N_m \) and \( x \ast y \notin N_{m-1} \). This means that \( x \ast y \in N'_m \). Hence, \( \mu(x \ast y) = \alpha_m \geq \alpha_n = \min\{\mu(x), \mu(y)\} \).

Similarly, \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \).

Case 2. If \( x \in N'_n \) and \( y \in N_m \) where \( 0 \leq m < n \leq k \), then \( x \in N_n \) and \( y \in N_m \subset N_n \). Hence, \( x \ast y \in N_n \). By WOP, there exists a smallest \( p \) such that \( x \ast y \in N_p \) and \( x \ast y \notin N_{p-1} \). This means that \( x \ast y \in N'_p \). Since \( m < n \), \( \alpha_m > \alpha_n \). It follows that \( \mu(x \ast y) = \alpha_p \geq \alpha_n = \min\{\mu(x), \mu(y)\} \).

Similarly, \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \).

Case 3. If \( x \in N'_m \) and \( y \in N_n \) where \( 0 \leq m < n \leq k \), then similarly as in case 2, we obtain \( \mu(x \ast y) \geq \min\{\mu(x), \mu(y)\} \) and \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \).

Therefore, \( \mu \) is a fuzzy sub JB-semigroup of \( X \). \( \square \)

Theorem 2.15 Let \( \mu \) be a fuzzy sub JB-semigroup of \( X \) such that \( \text{Im}(\mu) = \{\alpha_i : i \in \mathcal{A}\} \), where \( \mathcal{A} \) is an arbitrary index set. Then

i. \( X_\mu = \bigcap_{i \in \mathcal{A}} U(\mu; \alpha_i) = U(\mu; \alpha_{i_0}) \), where \( \mu(0) = \alpha_{i_0} \).

ii. \( X = \bigcup_{i \in \mathcal{A}} U(\mu; \alpha_i) \).

3 Fuzzy JB-ideal and Fuzzy JB\(_s\)-ideal

This section introduces the notions of fuzzy JB-ideals and fuzzy JB\(_s\)-ideals. It also provides some related properties.

Definition 3.1 A fuzzy JB-semigroup \( \mu \) of \( X \) is called a fuzzy JB-ideal of \( X \) if it satisfies the following conditions: For all \( a, b, x, y \in X \),

i. \( \mu((x \ast a) \ast (y \ast b)) \geq \min\{\mu(x \ast y), \mu(a \ast b)\} \), and

ii. \( \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \).

Example 3.2 The fuzzy set \( \mu \) in Example 2.2 is a fuzzy JB-ideal of \( X \).

Theorem 3.3 Every fuzzy JB-ideal is a fuzzy sub JB-semigroup.
Theorem 3.4 The intersection of any nonempty family of fuzzy JB-ideals of $X$ is also a fuzzy JB-ideal of $X$.

Definition 3.5 A fuzzy JB-semigroup $\mu$ of $X$ is called a fuzzy $JB_s$-ideal of $X$ if it satisfies the following conditions: For all $a, b, x, y \in X$,

i. $\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\}$,

ii. $\mu(x \cdot y) \geq \mu(x)$ and $\mu(x \cdot y) \geq \mu(y)$.

Theorem 3.6 Every fuzzy $JB_s$-ideal of $X$ is a fuzzy JB-ideal of $X$.

Theorem 3.7 The intersection of any nonempty family of fuzzy $JB_s$-ideals of $X$ is also a fuzzy $JB_s$-ideal of $X$.

Theorem 3.8 If $\mu$ is a fuzzy $JB_s$-ideal of $X$, then $U(\mu; t)$ is a JB-ideal of $X$ for every $0 \leq t \leq 1$, where $U(\mu; t) \neq \emptyset$.

