On Properties of Fractional Probability Measure

S. A. El-Shehawy

Department of Mathematics, Faculty of Science
Menoufia University, Shebin El-Kom, ZIP-Code 32511, Egypt

Abstract

This paper aims to present a study about some properties of the fractional probability measure. We define the fractional moments generating function and deduce its properties. We provide various results and show their application in an illustrated example. Moreover, this study confirms that the theorem of continuity of probability measure can not be achieved for the fractional probability measure.

Mathematics Subject Classification: 60E05, 60E99, 62E10, 62E20, 62H05, 62H10

Keywords: Fractional calculus, Probability density of fractional order \( \alpha \); fractional probability measure; fractional moments, fractional moments generating function; probability measure continuity

1 Introduction

Nowadays, the fractional calculus is sufficiently used to investigate many complex phenomena [1 – 20]. It has many important applications and uses in different research areas. The fractional probability theory can be considered as one of these areas [2, 4, 12, 21]. In [12], the first step in expanding a fractional probability theory has been provided and also the probability density of fractional order \( f_\alpha(x) \), \( f_\alpha(x) \geq 0 \),

\[
P \{ x \leq X < x' \} = F(x, x') = \frac{\Gamma(\alpha)}{\Gamma(n+1)} \int f_\alpha(\xi) d\xi^n
\]

has been defined by using fractional calculus. By using the uniform fractional probability density function, which can be explained by the probability density of fractional order, the classical probability axioms are validated for a fractional probability measure [12, 21]. Furthermore the evaluations of the fractional probability
principles are results in the fractional probability space \((\Omega, F, P_\alpha)\). Recently, some properties of the fractional probability measure in regards to the corresponding properties of classical probability measure have been discussed in \([2, 4 – 5, 8, 12, 21]\). One important theorem in the classical probability theory is the probability measure continuity \((P(\lim_{n \to \infty} X_n(\omega)) = \lim_{n \to \infty} P(X_n(\omega)))\) [22], which supports in proving some theorems in the probability theory.

In this paper we will continue the studies about properties of the fractional probability measure. We define the appropriate moment generating function in regards to the fractional probability space \((\Omega, F, P_\alpha)\). We provide proofs for some characteristics of this function. Moreover, one of our aims here is to verify whether the theorem of continuity of probability measure is satisfied in the fractional probability space or not (i.e., is it possible to set \(P(\lim_{n \to \infty} X_n(\omega)) = \lim_{n \to \infty} P(X_n(\omega))\)?).

2 Axioms and Properties of Fractional Probability

Here, we start with some definitions about the concept of fractional probability.

Definition 1 [21]
A fractional probability space is a triple \((\Omega, F, P_\alpha)\) where \(\Omega\) is the sample space corresponding to outcomes of some experiment, \(F\) is the \(\sigma\)–algebra of subsets (events) of \(\Omega\), and \(P_\alpha: F \to [0, 1], 0 < \alpha < 1\) is a fractional probability measure.

Definition 2 [12, 21]
Let \(X\) denote a real-valued random variable with the probability density \(f_\alpha(x)\), \(f_\alpha(x) \geq 0\). \(X\) is referred to as a random variable with fractional probability density of order \(\alpha, 0 < \alpha < 1\). Whenever one has the form (1) with normalizing condition
\[
\int_{-\infty}^{\infty} f_\alpha(x) \, (d\,x)^\alpha = 1.
\]

