The Linear Refinement of
a Quasi-Uniformity via Szpilrajn

John Mastellos

University of Patras, Greece

Abstract

A quasi-uniformity is a structure which gives to the underlying set a topology and, generally, an order. We apply to a quasi-uniformity the Szpilrajn’s procedure which extends an order to a linear one. We prove from a quasi-uniformity all the propositions that Szpilrajn proved in [2].

Keywords: Linear refinement, specialization order, Szpilrajn’s procedure

Introduction

We consider a quasi-uniform structure $(X, U)$, $X$ a topological space, $U$ a quasi-uniformity on $X$, that is a structure such that the relation

$$(x, y) \in \cap\{U | U \in U\}$$

is reflexive and transitive. The Szpilrajn’s procedure means that the order of an ordered set may become a linear one and naturally reflexive and transitive. We transfer the propositions, which Szpilrajn demonstrated in 1930 and which preserve the reflexivity and transitivity of the above relation.

In the second paragraph the relation gets the ordered character and demonstrated all the propositions which Szpilrajn proved. The order of the antisymmetric property is the specialization order, that is the one such that

if $x \leq y$, then and only then $cl\{x\} \subseteq cl\{y\}$. 
The quasi-uniformity \((X, U)\) is a relation which is reflective and for each \(U \in U\) there exists a \(V \in U\) such that \(V \circ V \subseteq U\). We denote by \(U^{-1}\) the set \(\{U^{-1}|U \in U\}\), where \(U^{-1} = \{(x,y) \in X \times X|(y,x) \in U\}, U \in U\). Finally, the term linear refinement is more preferable than the term linear extension.

1 The propositions for a quasi-uniformity

Let \((X, U)\) be a quasi-uniform space, \(X\) is the underlying set, \(U\) the quasi-uniformity on it. It is more convenient for us to assume that the quasi-uniformity \(U\) has a base. The relation \(G = \bigcap\{U|U \in U\}\) is reflexive and transitive. The elements of \(X\), say \(x\) and \(y\), such that \((x,y) \notin G\) and \((y,x) \notin G\) are said non-comparable and they are symbolized by \(x||y\).

**Lemma 1.1.** Let \((X, U)\) be a quasi-uniform space and \(a||b\). For every \(U \in U\), we put:

\[ F_{ab}^U = \{(x,y) \in X \times X|(x,y) \in U \text{ or } (x,a) \in U \text{ and } (b,y) \in U\} \]

and

\[ F_{ab} = \{F_{ab}^U|U \in U\}. \]

Then

1. The couple \((a,b)\) belongs to all \(F_{ab}^U, U \in U\). Moreover \(U \subseteq F_{ab}^U\) and the filter \(F_{ab}\) induces a quasi-uniformity on \(X\).
2. There holds: \(G \subseteq \bigcap F_{ab} = \{F_{ab}^U|U \in U\}\).
3. \(U^* = F_{ab} \vee F_{ab}^{-1}\) (\(U^*\) is the supremum of \(U^* = F_{ab}\) and \(F_{ab}^{-1}\)) is a uniform space.

**Proof**

1. Firstly, we remark that \((a,b) \in F_{ab}^U\) for every \(U \in U\), since \((a,a) \in U, (b,b) \in U\). Next we prove the transitive property for \(F_{ab}^U\). We have: given \(U \in U\), let \(V \in U\) such that \(V \circ V \circ V \subseteq U\). Then, \((a,b)\) and the couple \((b,a)\) is an element which does not belong to \(U\). Since \(a, b\) are not comparable, there is \(W \in U\) such that \((a,b) \notin W\) and \((b,a) \notin W\). In fact \((a,b)\) and \((b,a)\) do not belong to every \(W\). If, now, \((x,y) \in F_{ab}^V \circ F_{ab}^V\), there is a \(z \in X\) such that \((x,y) \in F_{az}^V\) and \((z,y) \in F_{az}^V\). We distinguish the cases:

   i) \((x,z) \in V, (z,y) \in V,\) hence \((x,y) \in F_{ab}.\)
   
   ii) \((x,z) \in V, (z,a) \in V,\) and \((b,y) \in V,\) hence \((x,a) \in V, (b,y) \in V,\) hence \((x,a) \in U\) and \((b,y) \in U,\) that is \((x,y) \in F_{ab}.\)
(iii) \((x, a) \in V\) and \((b, z) \in V\), \((z, y) \in V\), hence \((x, a) \in V, (b, y) \in V \circ V\), hence \((x, a) \in U\) and \((b, y) \in U\), that is \((x, y) \in F_{ab}^U\).
(iv) \((x, a) \in V\) and \((b, z) \in V\), \((z, a) \in V\) and \((b, y) \in V\) which means that \((b, a) \in V \circ V\) an absurd because of supposition.

