MCDM Method to Compare Estimators of Scale Parameter in the 2-Parameter Exponential Distribution with Prior Information

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Abstract

This paper proposes a problem of estimation of the scale parameter $\theta$ in 2-parameter exponential distribution with prior information $\theta_0$. The estimators of $\theta$ are maximum likelihood estimator and a class of shrinkage estimators. The comparison of these estimators and also ranking all of them together are based on Multiple Criteria Decision Making (MCDM) method. The result turns out that $\hat{\theta}(1)$ is the best estimator while $\hat{\theta}(-2), \hat{\theta}(-1), \hat{\theta}(MLE)$ and $\hat{\theta}(2)$ are lower in rank respectively for $n \geq 4$.

Keywords: multiple criteria decision making, 2-parameter exponential distribution, scale parameter

1 Introduction

The problem on estimation of parameters of any distribution is an interesting. One of this problem is on the estimation of scale parameter of 2-parameter exponential distribution. This distribution is formulated as

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{x-\gamma}{\theta}}, \text{ for } x \geq \gamma, \theta > 0,$$

where $\theta$ and $\gamma$ are scale and location parameter, respectively. This distribution has mean $\gamma + \theta$ and variance $\theta^2$. Let $x_1, x_2, \ldots, x_n$ be a random sample of size
n from an exponential population. The maximum likelihood estimator (MLE) of θ is

\[ \hat{\theta}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{(1)}) \]

where \( x_{(1)} = \min\{x_i \mid i = 1, 2, \ldots, n \} \).

Kourouklis [1] proposed a class of shrinkage estimators \( \hat{\theta}_p \) for the scale parameter of a 2-parameter exponential distribution, given a prior estimate of the scale parameter \( \hat{\theta}_0 \) is

\[ \hat{\theta}_p = \theta_0 + \alpha(p)(\hat{\theta} - \theta_0), \]

where \( \hat{\theta} \) is an unbiased estimator of \( \theta \), \( \alpha(p) = \frac{\Gamma(n-1-p)}{\Gamma(n-2-p)(n-1)^p} \) for \( p \in (-\infty, 1/2(n-1)) \).

In this paper the estimators of scale parameter θ in 2-parameter exponential distribution are compared and ranked in terms of mean square errors (MSEs) using Multiple Criteria Decision Making (MCDM) method. MCDM method is briefly described in Section 2. Section 3 contains the mean square errors of each estimator. Result of comparing and ranking of the estimators is shown in theorem in Section 4. The conclusion is presented in the last section.

2 A brief description of MCDM method

In the context of a ‘discrete’ risk matrix \( X = (x_{ij}) : K \times N \) where \( x_{ij} \)'s represent ‘risk’ of \( i^{th} \) ‘estimator’ for \( j^{th} \) ‘parameter point’, and we need to compare \( K \) estimators simultaneously with respect to all the \( N \) parameter points, MCDM is a novel statistical procedure to integrate the multiple risks \( (x_{i1}, \ldots, x_{iN}) \) for the \( i^{th} \) estimator into a single risk factor meaningful [2,3]. The \( K \) estimators are then compared on the basis of these integrated risk factors. Integration of risks is done by defining an Ideal Row (IDR) with the smallest observed value for each column denoted as

\[ IDR = (\min_i x_{i1}, \ldots, \min_i x_{iN}) = (u_1, \ldots, u_N), \quad i = 1, 2, \ldots, K \]

and a Negative Ideal Row (NIDR) with the largest observed value for each column denoted as

\[ NIDR = (\max_i x_{i1}, \ldots, \max_i x_{iN}) = (v_1, \ldots, v_N), \quad i = 1, 2, \ldots, K. \]

For any given row \( i \), we now compute the distance of each row from Ideal Row and from Negative Ideal Row based on a suitably chosen norm. Under \( L_1 \)-norm [4], we compute

\[ L_1(i, IDR) = \sum_{j=1}^{N} (x_{ij} - u_j)w_j \]
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$L_1(i, NIDR) = \sum_{j=1}^{N} (w_j - x_j) w_j$,

where $w_j$'s are appropriate weights. For details of weights, we refer to Maitra [3]. Likewise, under $L_2$-norm [4], we compute

$L_2(i, IDR) = \left( \sum_{j=1}^{N} (x_{ij} - u_j)^2 w_j \right)^{\frac{1}{2}}$,

$L_2(i, NIDR) = \left( \sum_{j=1}^{N} (x_{ij} - v_j)^2 w_j \right)^{\frac{1}{2}}$.

The various rows are now compared and ranked based on an overall index computed as

$L_1(\text{Index}_i) = \frac{L_1(i, IDR)}{L_1(i, IDR) + L_1(i, NIDR)}, \quad i = 1, \ldots, K$

for $L_1$-norm and

$L_2(\text{Index}_i) = \frac{L_2(i, IDR)}{L_2(i, IDR) + L_2(i, NIDR)}, \quad i = 1, \ldots, K$

for $L_2$-norm.