If it is assumed that the fractional probability density \(f_\alpha(x)\) is a fractional probability measure, according to the corresponding classical probability axioms \((P(A) \geq 0\) for all \(A \in F\) and \(P(\Omega) = 1\)), in Definition 2 the expressions \(f_\alpha(x) \geq 0\) and \(\int_{-\infty}^{\infty} f_\alpha(x) \, (d\,x)^\alpha = 1\) can be considered as two initial axioms of fractional probability measure. Therefore, the first axiom of fractional probability measure can be defined as \(P_\alpha(A) \geq 0\) for each event \(A \subset \Omega\) and the second axiom can be defined as \(P_\alpha(\Omega) = \int_{-\infty}^{\infty} f_\alpha(x) \, (d\,x)^\alpha = 1\). But, the third axiom of classical probability measure (i.e. for all \(A_i \in F\), if \(\{A_i\}\) are pairwise disjoint, then \(P(\bigcup A_i) = \sum_i P(A_i)\)) is not satisfied
in fractional probability space. To achieve this statement, consider the uniform probability density of fractional order $\alpha$ on the interval $[a,b]$ with the probability density
\[ f_a(x) = (b-a)^{-\alpha} \quad a \leq x \leq b, \]
(2)
see [12, 21]. Let $A_1 = [a,c]$ and $A_2 = [c,b]$ be two disjoint events. Then
\[ P_\alpha(a < x < c) = \frac{(c-a)^\alpha}{(b-a)^\alpha}, \quad P_\alpha(c < x < b) = \frac{(b-c)^\alpha}{(b-a)^\alpha} \quad \text{and} \quad P_\alpha(a < x < b) = 1. \]
By referring to Minkovski’s inequality we find that
\[ 1 \leq \frac{(c-a)^\alpha}{(b-a)^\alpha} + \frac{(b-c)^\alpha}{(b-a)^\alpha}, \quad [21 – 23]. \]
Therefore $P_\alpha(a < x < b) < P_\alpha(a < x < c) + P_\alpha(c < x < b)$.
This implies that, the third axiom of classical probability measure is not satisfied in fractional probability space. So, the axioms of fractional probability measure can be described as follows.

**Definition 3** [21] (Axioms of Fractional Probability Measure)
Given a sample space $\Omega$ and an associated $\sigma$–field $F$, a fractional probability measure of order $0 < \alpha < 1$, is a set function $P_\alpha: F \rightarrow [0,1], 0 < \alpha < 1$, that satisfies
1. $P_\alpha(A) \geq 0$ for all $A \in F$,
2. $P_\alpha(\Omega)=1$,
3. for all $A_i \in F$, if $\{A_i\}$ are pairwise disjoint, then $P_\alpha(\bigcup A_i) \leq \sum P_\alpha(A_i)$.
In the following theorem, we provide some properties of the fractional probability measure.

**Theorem 1**
Let $(\Omega, F, P_\alpha)$ be a fractional probability space, then one has
(i) $P_\alpha(\emptyset) = 0$,
(ii) If $A, B$ are two events such that $A \subset B$, then $P_\alpha(A) \leq P_\alpha(B)$,
(iii) $1 - P_\alpha(A^c) \leq P_\alpha(A) \leq 1$,
(iv) $P_\alpha(A - B) \leq P_\alpha(A) - P_\alpha(A \cap B)$,
(v) $P_\alpha(B - A) \leq P_\alpha(B) - P_\alpha(A \cap B)$,
(vi) $P_\alpha(A \cup B) \leq P_\alpha(A) + P_\alpha(B) - P_\alpha(A \cap B) \leq 1$,

**Proof:**
To prove (i), (ii) and (iii), see [21].
(iv) We can write the event $A$ as $A = (A - B) \cup (B \cap A)$, where $(A - B) \cap (A \cap B) = \emptyset$.
By applying axiom (iii) we find that
\[ P_\alpha(A) = P_\alpha((A - B) \cup (A \cap B)) \leq P_\alpha(A - B) + P_\alpha(A \cap B). \]
This implies that $P_\alpha(A - B) \geq P_\alpha(A) - P_\alpha(A \cap B)$. 

(v) We can write \(B = (B - A) \cup (A \cap B)\), where \((B - A) \cap (B \cap A) = \varnothing\). By applying axiom (iii) we find that \(P_a(B) = P_a((B - A) \cup (B \cap A)) \leq P_a(B - A) + P_a(B \cap A)\).

This implies that \(P_a(B - A) \geq P_a(B) - P_a(B \cap A)\).

(vi) We can write \(A \cup B = A \cup (B - A)\), where \(A \cap (B - A) = \varnothing\).

By applying axiom (iii) we find that \(P_a(A \cup B) = P_a(A \cup (B - A)) \leq P_a(A) + P_a(B - A) \leq P_a(A) + P_a(B) - P_a(A \cap B) \leq 1\).

