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A Family of Nonnegative Matrices with Prescribed Spectrum and Elementary Divisors[1]

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Abstract

A perturbation result, due to Rado, shows how to modify r eigenvalues of a matrix of order n, via a perturbation of rank $r \leq n$, without changing any of the $n - r$ remaining eigenvalues. This result extended a previous one, due to Brauer, on perturbations of rank 1. Both results have been exploited in connection with the nonnegative inverse eigenvalue problem and the nonnegative inverse elementary divisors problem. In this paper, we use the Rado result from a more general point of view, constructing a family of matrices with prescribed spectrum and elementary divisors, generalizing previous results. We also apply our results to the nonnegative pole assigment problem.

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1 Introduction

The origin of the present paper is a perturbation result due to Brauer [1], which shows how to modify one single eigenvalue of a matrix, via a rank-one perturbation, without changing any of the remaining eigenvalues. This result was first used in connection with the *nonnegative inverse eigenvalue problem* (NIEP) by Perfect [8], and later in [10], giving rise to a number of realizability criteria (see [14] and the references therein). Closer to our approach in this paper is Rado's extension of Brauer's result, Theorem 2.2 below, which was first used by Perfect in [9] and later in [11] to derive sufficient conditions for the NIEP to have a solution. Both, Brauer and Rado results, have been also used to derive efficient sufficient conditions for the real and symmetric nonnegative inverse eigenvalue problem [11, 12], as well as for the *nonnegative inverse elementary divisors problem* (NIEDP) [3, 15, 16]. In this paper we use the Rado result (Theorem 2.1 below) in a more general context, to construct in Section 2, a family of nonnegative matrices with prescribed spectrum. In Section 3 we construct a family of nonnegative matrices with prescribed elementary divisors. In Section 4 we apply our results to the pole assigment problem. We denote the spectrum of a matrix A as $\sigma(A)$.

2 Rank-r perturbations

The following result, due to R. Rado and introduced by Perfect [9] in 1955, is an extension of the above mentioned Brauer result. It shows how to change r eigenvalues of an $n \times n$ matrix A via a perturbation of rank r, without changing any of the remaining $n - r$ eigenvalues.

Theorem 2.1 *(Rado) Assume that for $r \leq n$ all following are given:*
i) let $A \in \mathbb{C}^{n\times n}$ be an arbitrary matrix with $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$;
ii) let $U = [u_1 \mid \cdots \mid u_r] \in \mathbb{C}^{n\times r}$ be a matrix of rank r such that $Au_i = \lambda_i u_i$ for each $i = 1, ..., r$;
iii) let $V = [v_1 \mid \cdots \mid v_r] \in \mathbb{C}^{n\times r}$ be an arbitrary matrix;
iv) let $\Omega = diag(\lambda_1, ..., \lambda_r) \in \mathbb{C}^{r\times r}$;
Then $\sigma(A + UV^T) = \sigma(\Omega + V^T U) \cup \{\lambda_{r+1}, ..., \lambda_n\}$.

We state an interesting consequence of Theorem 2.1

Corollary 2.1 *Assume that for $r \leq n$ all following are given:*
i) let $A \in \mathbb{C}^{n\times n}$ be an arbitrary matrix with $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$;
ii) let $U = [u_1 \mid \cdots \mid u_r] \in \mathbb{C}^{n\times r}$ be a matrix of rank r such that $Au_i = \lambda_i u_i$ for each $i = 1, ..., r$;
iii) let $W = [w_1 \mid \cdots \mid w_r] \in \mathbb{C}^{n\times r}$ be an arbitrary matrix with $W^T U = I_r$;

iv) let $C \in \mathbb{C}^{r\times r}$ be an arbitrary matrix and let $\Omega = diag(\lambda_1, ..., \lambda_r) \in \mathbb{C}^{r\times r}$;
Then $\sigma(A + UCW^T) = \sigma(\Omega + C) \cup \{\lambda_{r+1}, ..., \lambda_n\}$.

Proof. If we take $V = WC^T$ in Theorem 2.1, then $\Omega + V^TU = \Omega + C$ and $A + UV^T = A + UCW^T$. ■

The advantage of this result with respect to Theorem 2.1 is that Corollary 2.1 permits to construct a family of matrices with a prescribed spectrum since the set of eigenvalues $\{\mu_1, \mu_2, ..., \mu_r\}$ of $\Omega + C$ does not depend on the matrix W: namely, for any W such that $W^TU = I_r$ we can construct a new matrix $A + UCW^T$ with the desired spectrum $\{\mu_1, ..., \mu_r, \lambda_{r+1}, ..., \lambda_n\}$.

