Inextensible Curve Flows According to Bishop 2-Type Frame in Minkowski 3-Space

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Abstract

In this paper, we derive necessary and sufficient conditions for inextensible curve flow of timelike curves and spacelike curves with timelike normal according to Bishop 2-type frame in the Minkowski-3 space \( R^3_1 \).

Keywords: Bishop 2-type, Minkowski 3-space, inextensible

1 Introduction

The flow of a curve is called to be inextensible if its arclength is preserved [5]. Kwon, Park and Chi investigated inextensible flows of curves and developable surfaces in Euclidean 3-space [6]. Gürbüz study inextensible curve flows for null and non-null curves in Euclidean 3-space [1-3]. Yıldız, Tosun and Karakuş investigated inextensible flows of curves in \( E^n \) [10]. New version of Bishop frame was studied by Yılmaz [8]. In [9], authors studied characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space. Kızıltuğ was investigated characterization of inextensible flows of curves according to Type-2 Bishop frame Euclidean 3-space [4]. In this paper we study inextensible non-null curve flows according to Bishop 2-type frame in Minkowski 3-space.

Minkowski 3-space \( R^3_1 \) is given the following metric [7]:

\[
\langle \cdot, \cdot \rangle_M = -dx_1^2 + dx_2^2 + dx_3^2
\]

Frenet-Serret frame \( \{ T, N, B \} \) derivative formulas are given in \( R^3_1 \) as following:
\[ T_s = \epsilon_2 \kappa N, \]
\[ N_s = -\epsilon_1 \kappa T + \epsilon_3 \tau B, \]
\[ B_s = -\epsilon_2 \tau N. \]

Here \( \kappa, \tau \) Frenet curvatures with
\[ \langle T, T \rangle_M = \epsilon_1, \langle N, N \rangle_M = \epsilon_2, \langle B, B \rangle_M = \epsilon_3, \epsilon_i = \pm 1 \]
For an arbitrary vector \( x = (x_1, x_2, x_3) \) in \( \mathcal{R}_1^3 \), if \( \langle x, x \rangle_M > 0 \), \( x \) is spacelike, if \( \langle x, x \rangle_M < 0 \), \( x \) is timelike. Spacelike and timelike vectors are called non-null vectors.

Let \( \{V_1, V_2, B\} \) be Bishop 2-type frame with
\[ \langle V_1, V_1 \rangle_M = \epsilon_1, \langle V_2, V_2 \rangle_M = -\epsilon_2, \langle B, B \rangle_M = \epsilon_3. \]
Bishop 2-type frame derivative formulas are presented for timelike curves and spacelike curves with timelike normal in \( \mathcal{R}_1^3 \) as different from [9] in [3] as following:

**Theorem 1.1** [3] Let \( \{T, N, B\} \) be Frenet frame for a spacelike curve with timelike normal
\[ \langle T, T \rangle_M = 1, \langle N, N \rangle_M = -1, \langle B, B \rangle_M = 1 \]
and let \( \{V_1, V_2, B\} \) be Bishop 2-type frame with
\[ \langle V_1, V_1 \rangle_M = 1, \langle V_2, V_2 \rangle_M = -1, \langle B, B \rangle_M = 1 \]
Bishop 2-type frame derivative formulas are given by as following in \( \mathcal{R}_1^3 \) :
\[
\begin{bmatrix}
    V_{1s} \\
    V_{2s} \\
    B_s
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & -k_1 \\
    0 & 0 & k_2 \\
    k_1 & k_2 & 0
\end{bmatrix}
\begin{bmatrix}
    V_1 \\
    V_2 \\
    B
\end{bmatrix}
\]
(2)

where \( k_1, k_2 \) are curvatures according to Bishop 2-type frame in the Minkowski 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:
\[
\begin{bmatrix}
    T \\
    N \\
    B
\end{bmatrix} =
\begin{bmatrix}
    \cosh \theta & \sinh \theta & 0 \\
    \sinh \theta & \cosh \theta & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    V_1 \\
    V_2 \\
    B
\end{bmatrix}
\]

First and second Bishop 2-type curvatures are \( k_1 = -\langle V_{1s}, B \rangle_M = \tau \sinh \theta \), \( k_2 = \langle V_{2s}, B \rangle_M = \tau \cosh \theta \). Also \( \theta_s = -\kappa, \tau = |k_1^2 - k_2^2|^{1/2} \).