3 The \(k\)th Moment of Fractional Order

**Definition 4** [4, 12, 21]

For any \(k\) positive integer, \(k\)th moment of fractional order \(\alpha, 0 < \alpha < 1\), of random variable \(X\) is defined by the expression

\[m_{\alpha} := E_a[X] = \int_{\mathbb{R}} x^{\alpha} f_a(x) \, dx.\]  (3)

The first moment of fractional order \(\alpha\),

\[m_\alpha = \int_{\mathbb{R}} x^\alpha f_a(x) \, dx\]  (4)

which is the expected value of fractional order \(\alpha, 0 < \alpha < 1\) and the fractional variance of order \(\alpha\) results from the expression

\[\sigma_\alpha^2 := m_{2\alpha} - (m_\alpha)^2\]  (5)

**Theorem 3** [21]

For any real constant \(a\) and \(b\),

(i) \(E_a[aX] = a^n E_a[X]\), \(E_a[b] = b^n\).

(ii) If \(g(x) \geq 0\), for all \(x\), then \(E_a[g(X)] \geq 0\).

(iii) If \(g_1(x) \geq g_2(x)\), for all \(x\), then \(E_a[g_1(X)] \geq E_a[g_2(X)]\).

(iv) If \(a \leq g_1(x) \leq b\), for all \(x\), then \(a^n \leq E_a[g_1(X)] \leq b^n\).

(v) \(E_a[aX + bY] \leq a^n E_a[X] + b^n E_a[Y]\).

Here, we define the moments generating function in the case of the fractional probability space and this function is called the \(\alpha\)-moments generating function. This function is denoted by "\(M_{\alpha} (t)\)". Moreover, we will prove some properties of this function.

**Definition 5**

Let \(X\) denote a real-valued random variable with the fractional probability density \(f_a(x)\), \(f_a(x) \geq 0\) of order \(\alpha, 0 < \alpha < 1\). The \(\alpha\)-moments generating function of \(X\) is defined by the following expression

\[M_{\alpha} (t) := E_a[e^{tX}] = \int_{\mathbb{R}} e^{tx} f_a(x) \, dx\]  (6)
Theorem 4

Let $X$ denote a real-valued random variable with the fractional probability density $f_\alpha(x)$, $f_\alpha(x) \geq 0$ of order $\alpha$, $0 < \alpha < 1$. Then,

(i) the $k$th moments of order $\alpha$ can be obtain from the relation

$$m_k = \left. \frac{d^k}{dt^k} M_{X,\alpha}(t) \right|_{t=0}$$

(ii) for any real constant $a$ and $b$,

$$M_{aX+bX,\alpha}(t) = e^{b^\alpha t} M_{X,\alpha}(at)$$

Proof:

(i) From the form (6) of the $\alpha$–moments generating function of $X$ we find that

$$\frac{d^k}{dt^k} M_{X,\alpha}(t) = \frac{d^k}{dt^k} \int_a^b e^{tx^\alpha} f_\alpha(x) (dx)^\alpha = \int_a^b \frac{d^k}{dt^k} e^{tx^\alpha} f_\alpha(x) (dx)^\alpha = \int_a^b x^k \alpha e^{tx^\alpha} f_\alpha(x) (dx)^\alpha.$$

Substitute $t=0$ in the two sided, we obtain

$$\left. \frac{d^k}{dt^k} M_{X,\alpha}(t) \right|_{t=0} = \int_a^b x^k \alpha e^{0X^\alpha} f_\alpha(x) (dx)^\alpha = \int_a^b x^k f_\alpha(x) (dx)^\alpha.$$

By comparing the last result with the form (3) of the $k$th moments of order $\alpha$, where $k = 1, 2, 3, ...$, we find that the proof is completed.

(ii) By applying the form of the $\alpha$–moments generating function we find that,

$$M_{aX+bX,\alpha}(t) = E_x e^{(aX+b)t} f_\alpha(x) (dx)^\alpha = e^{b^\alpha t} \int_a^b x^k \alpha e^{tx^\alpha} f_\alpha(x) (dx)^\alpha = e^{b^\alpha t} M_{X,\alpha}(at).$$

In the following illustrated example we will explain one of the available simple fractional probability distribution and deduce some general forms for the $\alpha$–moments generating function, the $\alpha$–moments and $\alpha$–variance. Moreover, we provide special cases of the studied fractional probability distribution.