Corollary 2.2 *Assume all the following are given:*
i) let $S = [s_{ij}]_{i,j=1}^r \in \mathbb{C}^{r\times r}$ be an arbitrary matrix with $\sigma(S) = \{\rho_1, ..., \rho_r\}$;
ii) for each $i = 1, ..., r$ let $A_i \in \mathbb{C}^{n_i\times n_i}$ be an arbitrary matrix with an eigenvalue s_{ii};
iii) for each $i = 1, ..., r$ let $u_i \in \mathbb{C}^{n_i}$ be an eigenvector of A_i associated with the eigenvalue s_{ii};
iv) for each $i = 1, ..., r$ let $w_i \in \mathbb{C}^{n_i}$ be a vector such that $w_i^Tu_i = 1$.
Then the matrix

$$\begin{bmatrix} A_1 & s_{12}u_1w_2^T & \cdots & s_{1r}u_1w_r^T \\ s_{21}u_2w_1^T & A_2 & \cdots & s_{2r}u_2w_r^T \\ \vdots & \vdots & \ddots & \vdots \\ s_{r1}u_rw_1^T & s_{r2}u_rw_2^T & \cdots & A_r \end{bmatrix} \tag{1}$$

has spectrum $\{\sigma(A_1) - \{s_{11}\}\} \cup ... \cup \{\sigma(A_r) - \{s_{rr}\}\} \cup \{\rho_1, ..., \rho_r\}$.

Proof. We are going to apply Corollary 2.1. Therefore:
Let

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_r \end{bmatrix} \in \mathbb{C}^{(n_1+...+n_r)\times(n_1+...+n_r)};$$

Let

$$U = \begin{bmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_r \end{bmatrix} \in \mathbb{C}^{(n_1+...+n_r)\times r}$$

and observe that for each $i = 1, ..., r$, the ith column of U is an eigenvector of A with eigenvalue s_{ii};

Let

$$W = \begin{bmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_r \end{bmatrix} \in \mathbb{C}^{(n_1+\dots+n_r)\times r}$$

and observe that $W^T U = I_r$.
Let

$$C = \begin{bmatrix} 0 & s_{12} & \dots & s_{1r} \\ s_{21} & 0 & \dots & s_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ s_{r1} & s_{r2} & \dots & 0 \end{bmatrix}.$$

Then $A+UCW^T$, as given in (1), has spectrum $\{\rho_1, ..., \rho_r\} \cup \{\sigma(A) - \{s_{11}, ..., s_{rr}\}\}$.
■

Remark 2.1 *Assume in Corollary 2.2 that the matrices $A_i \in \mathbb{C}^{n_i \times n_i}$ are all nonnegative with constant row sums equal to s_{ii} (the Perron eigenvalue). Then $u_i = e = (1, 1, ..., 1)^T, i = 1, ..., r$. Let C an $r \times r$ nonnegative matrix. Moreover, assume that ω_i is nonnegative, $i = 1, ..., r$. The matrix $A+UCW^T$ in (1) is nonnegative with the spectrum $\{\sigma(A_1) - \{s_{11}\}\} \cup \dots \cup \{\sigma(A_r) - \{s_{rr}\}\} \cup \{\rho_1, ..., \rho_r\}$. Hence, under these conditions we may construct a family of nonnegative matrices with a prescribed spectrum.*

Example 2.1 *Let $\Lambda = \{6, 3, 3, -5, -5\}$. We look for a family of nonnegative matrices with spectrum Λ. Then we take the partition*

$$\begin{aligned} &\Lambda_1 = \{6, -5\},\ \Lambda_2 = \{3, -5\},\ \Lambda_3 = \{3\} \quad \textit{with} \\ &\Gamma_1 = \{5. - 5\},\ \Gamma_2 = \{5. - 5\},\ \Gamma_3 = \{2\} \quad \textit{being} \end{aligned}$$

realizable by

$$A_1 = A_2 = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \quad A_3 = [2].$$

Then the matrix

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

has eigenvalue $5, 5, 2, -5, -5$. Moreover

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; W = \begin{bmatrix} \alpha & \gamma & \eta \\ 1-\alpha & -\gamma & -\eta \\ \beta & \delta & \theta \\ -\beta & 1-\delta & -\theta \\ 0 & 0 & 1 \end{bmatrix},$$

with $W^T U = I$. *Next we compute*

$$B = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{bmatrix}, \quad \text{with } C = B - diagB = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}.$$

B *has spectrum* $\{6, 3, 3\}$ *and diagonal entries* $\{5, 5, 2\}$.
Thus, for $\eta = \theta = \gamma = \beta = 0$; $0 \leq \alpha, \delta \leq 1$, *the matrix*

$$A + UCW^T = \begin{bmatrix} \eta & 5-\eta & \theta & -\theta & 1 \\ 5+\eta & -\eta & \theta & -\theta & 1 \\ \alpha & -\alpha+1 & \beta & -\beta+5 & 0 \\ \alpha & -\alpha+1 & \beta+5 & -\beta & 0 \\ 4\gamma & -4\gamma & 4\delta & -4\delta+4 & 2 \end{bmatrix}$$

gives a family of nonnegative matrices with spectrum $\Lambda = \{6, 3, 3-5, -5\}$.