**Proof**. The tangent vector \( T \) can be written by
Inextensible curve flows according to Bishop 2-type frame

\[ \mathcal{T} = \cosh \theta V_1 + \sinh \theta V_2 \]  

(3)

Taking derivative of (3), substituting \( V_{1s} = -k_1 B, V_{2s} = k_2 B \) we obtain

\[ \mathcal{N} = \sinh \theta V_1 + \cosh \theta V_2, \quad \theta = \arg \tanh \frac{k_1}{k_2}, \quad \theta_s = -\kappa \]

From derivative of binormal,

\[ B_s = k_1 V_1 + k_2 V_2 = \tau \mathcal{N} \]  

(4)

Taking norm of (4), we have \( \tau = |k_1^2 - k_2^2|^{1/2} \).

**Theorem 1.2** [3] Let \( \{\mathcal{T}, \mathcal{N}, B\} \) be Frenet frame with \( \langle \mathcal{T}, \mathcal{T} \rangle_M = -1, \langle \mathcal{N}, \mathcal{N} \rangle_M = 1, \langle B, B \rangle_M = 1 \) and let \( \{V_1, V_2, B\} \) be Bishop 2-type frame with \( \langle V_1, V_1 \rangle_M = -1, \langle V_2, V_2 \rangle_M = 1, \langle B, B \rangle_M = 1 \). Bishop 2-type frame derivative formulas for timelike curves are given as following:

\[
\begin{bmatrix}
V_{1s} \\
V_{2s} \\
B_s
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -k_1 \\
0 & 0 & k_2 \\
-k_1 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
B
\end{bmatrix}
\]

where \( k_1, k_2 \) are curvatures according to Bishop 2-type frame in Minkowski 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:

\[ \mathcal{T} = \cosh \theta V_1 + \sinh \theta V_2 \]

\[ \mathcal{N} = \sinh \theta V_1 + \cosh \theta V_2 \]

\[ B = B \]

First and second Bishop 2-type curvatures are \( k_1 = \tau \sin \theta, k_2 = \tau \cos \theta \). Here \( \theta_s = \kappa \). Proof is obtained as similar with Theorem 1.1. As result, Bishop 2-type frame derivative formulas for timelike curves and spacelike curves with timelike normal are given as following [3]:

\[
\begin{bmatrix}
V_{1s} \\
V_{2s} \\
B_s
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -k_1 \\
0 & 0 & k_2 \\
\varepsilon_1 k_1 & \varepsilon_1 k_2 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
B
\end{bmatrix}
\]

(5)

where

\[ \langle V_1, V_1 \rangle_M = \varepsilon_1, \langle V_2, V_2 \rangle_M = \varepsilon_2, \langle B, B \rangle_M = \varepsilon_3, \varepsilon_i = \pm 1 \]

Here \( k_1, k_2 \) are first and second Bishop 2-type curvatures in Minkowski 3-space.
2 Inextensible flows of non-null curves according to Bishop 2-type frame in Minkowski 3-space

We consider \( P(\xi,t) : [0,l] \times [0,t] \to \mathcal{R}^3 \) one parameter family. Flow of spacelike curves with timelike normal and timelike curves according to Bishop 2-type frame is expressed as following

\[
\frac{\partial P(\xi,t)}{\partial t} = fV_1 + gV_2 + hB
\]

where \( f, g, h \) are components of flow of non null curve family. Let

\[
v = \left| \left< \frac{\partial P}{\partial \xi}, \frac{\partial P}{\partial \xi} \right>_M \right|^{1/2}
\]

be curve speed and arc length is \( s = \int_0^\xi v d\xi \) according to Bishop 2-type frame.

**Definition 2.1.** \( \frac{P(\xi,t)}{\partial t} \) are defined to be inextensible according to Bishop 2-type frame if

\[
\frac{\partial}{\partial t} \sqrt{\left| \left< \frac{\partial P}{\partial \xi}, \frac{\partial P}{\partial \xi} \right>_M \right|} = 0
\]

**Lemma 2.1.**

\[
\frac{\partial v}{\partial t} = \varepsilon_1 \cosh \theta \left( \frac{\partial f}{\partial t} + vhk_1 \right) + \varepsilon_2 \sinh \theta \left( \frac{\partial g}{\partial t} + vhk_2 \right)
\]

**Proof.** From (6), it is obtained

\[
2v \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left| \left< \frac{\partial P}{\partial \xi}, \frac{\partial P}{\partial \xi} \right>_M \right| = 2v \left| \left< \cosh \theta V_1 + \sinh \theta V_2, \frac{\partial}{\partial t} (fV_1 + gV_2 + hB) \right>_M \right|
\]

From (8), we obtain (7).