Example 1

For a uniform random $\alpha$- variable $X$ on the interval $[a,b]$ with probability density given in (2), the form of the $\alpha$–moments generating function of $X$ can be obtained as follows:

$$M_a(t) = \int_a^b e^{tx^\alpha} (b-a)^-\alpha (dx)^\alpha = \int_a^b e^{tx^\alpha} (dx)^\alpha.$$

To find the integral $\int_a^b e^{tx^\alpha} (dx)^\alpha$, we apply the following relation of the fractional derivatives $D^\alpha e^x = t^\alpha e^x$ (see [1, 5, 6, 9, 11, 15]) to $e^{tx^\alpha}$ and using the notation $\Gamma(1+\alpha) = \alpha !$, see [12, 21], we find that $\frac{1}{\alpha !} t^\alpha D^\alpha e^x = e^{tx^\alpha}$. 

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Then, from the relations between the fractional derivatives and integrations we find that

\[ \int_a^b e^{t\alpha} \, (d x)^\alpha = \frac{1}{\alpha!} \left( e^{t\alpha} e^t - e^{t\alpha} \right) \]  

(10)

From (9) and (10), we obtain the \( \alpha \)–moment in the following form:

\[ M_{\alpha}(t) = \frac{\left( e^{t\alpha} e^t - e^{t\alpha} \right)}{\alpha!} \]  

(11)

To deduce the fractional \( \alpha \)–moment \( m_{\alpha} \) and the \( \alpha \)–variance \( \sigma_{\alpha}^2 \) for this uniform random \( \alpha \) - variable \( X \) on the interval \( [a,b] \) we start from the general formula of the \( \alpha \)–moment as follows:

\[ m_{\alpha} = \int_a^b x^{k\alpha} (d x)^\alpha = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-a+1)} x^{k\alpha-a} = \frac{(k\alpha)!}{(k\alpha-a)!}, \quad \text{see [4 – 5, 8, 12, 21]} \]

(12)

Then, from the relations between the fractional derivatives and integrations we find that

\[ \int_a^b x^{k\alpha} (d x)^\alpha = \frac{(k\alpha)!}{(k\alpha-a)!} (b^{\alpha-a} - a^{\alpha-a}) \]  

(13)

From (12) and (13), we obtain the \( \alpha \)–moment, where \( 0 < \alpha < 1 \), in the following form:

\[ m_{\alpha} = \frac{(k\alpha)!}{(k\alpha-a)!} \frac{(b^{\alpha-a} - a^{\alpha-a})}{(b-a)^a} \]  

(14)

The \( \alpha \)–variance (the fractional variance of order \( \alpha \), \( 0 < \alpha < 1 \)) results from the given expression in (5), where

\[ m_a = \frac{\alpha!}{(2\alpha)!} (b^a - a^a) \quad \text{and} \quad m_{2a} = \frac{(2\alpha)!}{(3\alpha)!} \frac{(b^{3a} - a^{3a})}{(b-a)^a} \, , \]

are the first and the second \( \alpha \)–moment respectively. Therefore, the \( \alpha \)–variance (the fractional variance of order \( \alpha \)) results from the following expression

\[ \sigma_{\alpha}^2 := m_{2a} - (m_a)^2 \]  

(15)

This implies that, the \( \alpha \)–variance (the fractional variance of order \( \alpha \)) is

\[ \sigma_{\alpha}^2 = \frac{(2\alpha)!}{(3\alpha)!} \frac{(b-a)^a (b^{\alpha-a} - a^{\alpha-a}) - (3\alpha)! (\alpha!)^2 (b^{3a} - a^{3a})^2}{(b-a)^a} \]  

(16)
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In this example of the fractional uniform distribution of order $\alpha$, $0 < \alpha < 1$, on the interval $[a, b]$, it is noted that the obtained results of the first $\alpha$-moment, the second $\alpha$-moment and the $\alpha$-variance identify with the first moment, the second moment and the variance of the uniform distribution on the interval $[a, b]$ in the integer case (i.e. $\alpha \in \{1\}$), $m_1 = \frac{b + a}{2}$, $m_2 = \frac{b^2 + \alpha b + a^2}{3}$, and $\sigma^2 = \frac{(b - a)^2}{12}$, respectively.