We can also obtain a family of persymmetric nonnegative matrices with prescribed spectrum. Persymmetric matrices are common in physical sciences and engineering. They arise, for instance, in the control of mechanical and electric vibrations, where the eigenvalues of the Gram matrix, which is symmetric and persymmetric, play an important role. As the superscript T, in A^T, denotes the transpose of A, the superscript F, in A^F, denotes the flip-transpose of A, which flips A across its skew-diagonal. If $A = (a_{ij})_{mn}$, then $A^F = (a_{n-j+1,m-i+1})_{nm}$. A matrix A is said to be *persymmetric* if $A^F = A$, that is, if it is symmetric across its lower-left to upper-right diagonal. In [4], the authors give a persymmetric version of Theorem 2.1, which we use in the following

Corollary 2.3 *Assume that for* $r \leq n$ *all following are given:*
i) let $A \in \mathbb{C}^{n\times n}$ *be a persymmetric nonnegative matrix with* $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$;
ii) let $U = [u_1 \mid \cdots \mid u_r] \in \mathbb{C}^{n\times r}$ *be a matrix of rank* r *such that* $Au_i = \lambda_i u_i$ *for each* $i = 1, ..., r$;
iii) let $C \in \mathbb{C}^{r\times r}$ *be persymmetric nonnegative matrix and let* $\Omega = diag(\lambda_1, ..., \lambda_r) \in \mathbb{C}^{r\times r}$;
Then $A + UCU^F$ *is persymmetric nonnegative matrix with spectrum* $\sigma(A + UCU^F) = \{\mu_1, \ldots, \mu_r\} \cup \{\lambda_{r+1}, ..., \lambda_n\}$, *where* $\mu_1, \ldots, \mu_r$ *are the eigenvalues of* $\Omega + CU^F U$ ($\Omega + C$ *if the columns of* U *are taken as orthonormal).*

Proof. It is clear that UCU^F is persymmetric nonnegative and then $A + UCU^F$ es also persymmetric nonnegative. From Theorem 2.1 and Remark 2.1, $A + UCU^F$ has the spectrum $\sigma(A)$. ■

Example 2.2 *We construct a family of persymmetric nonnegative matrices with spectrum* $\Lambda = \{6, 4, -2, -2, -3, -3\}$. *We take the partition* $\Lambda = \Lambda_1 \cup \Lambda_2$ *with*

$$\Lambda_1 = \{6, -2, -3\}, \quad \Lambda_2 = \{4, -2, -3\}$$

and the associated realizable lists

$$\Gamma_1 = \{5, -2. - 3\}, \quad \Gamma_2 = \{5, -2, -3\}.$$

Let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^F \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{5} & \sqrt{5} & 0 & 0 & 0 \\ \sqrt{5} & 0 & 3 & 0 & 0 & 0 \\ \sqrt{5} & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & \sqrt{5} \\ 0 & 0 & 0 & 3 & 0 & \sqrt{5} \\ 0 & 0 & 0 & \sqrt{5} & \sqrt{5} & 0 \end{bmatrix},$$

where A_1 *and* A_1^F *realizan* Γ_1 *and* Γ_2 *respectively. Let* U *be the matrix of the normalized eigenvectors of* A *corresponding to the eigenvalues* $5, 5:$

$$U = \begin{bmatrix} \frac{2}{\sqrt{14}} & 0 \\ \frac{\sqrt{5}}{\sqrt{14}} & 0 \\ \frac{\sqrt{5}}{\sqrt{14}} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{14}} \\ 0 & \frac{\sqrt{5}}{\sqrt{14}} \\ 0 & \frac{2}{\sqrt{14}} \end{bmatrix} \text{ and let } C = \begin{bmatrix} 0 & y \\ x & 0 \end{bmatrix}.$$