(2) It is evident.
(3) It is enough to be proved that for any \(U \in U\), there is a \(V \in U\) such that \(F_{ab}^V \circ F_{ab}^V \subseteq U \cap U^{-1}\). We assume that \((a, b)\) as well as \((b, a)\) do not belong to \(U\) and we pick up a \(V\) such that \(V \circ V \circ V \subseteq U\).

We distinguish the cases:

(i) \((x, y) \in V\), \((x, y) \in V\) hence \((x, y) \in U \cap U^{-1}\), that is \(x = y\).
(ii) \((x, y) \in V\), \((y, a) \in V\) and \((b, x) \in V\), hence \((b, a) \in V^3\), impossible since \((a, b)\) and \((b, a)\) are non comparable.
(iii) \((x, a) \in V\), \((b, y) \in V\) and \((y, x) \in V\), hence \((b, a) \in V^3\), impossible as well.
(iv) \((x, a) \in V\), \((b, y) \in V\), \((y, a) \in V\) and \((b, x) \in V\), hence \((b, a) \in V^2\), impossible.

The proof is over. \(\square\)

**Lemma 1.2.** Let \((X, U)\) be a quasi-uniform space, \((U\) a base of the quasi-uniformity), and \(i\) be an ordinal limit number. We assume the following:

(1) For every \(a\) there is a base \(F_a\) of a quasi-uniformity on \(X\) and for every \(U \in U\) and for every pair \(a, b\) of ordinals such that \(a \leq b < i\), there are uniquely assigned entourages \(F_a^U \in F_a\) and \(F_b^U \in F_b\) such that \(U \subseteq F_a^U \subseteq F_b^U\). (We have \(F_a = \{F_a^U | U \in U\} \) for every \(a\)).

(2) We put \(G = \cap U\) and \(G_a = \cap F_a\). Then, if \(a \leq b < i\), implies \(G \subseteq G_a \subseteq G_b\).

(3) For every \(U \in U\) and every \(a\), there holds: \(F_{ab}^U \cap F_{ab}^U^{-1} = U^*\), in the supremum of \(U\) and \(U^{-1}\).

Then, there is a base \(F_i\) of a quasi-uniformity corresponded to \(i\), for which there hold the properties (1)-(3), (i.e. the properties are extended to every ordinal number).

**Proof.** (We preserve the above notation). For every \(U \in U\), we consider the set \(F_{ab}^U = \cup \{F_a^U | a < i\}\), where \(F_a^U\) is the entourage of \(F_a\) the assigned to \(U\). We prove that the family \(F_i = \{F_i^U | U \in U\}\) constitutes a base for the desired quasi-uniformity.

We have the following:

(1) For every \(a < i\) and every \(F_a^U \in F_a\) such that \(F_a^V \circ F_a^V \subseteq F_a^U\). Without loss of generality we may assume that if \(\gamma < \delta < i\), then \(F_\gamma^V \subseteq F_\delta^V\). Let \(\kappa_{\gamma, \delta} = sup\{\gamma, \delta\}\).

Then:
\[(\bigcup_{a<i} F^V_a) \circ (\bigcup_{a<i} F^V_a) = \bigcup_{a<i} (F^V_a \circ \bigcup_{b<i} F^V_b) = \bigcup_{a<i} \bigcup_{b<i} (F^V_a \circ F^V_b) \subseteq \bigcup_{a<i} \bigcup_{b<i} F^V_{\kappa_{ab}} \subseteq \bigcup_{a<i} \bigcup_{b<i} (F^U_a \circ F^U_b) = \bigcup_{a<i} F^U_a. \]  

The sets, thus, \(F^U_i\), where \(U\) runs through \(U\), constitute a base for a quasi-uniformity \(F_i\).

Moreover for every \(U \in U\) with \(a < i\), there holds \(F^U_a \subset F^U_i\).

(2) If \(G = \cap F_i\), then \(G_i = \bigcup G_a\). In fact:

\[\bigcap_{U \in U} (\bigcup_{a<i} F^U_a) = \bigcup_{a<i} \bigcap_{U \in U} F^U_a = \bigcup_{a<i} G_a.\]

(3) By supposition for every \(U \in U\) and every \(a < i\), there holds:

\[F^U_a \cap (F^U_a)^{-1} = U \ast \in U \lor U^{-1}.\]

Besides, for every \(a, b\) smaller than \(i\) we have \(F^U_a \cap (F^U_a)^{-1} \subseteq F^U_{\kappa_{ab}} \cap (F^U_{\kappa_{ab}})^{-1}, \kappa_{ab} = \sup\{a, b\}\).