For the case $a = 0$ in this example of the fractional uniform distribution of order $\alpha$, $0 < \alpha < 1$, on the interval $[a, b]$, we find that the probability density function is $f_\alpha(x) = b^{-\alpha}, 0 \leq x \leq b$.

The fractional $\alpha$-moment and the $\alpha$-variance for the random $\alpha$-variable $X$ can be obtained directly by putting $a = 0$ in (14) and (16) respectively. The results in this case take the following forms of the fractional $\alpha$-moment and the $\alpha$-variance,

$$m_\alpha = \frac{(k \alpha)!}{(k \alpha + \alpha)!} b^k \alpha$$  \hspace{0.5cm} (17)

and

$$\sigma^\alpha = \left(\frac{2\alpha)!}{(3\alpha)!} \left(\frac{\alpha!}{(2\alpha)!}\right)^2 \right) b^{2\alpha}$$  \hspace{0.5cm} (18)

respectively.

4 Continuity of Fractional Probability Measure

In the case of the classical probability space $(\Omega, F, P)$, the theorem of continuity of probability measure $(P(\lim_{n \to \infty} X_n(\omega)) = \lim_{n \to \infty} P(X_n(\omega)))$ have been provided with its proof [22]. Based on some results in [21], the study of this theorem for the case of the fractional probability principles throughout the following counter illustrated example shows that it is not satisfied.

Example 2
Consider the fractional probability space $([0, b], B([0, b]), P_\alpha)$ and a probability density function $f_\alpha(x) = l([0, b]))^{-\alpha} = b^{-\alpha}, b > 1, 0 < \alpha < 1$, \hspace{0.5cm} (19)

see [12, 21]. We define the sequence of functions (fractional random $\alpha$-variables) $X_n$ on $[0, b]$ as follows:

$$X_n(\omega) = n^{-\alpha}, \omega \in [0, b].$$  \hspace{0.5cm} (20)

From (19) and (20), it is clear that

$$P_\alpha(\lim_{n \to \infty} X_n(\omega)) = P_\alpha(0) = P_\alpha(0 \leq \omega \leq b) = b^{-\alpha},$$  \hspace{0.5cm} (21)

and also
Based on the fractional probability density and using transformation of fractional probability density, the fractional probability density function $f_{\alpha,n}(\omega)$ of the fractional random variable $X_\alpha$ is calculated as bellow:

$$
\int_0^{b^{-1}} P_\alpha(n X_\alpha) (n d X_\alpha)^\alpha = 1 , \ 0 \leq X_\alpha(\omega) \leq b^{-1} \tag{23}
$$

where $X_\alpha(\omega) = \omega n^{-1}$, $d\omega = n dX_n$ and $\int_0^{b^{-1}} (d\omega)^\alpha = 1$. Since $P_\alpha(n X_\alpha) = b^{-\alpha}$, we find that

$$
\int_0^{b^{-1}} b^{-\alpha} n^{-\alpha} (d X_\alpha)^\alpha = 1 , \ 0 \leq X_\alpha(\omega) \leq b^{-1} . \tag{24}
$$

So,

$$
P_\alpha(X_\alpha(\omega)) = n^{\alpha} b^{-\alpha}. \tag{25}
$$

The equation (25) implies that

$$
\lim_{n \to \infty} P_\alpha(X_\alpha(\omega)) = \infty . \tag{26}
$$

From (21) and (26), it is noted that

$$
P_\alpha(\lim_{n \to \infty} X_\alpha(\omega)) \neq \lim_{n \to \infty} P_\alpha(X_\alpha(\omega)) . \tag{27}
$$

This implies that the continuity of the probability measure is not satisfied in the case of the fractional probability space.

5 Conclusion

In this study, some important properties on fractional probability space $(\Omega, F, P_\alpha)$ and fractional probability measure $P_\alpha$ were proposed, in order to expand a probability theory of fractional order completely parallel to the classical probability theory. Moreover, the continuity of fractional probability measure was discussed. Some proofs of the studied properties were provided and the results were illustrated in some examples throughout a simple fractional probability distribution.

As future work, we plan to study some other theorems of convergence concept and also to provide more examples and applications to explain more properties for the fractional probability measure.

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https://doi.org/10.1016/j.asej.2015.04.001


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Received: October 21, 2016; Published: December 14, 2016