Then for all $x, y \geq 0$,

$$A + UCU^F = \begin{bmatrix} 0 & \sqrt{5} & \sqrt{5} & \frac{1}{7}y\sqrt{5} & \frac{1}{7}y\sqrt{5} & \frac{2}{7}y \\ \sqrt{5} & 0 & 3 & \frac{5}{14}y & \frac{5}{14}y & \frac{1}{7}y\sqrt{5} \\ \sqrt{5} & 3 & 0 & \frac{5}{14}y & \frac{5}{14}y & \frac{1}{7}y\sqrt{5} \\ \frac{1}{7}x\sqrt{5} & \frac{5}{14}x & \frac{5}{14}x & 0 & 3 & \sqrt{5} \\ \frac{1}{7}x\sqrt{5} & \frac{5}{14}x & \frac{5}{14}x & 3 & 0 & \sqrt{5} \\ \frac{2}{7}x & \frac{1}{7}x\sqrt{5} & \frac{1}{7}x\sqrt{5} & \sqrt{5} & \sqrt{5} & 0 \end{bmatrix}$$

is persymmetric nonnegative. In particular for $x = y = 1$, $A + UCU^F$ *has the spectrum* Λ.

3 Inverse elementary divisors problem

In this section we show how to construct a family of nonnegative matrices with prescribed elementary divisors. Here we exploit Corollary 2.2 for the nonnegative case (as in Remark 2.1).

Let $A \in \mathbb{C}^{n\times n}$ and let

$$J(A) = S^{-1}AS = \begin{bmatrix} J_{n_1(\lambda_1)} & & & \\ & J_{n_2(\lambda_2)} & & \\ & & \ddots & \\ & & & J_{n_k(\lambda_k)} \end{bmatrix}$$

be the *Jordan canonical form* of A. The $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, 2, \ldots, k$$

are the *Jordan blocks* of $J(A)$. The *elementary divisors* of A are the polynomials $(\lambda - \lambda_i)^{n_i}$, that is, the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \ldots, k$.

The *nonnegative inverse elementary divisor problem* (*NIEDP*) is the problem of determining necessary and sufficient conditions under which the polynomials $(\lambda - \lambda_1)^{n_1}, (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}$, $n_1 + \cdots + n_k = n$, are the *elementary divisors* of an $n \times n$ nonnegative matrix A [5, 6, 7]. The *NIEDP* contains the *NIEP* and both problems remain unsolved (they have been solved only for $n \leq 4$).

The following result gives a sufficient condition for the existence and construction of a nonnegative matrix with prescribed spectrum and elementary divisors.

Corollary 3.1 *Let $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ be a list of complex numbers, with $\Lambda = \overline{\Lambda}$, $\lambda_1 \geq \max_i |\lambda_i|$, $i = 2, \ldots, n$, and $\sum_{i=1}^{n} \lambda_i \geq 0$. Assume that there exists a partition $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_r$, where some of the lists Λ_i, $i = 1, \ldots, r$, can be empty, such that:*
i) let $S = [s_{ij}]_{i,j=1}^{r}$ is an $r \times r$ nonnegative matrix with spectrum $\sigma(S) = \Lambda_0 = \{\lambda_1, ..., \lambda_r\}$;
ii) for each $i = 1, ..., r$, there exists a $(p_i + 1) \times (p_i + 1)$ nonnegative matrix A_i, with constant row sums equal to s_{ii}, spectrum $\Gamma_i = \{s_{ii}, \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ip_i}\}$, and prescribed elementary divisors associated with Γ_i.

iii) for each $i = 1, ..., r$ *let* $\mathbf{u}_i = \mathbf{e}$ *be an eigenvector of* A_i *corresponding to the eigenvalue* s_{ii}.
iv) Let A *be the block diagonal matrix* $A = diag\{A_1, \ldots, A_r\}$ *and let* $U = [\widehat{\mathbf{u}}_1 \mid \cdots \mid \widehat{\mathbf{u}}_r]$ *such that* $A\widehat{\mathbf{u}}_i = s_{ii}\widehat{\mathbf{u}}_i$.
v) for each $i = 1, ..., r$ *let* $\mathbf{w}_i$ *be a nonnegative vector such that* $\mathbf{w}_i^T\widehat{\mathbf{u}}_i = 1$, *and let* $W = [\mathbf{w}_1 \mid \cdots \mid \mathbf{w}_r]$.
Then the nonnegative matrix $A+UCW^T$, *where* $C = S-diag\{s_{11}, s_{22}, \ldots, s_{rr}\}$, *has spectrum* Λ *and the prescribed elementary divisors associated with* Λ_i, $i = 1, ..., r$.

