An Approach of Some Convergence Type by Filter

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Abstract

This paper is concerned with the giving filter approaching of several type of limit and cluster point notions defined and cited by thousands of papers. By using elementary topological properties of filters some of main results of these papers is easily obtained.

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1. INTRODUCTION

The theory of statistical convergence has been introduced in [4]. Fridy progressed with the concept of statistically Cauchy sequence in [5] and proved that it is equivalent to statistical convergence. Besides, in [6] the notion of the statistical limit point is defined by him. This definition become very efficient tool for some summability problems and studied from hundreds of writer.

The theory of time scales was first constructed by Hilger in his Ph. D. thesis in [1]. The concept of time scale is based on the aspect of unite discreet analysis and continuous analysis. The time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In fact, $\mathbb{T}$ is a complete metric space with the usual metric. Throughout this paper we consider a time scale $\mathbb{T}$ with the topology that inherits from the real numbers with the standard topology. For detailed information about time scale theory, one can see [8] and [9]. Measure theory on time scales has been introduced in [2] then Deniz-Ufuktepe defined
Lebesgue-Stieltjes $\Delta$ and $\nabla$-measures and by using these measures, they defined an integral which is adaptable to time scale, specifically Lebesgue-Stieltjes $\Delta$-integral, in [3]. In the light of these studies, let us introduce some time scale and measure theoretic notations. The forward jump operator $\sigma : T \to T$, for each $t \in T$ by via formula,

$$\sigma(t) := \inf\{s \in T: s > t\}$$

For $a, b \in T$ with $a \leq b$ we define the interval $[a, b]$ in $T$ by

$$[a, b] = \{t \in T: a \leq t \leq b\}.$$ 

Open intervals and half-open intervals are defined similarly. Let $S$ be semi-ring of left-closed and right-open intervals and $m^*$ be Caratheodory extension of the Lebesgue set function $m$ which is defined by $m([a, b)) = b - a$, associated with the family $S$ in the time scale $T$ as in the real case. Also let $\mathcal{M}(m^*)$ be the $\sigma$-algebra of all $m^*$-measurable sets. Recall that $\mathcal{M}(m^*)$ consists of such a subset $E$ has the property that $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$ for all $A \subset T$. It is well known that the restriction of $m^*$ to $\mathcal{M}(m^*)$ which we denote by $\mu_\Delta$ is a countably additive measure on $\mathcal{M}(m^*)$. This measure called Lebesgue $\Delta$-measure. The measurable subsets of $T$ is called $\Delta$-measurable and a function $f : T \to \mathbb{R}$ is called measurable function, if $f^{-1}(O) \in \mathcal{M}(m^*)$ for every open subset $O$ of $\mathbb{R}$. From [2] we know that If $a, b \in T$ and $a \leq b$, then

$$\mu_\Delta((a, b)) = b - a, \quad \mu_\Delta((a, b)) = b - \sigma(a).$$

If $a, b \in T \setminus \{\max T\}$ and $a \leq b$, then

$$\mu_\Delta((a, b]) = \sigma(b) - \sigma(a), \quad \mu_\Delta([a, b]) = \sigma(b) - a.$$ 

The central purpose of the present paper is approaching the various of convergence type which are consider as useful tools for the summability theory by using elementary properties of filters.

2. Generalization the notion of $\Delta$-Convergence by Filters and Some Results

It is natural to attempt to generalize natural density by adapting the ”$\Delta$-density” which is defined in [7]. Let us recall $\Delta$-density, $\Delta$ -convergence, $\Delta$-Cauchy definitions and some results taken from same paper in which necessarily for our purpose. Throughout this section let us take all time scales unbounded from above and have a minimum point.

Let $A$ be a $\Delta$-measurable subset of $T$ and $a = \min T$, the $\Delta$-density of $A$ in $T$ is defined by

$$\delta_\Delta(A) = \lim_{s \to \infty} \frac{\mu_\Delta(A(s))}{\sigma(s) - a}$$

(if this limit exists) where $A(s) = \{t \in A : t \leq s\}$. The $\Delta$-density function can be consider as a probabilistic finite additive measure on the algebra of subset of
An approach of some convergence type by filter

\( \mathbb{T} \) which have a \( \Delta \)-density. We will denote this space by \( \mathcal{M}_d \). Obviously subset of \( \mathbb{T} \) which have a zero \( \Delta \)-density is an element of \( \mathcal{M}_d \). This collection denoted by \( \mathcal{M}_d^0 \) and it has ring structure. By using the \( \Delta \)-density we obtained following new type of convergence which is a generalization of the natural statistical convergence and statistical Cauchy sequences definitions. Let us remember some of these notions. A \( \Delta \)-measurable function \( f \) is called \( \Delta \)-convergent to the number \( L \) if

\[
\delta_\Delta (f^{-1}((L - \epsilon, L + \epsilon))) = 1 \quad \text{for all } \epsilon > 0.
\]

A measurable set \( K \) is called a \( \Delta \)-non thin subset of \( \mathbb{T} \) if it may have a positive \( \Delta \)-density or may not have even a \( \Delta \)-density and a measurable set \( K \) is called a \( \Delta \)-null subset of \( \mathbb{T} \) if

\[
\delta_\Delta (K) = 0.
\]