**Theorem 2.1.** Spacelike curve with timelike normal and timelike curve flow \( \frac{\partial P}{\partial t} = fV_1 + gV_2 + hB \) according to Bishop 2-type frame is inextensible if and only if

\[
\varepsilon_1 \cosh \theta \left( \frac{\partial f}{\partial t} + vhk_1 \right) = -\varepsilon_2 \sinh \theta \left( \frac{\partial g}{\partial t} + vhk_2 \right).
\]

**Proof.** If spacelike curve with timelike normal and timelike curve not be subject to any elongation or compression, the condition

\[
\frac{\partial}{\partial t} s(\xi,t) = \int_0^\xi \frac{\partial v}{\partial t} d\xi = 0
\]
is satisfied. From Lemma 2.1,
\[
\frac{\partial}{\partial t}s(\xi, t) = \int_0^\xi \varepsilon_1 \cosh \theta (\frac{\partial f}{\partial t} + \varepsilon h k_1) + \varepsilon_2 \sinh \theta (\frac{\partial g}{\partial t} + \varepsilon h k_2) \, d\xi = 0 \tag{10}
\]
From (10) we obtain (9).

**Theorem 2.2.**
\[
\frac{\partial V_1}{\partial t} = [-\varepsilon \sinh \theta (\varepsilon (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1) - \varepsilon \sinh \theta \frac{\partial \theta}{\partial t}) - \varepsilon_2 \Psi_1] V_2 - \varepsilon_3 \Phi_1 B, \tag{11}
\]
\[
\frac{\partial V_2}{\partial t} = [-\varepsilon \cosh \theta (\varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2) - \varepsilon \cosh \theta \frac{\partial \theta}{\partial t}) + \varepsilon_1 \Psi_1] V_1 - \varepsilon_3 \Phi_2 B, \tag{12}
\]
\[
\frac{\partial B}{\partial t} = V_1 [\varepsilon_1 \Phi - \varepsilon_3 (\frac{\partial h}{\partial s} - f k_1 + g k_2) \cosh \theta] + V_2 [\varepsilon_2 \Phi_2 - \varepsilon_3 (\frac{\partial h}{\partial s} - f k_1 + g k_2) \sinh \theta]. \tag{13}
\]

**Proof.**
\[
\frac{\partial T}{\partial t} = \frac{\partial}{\partial s} \frac{\partial P}{\partial t} = (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1)V_1 + (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2)V_2 + (\frac{\partial h}{\partial s} - f k_1 + g k_2)B \tag{14}
\]
From (14),
\[
\left\langle \frac{\partial T}{\partial t}, V_1 \right\rangle_M = \varepsilon_1 (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1). \tag{15}
\]
Also
\[
\left\langle \frac{\partial T}{\partial t}, V_1 \right\rangle_M = \sinh \theta (\varepsilon_1 \frac{\partial \theta}{\partial t} + \left\langle \frac{\partial V_2}{\partial t}, V_1 \right\rangle_M). \tag{16}
\]
From (15), (16) we obtain
\[
\sinh \theta (\varepsilon_1 \frac{\partial \theta}{\partial t} + \left\langle \frac{\partial V_2}{\partial t}, V_1 \right\rangle_M) = \varepsilon_1 (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1). \tag{17}
\]
\[
\left\langle \frac{\partial V_1}{\partial t}, V_2 \right\rangle_M + \left\langle \frac{\partial V_2}{\partial t}, V_1 \right\rangle_M = 0 \Rightarrow \frac{\partial V_1}{\partial t} = -\varepsilon_2 \Psi_1 V_2, \tag{18}
\]
\[
\left\langle \frac{\partial V_1}{\partial t}, B \right\rangle_M + \left\langle \frac{\partial B}{\partial t}, V_1 \right\rangle_M = 0 \Rightarrow \frac{\partial V_1}{\partial t} = -\varepsilon_3 \Phi_1 B, \tag{19}
\]
where
\[
\Psi_1 = \left\langle V_1, \frac{\partial V_2}{\partial t} \right\rangle_M, \quad \Phi_1 = \left\langle V_1, \frac{\partial B}{\partial t} \right\rangle_M. \tag{20}
\]
With aid (17), (18), (19) we obtain (11). From (14)

$$\left\langle \frac{\partial T}{\partial t}, V_2 \right\rangle_M = \varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2),$$