Proof. From $ii) - iv)$ $A = diag\{A_1, \ldots, A_r\}$ is nonnegative with spectrum $\Gamma_1 \cup \cdots \cup \Gamma_r$ and $A\widehat{\mathbf{u}}_i = s_{ii}\widehat{\mathbf{u}}_i$, $i = 1, \ldots, r$. Then from $i)$ and Remark 2.1 the result follows. ■

Example 3.1 *Let* $\Lambda = \{7, 1, -2, -2, -1+3i, -1-3i\}$. *We want to construct a family of nonnegative matrices with spectrum* Λ, *and with elementary divisors* $(\lambda - 7), (\lambda - 1), (\lambda + 2)^2, \lambda^2 + 2\lambda + 10$. *Then we take the partition*

$$\Lambda_0 = \{7, -1+3i, -1-3i\},\ \Lambda_1 = \{-2, -2\},\ \Lambda_2 = \{1\},\ \Lambda_3 = \emptyset \quad \textit{with}$$
$$\Gamma_1 = \{4, -2, -2\},\ \Gamma_2 = \{1, 1\},\ \Gamma_3 = \{0\}.$$

We compute the nonnegative matrix

$$S = \begin{bmatrix} 4 & 0 & 3 \\ \frac{34}{7} & 1 & \frac{8}{7} \\ 0 & 7 & 0 \end{bmatrix} \quad \textit{with spectrum } \Lambda_0 \textit{ and diagonal entries } 4, 1, 0,$$

and the realizing matrices

$$A_1 = \begin{bmatrix} 4 & 0 & 0 \\ 7 & -2 & -1 \\ 6 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad A_3 = [0],$$

with spectra $\Gamma_1, \Gamma_2, \Gamma_3$, *respectively. Observe that the matrix* A_1 *has elementary divisors* $(\lambda - 4), (\lambda + 2)^2$. *Let*

$$W = \begin{bmatrix} \alpha & \gamma & \eta \\ 1-\alpha & -\gamma & -\eta \\ 0 & 0 & 0 \\ \beta & \delta & \theta \\ -\beta & 1-\delta & -\theta \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $W^TU = I$, *and*

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} + UCW^T$$

$$= \begin{bmatrix} 3\eta & 2-3\eta & 2 & 3\theta & -3\theta & 3 \\ 3\eta+3 & -3\eta & 1 & 3\theta & -3\theta & 3 \\ 3\eta+2 & 2-3\eta & 0 & 3\theta & -3\theta & 3 \\ \frac{34}{7}\alpha+\frac{8}{7}\eta & \frac{34}{7}-\frac{8}{7}\eta-\frac{34}{7}\alpha & 0 & \frac{8}{7}\theta+\frac{34}{7}\beta+1 & -\frac{8}{7}\theta-\frac{34}{7}\beta & \frac{8}{7} \\ \frac{34}{7}\alpha+\frac{8}{7}\eta & \frac{34}{7}-\frac{8}{7}\eta-\frac{34}{7}\alpha & 0 & \frac{8}{7}\theta+\frac{34}{7}\beta & 1-\frac{34}{7}\beta-\frac{8}{7}\theta & \frac{8}{7} \\ 7\gamma & -7\gamma & 0 & 7\delta & 7-7\delta & 0 \end{bmatrix}$$

is a family of matrices with spectrum Λ *and the desired elementary divisors. In particular for*

$$\eta = \gamma = \theta = \beta = 0, \text{ and } 0 \le \alpha, \delta \le 1$$

we have the nonnegative family

$$B = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 3 \\ 3 & 0 & 1 & 0 & 0 & 3 \\ 2 & 2 & 0 & 0 & 0 & 3 \\ \frac{34}{7}\alpha & \frac{34}{7}-\frac{34}{7}\alpha & 0 & 1 & 0 & \frac{8}{7} \\ \frac{34}{7}\alpha & \frac{34}{7}-\frac{34}{7}\alpha & 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 & 7\delta & 7-7\delta & 0 \end{bmatrix},$$

with spectrum Λ *and elementary divisors*

$$(\lambda-7), (\lambda-1), (\lambda+2)^2, \lambda^2+2\lambda+10.$$

We can also solve the inverse elementary divisors problem for a family of nonnegative matrices with constant row sums and spectrum $\Lambda = \{\lambda_1, ..., \lambda_n\}$ in the following cases, which have been solved for a single nonnegative matrix in [13, 15, 3]:

i) $\lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n \ge 0$
ii) $\lambda_1 > 0 \ge \lambda_2 \ge \cdots \ge \lambda_n$
iii) $Re\lambda_i \le 0, \ |Re\lambda_i| \ge |Im\lambda_i|$
iv) $Re\lambda_i \le 0, \ \left|\sqrt{3}Re\lambda_i\right| \ge |Im\lambda_i|$.