Hence:

\[F^U_i \cap (F^U_i)^{-1} = (\bigcup_{a<i} F^U_a) \cap (F^U_a)^{-1} = \bigcup_{a<i} (F^U_a \cap (F^U_a)^{-1} = \bigcup_{a<i} \bigcup_{b<i} (F^U_a \cap (F^U_b)^{-1} = \bigcup_{a<i} \bigcup_{b<i} U = U^* = U.\]

**Theorem 1.3.** For any quasi-uniform space there exists another one which is reflexive and transitive. Moreover, the uniformity the induced by the new quasi-uniformity coincides with the uniformity of the given one.

**Proof.** Let \((X, U)\) be a \(T_0\) quasi-uniform space, \(U^*\) is the supremum of \(U\) and \(U^{-1}\) and \(G\) the relation in (*). Put \(U = F_0\) and \(G = G_0\). We assume that there is a pair of non comparable elements \(a\) and \(b\); then, we construct a quasi-uniformity \(F_1\), exactly as in the Lemma 1.1 and we construct the set \(F_{ab}\) from \(U\). We know that if \(G_1\), the induced by \(F_1\), it is possible that \(aG_1b\) or \(bG_1a\) and if there are yet non comparable pairs of elements. If the process does not come to an end we continue the same procedure until the exhaustion of all non comparable pairs. We may assume that a well ordered set \(I\) enumerates the steps of the procedure. This enumeration is relative to the number of non comparable pairs of points although it is remarkable that after the ordering of a new pair of points, say \(a\) and \(b\), there are some other pairs which are automatically ordered as a result of the ordering of \(a\) and \(b\). We can easily arrange the procedure, as exactly Szpilrajn had done.

We refer to the construction in the step \(i \in I\), which, in fact, is contained in the demonstrations of the Lemmas 1.1 and 1.2.
If \( i = i_0 + 1, i_0 \in I \) and there is not a couple of non comparable elements, we consider that all the \( F_r, r \geq i \), are equal to \( F_{i_0} \) and the construction has been completed. Otherwise we construct from \( F_{i_0} \) the quasi-uniformity \( F_{i_0+1} \), exactly as we have constructed from \( U \) the quasi-uniformity \( F_{ab} \) in the Lemma 1.1.

If \( i \) is a limit ordinal, then the \( F_i \) consists of the entourages \( \bigcup \{ F_a^U | a < i \} \), when \( U \) runs through \( U \), as we have exactly constructed in the Lemma 1.2. In both cases the family \( F_i \) fulfils the properties (1)-(3) of the Lemma 1.2. Thus, for every \( i \in I \) we construct a quasi-uniformity \( F_i \), which induces a \( G_i \).

Now, we consider quasi-uniformities on \( X \), constructed as above, following different ways, which contain \( F_0 \) and induce \( G_0 \). We will prove that, with the acceptance of the choice axiom, there is a maximal quasi-uniformity. As in Szpilrajn’s paper, we make use of the Zermelo’s well-ordered Theorem, equivalent to Zorn’s Lemma. In order to prove it, let \( (F_i)_{i \in I} \) be a family of quasi-uniformities and \( (G_i)_{i \in I} \) the corresponding family the induced by \( F_i \) in the following meaning: the former contains \( F_0 \) and each \( F_i \) is constructed as above; for the latter there holds: \( G_0 \subseteq G_1 \subseteq \ldots \subseteq G_i \subseteq \ldots \) For every \( U \in U \), the sets \( F_a^U, a \in I \), belong to \( F_a \) and, thus, we take the set \( \bigcup \{ F_a^U | a \in I \} \). As we have demonstrated in the Lemmas 1.1 and 1.2 this set is an entourage for a quasi-uniformity, let us denote it by \( F \). It corresponds to \( \bar{G} = \bigcup \{ G_i / i \in I \} \) which is a reflexive and transitive relation. So, the suppositions of the Zermelo’s theorem are satisfied and there is a maximal quasi-uniformity, say \( F \), and its maximal \( \bar{G} \). If, now, a pair, say \( (p, q) \), does not belong to \( \bar{G} \), then the \( (p, q) \) does not belong to any of \( G_i \), hence, \( \bar{G} \) is not maximal. This finishes the proof of the theorem.