A measurable function \( f : \mathbb{T} \to \mathbb{R} \) is called \( \Delta \)-bounded if

\[
\delta_\Delta (\{ t \in \mathbb{T} : |f(t)| \leq r \}) = 1.
\]

A measurable function \( f \) is \( \Delta \)-a.a. \( t \) equal to the function \( g \) if the set of \( x \) satisfying \( f(x) \neq g(x) \) is a set of \( \Delta \)-density zero. Note that this definitions coincide with notions of statistical convergence and statistical Cauchy definitions for the real sequences whenever \( \mathbb{T} \) is taken as the natural numbers. The reason of this fact is that notions of \( \Delta \)-density and natural density coincide in the case \( \mathbb{T} = \mathbb{N} \). The rest of this section we will approach the results of the paper [7] by elementary properties of Filter. For detailed information about the concept of filter one can see [10].

**Proposition 2.1.** Let \( \mathbb{T} \) be a time scale. Followings are true.

1- The family

\[
\mathcal{F}_1 = \{ A \subset \mathbb{T} : \delta_\Delta (A) = 1 \}
\]

is a filter base on \( \mathbb{T} \).

2-Let \( (X, \tau) \) be a topological space and \( f : \mathbb{T} \to X \) be a function. Then the family

\[
\mathcal{F}_1^f = \{ f(A) \subset X : A \in \mathcal{F}_1 \}
\]

is a filter base on \( X \).

3-The family

\[
\mathcal{F}_2 = \{ \{ t \in \mathbb{T} : t > t_0 \} : t_0 \in \mathbb{T} \}
\]

is a filter base on \( \mathbb{T} \).

4-Let \( (X, \tau) \) be a topological space and \( f : \mathbb{T} \to X \) be a function. Then the family

\[
\mathcal{F}_2^f = \{ f(A) \subset X : A \in \mathcal{F}_2 \}
\]

is a filter base on \( X \).

5-The filter base \( \mathcal{F}_1 \) is finer than \( \mathcal{F}_2 \) and \( \mathcal{F}_1^f \) is finer than \( \mathcal{F}_2^f \).

From above we can easily summarize the notion of statically convergence or more general \( \Delta \)-convergence notion by using well known filter language.

**Theorem 2.2.** Let \( \mathbb{T} \) be a time scale and \( f : \mathbb{T} \to \mathbb{R} \) be a function. Then the followings are true.

1- \( \Delta \)- \( \lim_{t \to \infty} f(t) = L \) if and only if \( \mathcal{F}_1^f \to L \).

2- \( \lim_{t \to \infty} f(t) = L \) if and only if \( \mathcal{F}_2^f \to L \).
$3$. $f$ is $\Delta$-Cauchy if and only if $\mathbb{F}^f_1$ is Cauchy.

Following theorem gives main results of the paper [7]. Note that this paper is generalization of Fridy’s paper [5] which is cited around 300 times from several writers. Please note that proof of each claim is only a few words.

**Theorem 2.3.** Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$ be a function. Then the followings are true.

1. The limit of $\mathbb{F}^f_1$ is unique.
2. If $\mathbb{F}^f_2 \to L$ then $\mathbb{F}^f_1 \to L$.
3. Followings are equivalent.
   - $\mathbb{F}^f_1 \to L$.
   - $\mathbb{F}^f_1$ is Cauchy.
   - There exists a function $g : \mathbb{T} \to \mathbb{R}$ which is $\Delta$-a.a.t. equal to $f$ such that $\mathbb{F}^g_1 \to L$.

**Proof.**

1. $\mathbb{R}$ is Hausdorff.
2. $\mathbb{F}^f_1 \subset \mathbb{F}^f_2$.
3. For $i \iff ii$: $\mathbb{R}$ is complete. For $i \iff iii$: $\mathbb{F}^g_1 \to L \iff \mathbb{F}^f_1 \to L$.

We will close with another convergence notion which is known as cluster point of a sequence. The concept of statistically cluster point is defined in [6]. This notion can be seen as special case of $\Delta$-cluster point of a $\Delta$-measurable function $f$ defined in [11]. Let us recall that the number $L$ is called a $\Delta$-cluster point of a $\Delta$-measurable function $f$ if $\delta_{\Delta}(f^{-1}((L - \epsilon, L + \epsilon)))$ is a $\Delta$-non thin subset of $\mathbb{T}$.

**Theorem 2.4.** Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$ be a function. If $L$ is $\Delta$-cluster point of $f$ then $L$ is cluster point of the filter $\mathbb{F}^f_1$.

**Proof.** If $L$ is $\Delta$-cluster point of $f$ then $f^{-1}(L - \epsilon, L + \epsilon)$ is a $\Delta$-non thin subset of $\mathbb{T}$ for all $\epsilon > 0$. Assume that there exists $A \in \mathbb{F}_1$ such that no matter how $\epsilon$ is $f(A) \cap (L - \epsilon, L + \epsilon) = \emptyset$. This implies $A \subset \mathbb{T} - f^{-1}(L - \epsilon, L + \epsilon)$. Since $\delta_{\Delta}(A) = 1$ then $\delta_{\Delta}(f^{-1}(L - \epsilon, L + \epsilon)) = 0$. But that means the set $f^{-1}(L - \epsilon, L + \epsilon)$ is not a $\Delta$-non thin set.

**References**


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