For each one of these cases, we may apply the same techniques used in [13, 15, 3] for a single nonnegative matrix. For example, in the first case *i)* $\Lambda = \{5, 3, 3, 3\}$, we use from [8], the Perfect matrix

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

to obtain a positive matrix

$$A = PDP^{-1} = \begin{bmatrix} \frac{13}{4} & \frac{1}{4} & \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{13}{4} & \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{7}{2} & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 4 \end{bmatrix},$$

where $D = diag\{5, 3, 3, 3\}$. If $E_{i,j}$ is the matrix with 1 in position (i, j) and zeros elsewhere, then

$$M = A + aPE_{3,4}P^{-1} = \begin{bmatrix} \frac{1}{2}a + \frac{13}{4} & \frac{1}{4} - \frac{1}{2}a & \frac{1}{2} & 1 \\ \frac{1}{2}a + \frac{1}{4} & \frac{13}{4} - \frac{1}{2}a & \frac{1}{2} & 1 \\ \frac{1}{4} - \frac{1}{2}a & \frac{1}{2}a + \frac{1}{4} & \frac{7}{2} & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 4 \end{bmatrix},$$

with $0 \leq a \leq \frac{1}{2}$, is a nonnegative matrix with constant row sums equal to 5, and with elementary divisors $J_1(5), J_2(3), J_1(3)$.
For $J_1(5), J_3(3)$ we have with $E = aE_{3,4} + bE_{2,3}$,

$$M = A + aPEP^{-1} = \begin{bmatrix} \frac{1}{2}a + \frac{1}{4}b + \frac{13}{4} & \frac{1}{4}b - \frac{1}{2}a + \frac{1}{4} & \frac{1}{2} - \frac{1}{2}b & 1 \\ \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4} & \frac{1}{4}b - \frac{1}{2}a + \frac{13}{4} & \frac{1}{2} - \frac{1}{2}b & 1 \\ \frac{1}{4}b - \frac{1}{2}a + \frac{1}{4} & \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4} & \frac{7}{2} - \frac{1}{2}b & 1 \\ \frac{1}{4} - \frac{1}{4}b & \frac{1}{4} - \frac{1}{4}b & \frac{1}{2}b + \frac{1}{2} & 4 \end{bmatrix},$$

which for $0 \leq a, b \leq 1$, is nonnegative with the desired JCF.
For the second case $ii)$ $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we apply Brauer result:

$$M = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 12 & -1 & -1 & 0 & 0 \\ 11 & 0 & -1 & 0 & 0 \\ 15-a & a & 0 & -4 & -1 \\ 14-a & 0 & a & 0 & -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -10 & 1 & 1 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 4 & 4 \\ 2 & 0 & 0 & 4 & 4 \\ 1 & 1 & 0 & 4 & 4 \\ 5-a & a+1 & 1 & 0 & 3 \\ 4-a & 1 & a+1 & 4 & 0 \end{bmatrix},$$

which, for $0 \leq a \leq 4$, is nonnegative, with constant row sums $\lambda_1 = 10$, and elementary divisors $J_1(10), J_2(-1), J_2(-4)$. In the same way, for the list

$$\Lambda = \{4, -1+i, -1-i, -1+i. -1-i\}$$

we have

$$M = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 4 & -1 & 1 & 0 & 0 \\ 7 & -1 & -1 & -1 & 0 \\ 4 & 0 & 0 & -1 & 1 \\ 6 & 0 & 0 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -3-a & 1 & 1 & 1 & a \end{bmatrix}$$

$$= \begin{bmatrix} 1-a & 1 & 1 & 1 & a \\ 1-a & 0 & 2 & 1 & a \\ 4-a & 0 & 0 & 0 & a \\ 1-a & 1 & 1 & 0 & a+1 \\ 3-a & 1 & 1 & 0 & a-1 \end{bmatrix}, \quad 0 \leq a \leq 1,$$

with JCF

$$J(M) = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & -1-i & 0 & 0 & 0 \\ 0 & 1 & -1-i & 0 & 0 \\ 0 & 0 & 0 & -1+i & 0 \\ 0 & 0 & 0 & 1 & -1+i \end{bmatrix}$$

4 Nonnegative pole assigment problem

In [2] the authors show applications of Theorem 2.1 (Rado Theorem), to deflation techniques and the pole assigment problem for multi-imput multi-output (MIMO) systems defined by pairs of matrices (A, B). Corollary 2.1 may also be applied to deflation techniques and the pole assignment problem to obtain a nonnegative solution matrix. Let A be an $n \times n$ matrix, let B be an $n \times r$ matrix with $\sigma(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$. Let $\{\mu_1, \mu_2, ..., \mu_r\}$ be a list of numbers. The pole assignment problem for MIMO systems asks conditions on the pair (A, B) for the existence of a matrix F such that

$$\sigma(A + BF^T) = \{\mu_1, \mu_2, ..., \mu_r, \lambda_{r+1}, \lambda_{r+2}, ..., \lambda_n\}.$$