2 When the reflexive and transitive properties correspond to orders

The anti-symmetric property is the one which creates the ordering property from a relation which is reflexive and transitive. The \( T_0 \)-axiom is compatible to the anti-symmetric property. In fact, the \( T_0 \)-axiom coincides with the specialization property. So, if the spaces are \( T_0 \) automatically are defined ordered by the specialization ordering.

We symbolize by \( G \) the specialization ordering of the structure, that is

\[
G = \cap \{ U | U \in U, U \text{ is an ordering} \}.
\]

We have the following properties.
Lemma 2.1. Let $(X, U)$ be a $T_0$ quasi-uniform space with respect to the specialization order $G$ and a, b two non-comparable elements of $X$. For every $U \in U$, we put:

$$F_{ab}^U = \{(x, y) \in X \times X | (x, y) \in U \text{ or } [(x, a) \text{ and } (b, y) \in U]\}$$

and

$$F_{ab} = \{F_{ab}^U | U \in U\}.$$

Then

1. The couple $(a, b)$ belongs to all $F_{ab}^U$, $U \in U$. Moreover $U \subseteq F_{ab}^U$ and the filter $F_{ab}$ induces a $T_0$ quasi-uniformity on $X$.
2. There holds: $G \subseteq \{F_{ab}^U | U \in U\}$.
3. $U^* = F_{ab} \lor F_{ab}^{-1}$.

For $T_0$ quasi-uniformity the space is ordered and the things are as we have drawn them. The rest are unchanged.

Lemma 2.2. Let $(X, U)$ be a quasi-uniform space, $(U$ a base of quasi-uniformity$)$, and $i$ be an ordinal limit number. We assume the following:

1. For every $a < i$ there is a base $F_a$ of a quasi-uniformity on $X$ and for every $U \in U$ and for every pair $a, b$ of ordinals such that $a \leq b < i$, there are uniquely assigned entourages $F_a^U \in F_a$ and $F_b^U \in F_b$ such that $U \subseteq F_a^U \subset F_b^U$. (We have $F_a = \{F_a^U | U \in U\}$ for every $a < i$).
2. We put $G = \cap U$ and $G_a = \cap F_a$. Then $a \leq b < i$ implies $G \subseteq G_a \subset G_b$.
3. For every $U \in U$ and every $a < i$ there holds: $F_a^U \cap F_a^{-1} = U^*$, in the supremum of $U$ and $U^{-1}$.
4. For every $a < i$ the topology $\tau(F_a)$ is $T_0$ not $T_1$.

Then, there is a base $F_i$ of a quasi-uniformity corresponded to $i$, for which there hold the properties (1)-(4), (i.e. the properties are extended to the ordinal).

Proof. It remains only the (4): There holds (c.f. [1, Prop. 1.9]) that the space $(X, U)$ is $T_0$ if and only if $(X, U^*)$ is $T_2$ (see below to the Theorem 2.5). After the above (3) the space is $T_0$. Besides, for every $U \in U$, $F_i^U = \bigcup_{a < i} F_a^U$ and the $\tau(F_a), a < i$, is not $T_1$ and $F_i$ is not $T_1$, as well.

Theorem 2.3. For every $T_0$ quasi-uniform space there is another one whose the specialization order is a linear order and it is the linear refinement of the specialization order of the given space; moreover, the uniformity the induced by the new quasi-uniformity coincides with the uniformity of the given one.
Remarks 2.4

1) If $R_U$ is the specialization order which the $T_0$-property secures we assure that there are non-comparable elements $a$ and $b$ such that does neither hold $aR_Ub$ nor $bR_Ua$.

The proofs are the same, only the $T_0$-property means that we have order where we had reflective and transitive relations.

2) About the Lemmas 1.2 and 2.2: we consider $U$ as an $F_0$ and after that construct a quasi-uniformity $F_1$. In general we may consider that for a quasi-uniform space $F_a$ we arrive to another $F_{a+1}$. The two lemmas mean that there exists a well ordered set $I$ of ordinal numbers and we define the quasi-uniformity $F_i$ for a limit ordinal number $i$ from $a < i$.

As a direct consequence of the above we conclude the following:

**Theorem 2.4.** For every $T_2$ uniform space $(X, U^*)$ there are two conjugate quasi-uniform spaces $U_\infty$ and $U^{-1}_\infty$ which induce $U^*$ and whose their specialization order is linear (and the first is inverse to the second).

References


Received: July 11, 2016; Published: September 19, 2016