(21)

$$\left\langle \frac{\partial T}{\partial t}, V_2 \right\rangle_M = \cosh \theta \left( \left\langle \frac{\partial V_1}{\partial t}, V_2 \right\rangle_M + \varepsilon_2 \frac{\partial \theta}{\partial t} \right).$$

(22)

From (22),

$$- \cosh \theta \left( \left\langle \frac{\partial V_2}{\partial t}, V_1 \right\rangle_M + \varepsilon_2 \frac{\partial \theta}{\partial t} \right) = \varepsilon_2 \left( \frac{\partial g}{\partial s} + \varepsilon_1 h k_2 \right)$$

(23)

From (20),

$$\frac{\partial V_2}{\partial t} = \varepsilon_1 \Psi_1 V_1,$$

(24)

$$\left\langle \frac{\partial V_2}{\partial t}, B \right\rangle_M + \left\langle V_2, \frac{\partial B}{\partial t} \right\rangle_M = 0 \Rightarrow \frac{\partial V_2}{\partial t} = -\varepsilon_3 \Phi_2 B,$$

(25)

where

$$\left\langle V_2, \frac{\partial B}{\partial t} \right\rangle_M = \Phi_2$$

(26)

With aid (23), (24), (25) we obtain (12)

$$\left\langle \frac{\partial T}{\partial t}, B \right\rangle_M = \varepsilon_3 \left( \frac{\partial h}{\partial s} - f k_1 + g k_2 \right),$$

(27)

$$\left\langle \frac{\partial T}{\partial t}, B \right\rangle_M = \cosh \theta \left\langle \frac{\partial V_1}{\partial t}, B \right\rangle_M + \sinh \theta \left\langle \frac{\partial V_2}{\partial t}, B \right\rangle_M$$

(28)

From (26), (27), (28) we have (13).

**Theorem 2.3.** Assume spacelike curve with timelike normal flow and timelike curve flow $\frac{\partial P}{\partial t}$ is inextensible for Bishop 2-type. Necessary and sufficient conditions for inextensible non-null curve flow are given as a system of partial differential equation contain first and second Bishop 2-type curvatures as following:

$$\frac{\partial k_1}{\partial t} = k_2 \varepsilon_2 \Psi_1 + \varepsilon_2 \sinh \theta(\varepsilon_1 (\frac{\partial f}{\partial s} + \varepsilon_1 h k_1) - \varepsilon_1 \sinh \theta \frac{\partial \theta}{\partial t}) + \varepsilon_3 \frac{\partial \Phi_1}{\partial s}$$

(29)

$$\frac{\partial k_2}{\partial t} = -\varepsilon_3 \frac{\partial \Phi_2}{\partial s} - k_1 [\varepsilon_1 \Psi_1 - \varepsilon_1 \cosh \theta(\varepsilon_2 (\frac{\partial g}{\partial s} + \varepsilon_1 h k_2) - \varepsilon_2 \cosh \theta \frac{\partial \theta}{\partial t})]$$

(30)

**Proof.**

$$\frac{\partial}{\partial t} \frac{\partial V_1}{\partial s} = \frac{\partial}{\partial t}(-k_1 B) = -\frac{\partial k_1}{\partial t} B - k_1 \frac{\partial B}{\partial t},$$

(31)
\[
\frac{\partial}{\partial t} \frac{\partial V_1}{\partial s} = [k_2 \Omega - \varepsilon_3 \frac{\partial \Phi_1}{\partial s}] B + \left[ \frac{\partial \Omega}{\partial s} - \varepsilon_3 \varepsilon_1 k_2 \Psi_1 \right] V_2 - \varepsilon_1 \varepsilon_3 k_1 \Phi_1 V_1
\] (32)

where
\[
\Omega = -\varepsilon_2 \Psi_1 + \varepsilon_1 \varepsilon_2 \sinh \theta (\sinh \theta \frac{\partial \theta}{\partial t} - (\frac{\partial f}{\partial s} + h k_1)).
\]

and using compatibility conditions (31), (32), we obtain (29). From
\[
\frac{\partial}{\partial t} \frac{\partial V_2}{\partial s} = \frac{\partial}{\partial s} \frac{\partial V_2}{\partial t}
\]
we obtain (30).

References


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