For the nonnegative pole assigment problem we have the following result:

Corollary 4.1 *Let (A, B) be a given pair of matrices with A an $n \times n$ nonnegative matrix and B an $n \times r$ matrix. Let $\sigma(A) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and let $\mu = \{\mu_1, \mu_2, ..., \mu_r\}$ be a list of numbers. Let $X = [\alpha_1 x_1 \mid ... \mid \alpha_r x_r]$ be an $n \times r$ matrix such that $rankX = r$ and $A^T x_i = \lambda_i x_i$, $i = 1, 2, ..., r$. If there*

exists a column $b_j = [b_{1j}, b_{2j}, ..., b_{rj}]^T$ *of the matrix* B*, with all its entries being nonnegative such that* $b_j^T x_i \neq 0, i = 1, 2, ..., r$, $\sigma(\Omega + [b_j \mid ... \mid b_j]^T X) = \{\mu_1, \mu_2, ..., \mu_r\}$ *and* $\sum_{i=1}^r \alpha_i x_{si} \geq 0, s = 1, 2, ..., n$*, then there exists a nonnegative matrix* F *such that the nonnegative matrix* $A + BF^T$ *has spectrum* $\{\mu_1, ..., \mu_r, \lambda_{r+1}, ..., \lambda_n\}$.

Proof. Let the $n \times r$ matrix

$$C^T = [b_j \mid ... \mid b_j] = B[e_j \mid ... \mid e_j],$$

where e_j is the jth column of the identity matrix of order r. Then

$$\begin{aligned} C^T X^T &= B[e_j \mid ... \mid e_j] X^T \\ &= \begin{bmatrix} b_{1j} \sum_{i=1}^r \alpha_i x_{1i} & \cdots & b_{1j} \sum_{i=1}^r \alpha_i x_{ni} \\ \vdots & \ddots & \vdots \\ b_{nj} \sum_{i=1}^r \alpha_i x_{1i} & \cdots & b_{nj} \sum_{i=1}^r \alpha_i x_{ni} \end{bmatrix} \geq 0. \end{aligned}$$

If we take $F^T = [e_j \mid ... \mid e_j] X^T$ then

$$0 \leq A + C^T X^T = A + B[e_j \mid ... \mid e_j] X^T = A + BF^T$$

and for Theorem 2.1

$$\sigma(A + BF^T) = \sigma(A + C^T X^T) = \sigma(\Omega + [e_j \mid ... \mid e_j]^T B^T X) \cup \{\lambda_{r+1}, ..., \lambda_n\}.$$

■

Example 4.1 *Consider the pair* (A, B) *and* $\mu = \{8 + \sqrt{19}, 8 - \sqrt{19}\}$ *where*

$$A = \begin{bmatrix} 0 & 5 & 1 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 4 \\ 0 & 0 & 5 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 1 \end{bmatrix} = [b_1 \mid b_2],$$

with $\sigma(A) = \{6, 3, 3, -5, -5\}$*. We compute the eigenvector of* A^T :

$$x_{\lambda=6} = (1, 1, 1, 1, 1)^T, x_{\lambda=3} = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1)^T$$

$$x_{\lambda=-05} = \{(-5, 5, 0, 1, 0)^T, (\frac{35}{4}, -\frac{179}{20}, -\frac{7}{4}, 0, 1)^T\}.$$

Since for $b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$, $b_2^T x_{\lambda=6} \neq 0$, *and* $b_2^T x_{\lambda=3} \neq 0$, *we may change the eigenvalues* $\lambda = 6$ *and* $\lambda = 3$ *to* $\mu_1 = 8 + \sqrt{19}$ *and* $\mu_2 = 8 - \sqrt{19}$, *respectively. To do this, let*

$$C^T = [b_2 \mid b_2] = B \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = B\,[e_2 \mid e_2]$$

with

$$X^T = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ -\frac{1}{2}\alpha_2 & -\frac{1}{2}\alpha_2 & \frac{1}{4}\alpha_2 & \frac{1}{4}\alpha_2 & \alpha_2 \end{bmatrix}.$$

Since $\sigma(\Omega + [b_2 \mid b_2]^T X) = \{\mu_1, \mu_2\}$, *must be*

$$\begin{array}{lcl} 6 + 5\alpha_1 + 3 + 2\alpha_2 & = & 8 + \sqrt{19} + 8 - \sqrt{19} \\ (6 + 5\alpha_1)(3 + 2\alpha_2) - (5\alpha_1)(2\alpha_2) & = & (8 + \sqrt{19})(8 - \sqrt{19}) \end{array}.$$