Proof: Let $\mu$ is a fuzzy $JB_s$-ideal of $X$ and $U(\mu; t) \neq \emptyset$. Let $a, b, x, y \in U(\mu; t)$ with $x * y, a * b \in U(\mu; t)$. Then $\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\} \geq t$. Thus, $(x * a) * (y * b) \in U(\mu; t)$. Suppose that $a \in U(\mu; t)$ and $x \in X$. Then $\mu(a \cdot x) \geq \mu(a) \geq t$ and $\mu(x \cdot a) \geq \mu(a) \geq t$. Hence, $a \cdot x, x \cdot a \in U(\mu; t)$. Therefore, $U(\mu; t)$ is a JB-ideal of $X$. \qed

Theorem 3.9 Let $\emptyset \neq N \subseteq X$ and $\mu_N$ be a fuzzy set in $X$ defined by

$$
\mu_N(x) = \begin{cases} 
\alpha & \text{if } x \in N, \\
\beta & \text{otherwise}
\end{cases}
$$

for all $x \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. If $\mu_N$ is a fuzzy $JB_s$-ideal of $X$, then $N$ is a JB-ideal of $X$. In this case, $X_{\mu_N} = N$.

Proof: Let $a, b, x, y \in X$ such that $x * y, a * b \in N$. Then by Definition 3.5(i), $\mu_N((x * a) * (y * b)) \geq \min\{\mu_N(x * y), \mu_N(a * b)\} = \alpha$. It follows that $\mu_N((x * a) * (y * b)) = \alpha$. Hence, $(x * a) * (y * b) \in N$. Let $a \in N$ and $x \in X$. Then $\mu_N(a \cdot x) \geq \mu_N(a) = \alpha$ and so, $\mu_N(a \cdot x) = \alpha$, that is, $a \cdot x \in N$. Similarly, $x \cdot a \in N$. Therefore, $N$ is a JB-ideal of $X$. Moreover, by Theorem 2.11, $X_{\mu_N} = N$. \qed

Theorem 3.10 Let $\mu$ be a fuzzy set in $X$ with $\text{Im}(\mu) = \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{N_n : n = 0, 1, \ldots, k\}$ be a family of JB-ideals of $X$ such that

(i) $N_0 \subset N_1 \subset \cdots \subset N_k = X$,

(ii) $\mu(N_n') = \alpha_n$, where $N_n' = N_n \setminus N_{n-1}$ for $n = 0, 1, \ldots, k$ and $N_{-1} = \emptyset$.

Then $\mu$ is a fuzzy JB-ideal of $X$. 

Proof: Let $a, b, x, y \in X$. Consider the following cases.

Case 1. If $x * y, a * b \in N'_n$, then $x * y, a * b \in N_n$. Since $N_n$ is a JB-ideal, $(x * a) * (y * b) \in N_n$. By WOP, there exists a smallest $m$ such that $(x * a) * (y * b) \in N_m$ and $(x * a) * (y * b) \notin N_{m-1}$. Thus, $(x * a) * (y * b) \in N'_m$. Hence, $\mu((x * a) * (y * b)) = \alpha_m \geq \alpha_n = \min\{\mu(x * y), \mu(a * b)\}$.

Case 2. If $x * y \in N'_n$ and $a * b \in N'_m$ where $0 \leq m < n \leq k$, then $x * y \in N_n$ and $a * b \in N_m \subset N_n$. Hence, $(x * a) * (y * b) \in N_n$. By WOP, there exists a smallest $p$ such that $(x * a) * (y * b) \in N_p$ and $(x * a) * (y * b) \notin N_{p-1}$. This means that $(x * a) * (y * b) \in N'_p$. Since $m < n$, $\alpha_m > \alpha_n$. It follows that $\mu((x * a) * (y * b)) = \alpha_p \geq \alpha_n = \min\{\mu(x * y), \mu(a * b)\}$.

Case 3. If $x * y \in N'_m$ and $a * b \in N'_n$ where $0 \leq m < n \leq k$, then following the argument as in case 2, we get $\mu((x * a) * (y * b)) \geq \min\{\mu(x * y), \mu(a * b)\}$. Therefore, $\mu$ is a fuzzy JB-ideal of $X$. □

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