Then $\alpha_1 = 1, \alpha_2 = 1$
Taking $F^T = [e_2 \mid e_2]\, X^T$, *we have that*

$$A + BF^T = \begin{bmatrix} 0 & 5 & 1 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 4 \\ 2 & 2 & 10 & 5 & 8 \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{4} & \frac{5}{4} & 4 \end{bmatrix}$$

is nonnegative and

$$\sigma(A + BF^T) = \sigma(\Omega + [e_2 \mid e_2]^T B^T X) \cup \{3, -5, -5\}$$

where

$$\Omega + [e_2 \mid e_2]^T B^T X = \begin{bmatrix} 11 & 2 \\ 5 & 5 \end{bmatrix}$$

has the spectrum $8 + \sqrt{19}, 8 - \sqrt{19}$.

References

[1] A. Brauer, Limits for the characteristic roots of a matrix. IV: Aplications to stochastic matrices, *Duke Math. J.*, **19** (1952), 75-91. http://dx.doi.org/10.1215/s0012-7094-52-01910-8

[2] R. Bru, R. Canto, R. Soto, A.M. Urbano, A Brauer's theorem and related results, *Central European Journal of Mathematics*, **10** (2012), 312-321. http://dx.doi.org/10.2478/s11533-011-0113-0

[3] R.C. Díaz, R.L. Soto, Nonnegative inverse elementary divisors problem in the left half plane, *Linear and Multilinear Algebra*, **64** (2015), 258-268. http://dx.doi.org/10.1080/03081087.2015.1034640

[4] A.I. Julio, R.L. Soto, Persymmetrix and bisymmetric nonnegative inverse eigenvalue problem, *Linear Algebra and its Appl.*, **469** (2015), 130-152. http://dx.doi.org/10.1016/j.laa.2014.11.025

[5] H. Minc, Inverse elementary divisor problem for nonnegative matrices, *Proceedings of the American Mathematical Society*, **83** (1981), no. 4, 665-669. http://dx.doi.org/10.1090/s0002-9939-1981-0630033-x

[6] H. Minc, Inverse elementary divisor problem for doubly stochastic matrices, *Linear and Multilinear Algebra*, **11** (1982), 121-131. http://dx.doi.org/10.1080/03081088208817437

[7] H. Minc, *Nonnegative Matrices*, John Wiley & Sons, New York, 1988.

[8] H. Perfect, Methods of constructing certain stochastic matrices, *Duke Math. J.*, **20** (1953), 395-404. http://dx.doi.org/10.1215/s0012-7094-53-02040-7

[9] H. Perfect, Methods of constructing certain stochastic matrices II, *Duke Math. J.*, **22** (1955), 305-311. http://dx.doi.org/10.1215/s0012-7094-55-02232-8

[10] R.L. Soto, Existence and construction of nonnegative matrices with prescribed spectrum, *Linear Algebra and its Appl.*, **369** (2003), 169-184. http://dx.doi.org/10.1016/s0024-3795(02)00731-0

[11] R.L. Soto, O. Rojo, Applications of a Brauer theorem in the nonnegative inverse eigenvalue problem, *Linear Algebra and its Appl.*, **416** (2006), 844-856. http://dx.doi.org/10.1016/j.laa.2005.12.026

[12] R.L. Soto, O. Rojo, J. Moro, A. Borobia, Symmetric nonnegative realization of spectra, *Electronic J. of Linear Algebra*, **16** (2007), 1-18. http://dx.doi.org/10.13001/1081-3810.1178

[13] R.L. Soto, J. Ccapa, Nonnegative matrices with prescribed elementary divisors, *Electron. J. of Linear Algebra*, **17** (2008), 287-303. http://dx.doi.org/10.13001/1081-3810.1264

[14] R.L. Soto, A family of realizability criteria for the real and symmetric nonnegative inverse eigenvalue problem, *Numerical Linear Algebra with Appl.*, **20** (2011), 336-348. http://dx.doi.org/10.1002/nla.835

[15] R.L. Soto, R.C. Díaz, H. Nina, M. Salas, Nonnegative matrices with prescribed spectrum and elementary divisors, *Linear Algebra and its Appl.*, **439** (2013), 3591-3604. http://dx.doi.org/10.1016/j.laa.2013.09.034

[16] R.L. Soto, A.I. Julio, M. Salas, Nonnegative persymmetric matrices with prescribed elementary divisors, *Linear Algebra and its Appl.*, **483** (2015) 139-157. http://dx.doi.org/10.1016/j.laa.2015.05.032

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