A ‘continuous’ version of this setup which is relevant for our problem would involve $x_{ij}$'s representing mean square error where the index $j$ would vary ‘continuously’. In the context of the problem of comparing several the estimators of scale parameter $\theta$, that is $\hat{\theta}_{MLE}$ and a class of shrinkage estimator $\bar{\hat{\theta}}_{(p), p = \pm 1, \pm 2}$. Finally we have five estimators to compare, $x_{ij}$'s are functions of $r$. So $L_1$-norm and $L_2$-norm can be redefined as

$L_1(i, IDR) = \int_{\xi}^{\eta} (x_i(r) - u(r)) w(r) dr$

$L_1(i, NIDR) = \int_{\xi}^{\eta} (v(r) - x_i(r)) w(r) dr$

$L_2(i, IDR) = \left( \int_{\xi}^{\eta} (x_i(r) - u(r))^2 w(r) dr \right)^{\frac{1}{2}}$

$L_2(i, NIDR) = \left( \int_{\xi}^{\eta} (x_i(r) - v(r))^2 w(r) dr \right)^{\frac{1}{2}}$. 
where \( u(r) = \min_i \{ x_i(r) \} \) and \( v(r) = \max_i \{ x_i(r) \} \), \( i = 1, 2, 3, 4, 5 \) when \( r \leq r \leq \bar{r} \).

In the case of \( L_1 \)-norm, Lertprapai [5] has shown that some estimators to be compare with respect to their mean square errors, the \( i \)th estimator is better than the \( j \)th estimator if

\[
L_1(\text{index}_i) < L_1(\text{index}_j)
\]

that is

\[
\int_{\ell}^{\bar{r}} x_i(r)w(r)dr < \int_{\ell}^{\bar{r}} x_j(r)w(r)dr
\]

where \( w(r) \) is the weight function. Therefore we can use this inequality to rank the estimators instead. Such a result does not hold under \( L_2 \)-norm.

### 3 Mean Square Errors (MSEs)

From Kourouklis [1] the mean square errors of \( \hat{\theta}_{\text{MLE}} \) and \( \hat{\theta}_{(p)} \) are given as the followings.

\[
MSE(\hat{\theta}_{\text{MLE}}) = \frac{\theta^2}{n},
\]

\[
MSE(\hat{\theta}_{(p)}) = \theta^2 \left( (1 - \alpha(p))^2 r^2 + \frac{\alpha^2(p)}{n-1} \right)
\]

where \( p \) is a non-zero real number, \( \alpha(p) = \frac{\Gamma(n-1-p)}{\Gamma(n-1-p)(n-1-p)} \) and \( r = \frac{\theta_0}{\theta} - 1 \).

Since \( \theta^2 \) is a common term which can be omitted so mean square errors are in the form

\[
MSE(\hat{\theta}_{\text{MLE}}) = \frac{1}{n}
\]

and

\[
MSE(\hat{\theta}_{(p)}) = (1 - \alpha(p))^2 r^2 + \frac{\alpha^2(p)}{n-1}.
\]

Let \( p = \pm 1, \pm 2 \), so we get

\[
\alpha(-2) = \frac{(n-1)^2}{(n+1)(n+2)}, \quad \alpha(-1) = \frac{n-1}{n}, \quad \alpha(1) = \frac{n-3}{n-1}, \quad \alpha(2) = \frac{(n-4)(n+5)}{(n-1)^2}.
\]

### 4 Main Result

In Section 3, MSEs of each estimator are computed. After using inequality (2.1) and (2.2) with the range \(-1 < r < 1\) and \( n \geq 4 \), we can present the result in the following theorem.
**Theorem 4.1.** Let $\theta$ be a scale parameter in the 2- parameter exponential distribution with prior information $\theta_0$. If the estimators of scale parameter are maximum likelihood estimator ($\hat{\theta}_{MLE}$) and a class of shrinkage estimators ($\hat{\theta}(\rho)$). Then based on $MSE$s, $\hat{\theta}(1)$ is the best estimator while $\hat{\theta}(-2), \hat{\theta}(-1), \hat{\theta}_{MLE}$ and $\hat{\theta}(2)$ are lower in rank respectively for $n \geq 4$ under MCDM approach using $L_1$-norm and weight function defined by $w(r) = 1$.

**Proof:** Let $\hat{\theta}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{(1)})$, $x_{(1)} = \min \{x_i \mid i = 1, 2, \ldots, n\}$ and $\hat{\theta}(\rho) = \theta_0 + \alpha(\rho)(\hat{\theta} - \theta_0)$ with $\rho = \pm 1, \pm 2$.

Writing $MSE(\hat{\theta}_{MLE}) = x_1(r)$, $MSE(\hat{\theta}(-2)) = x_2(r)$, $MSE(\hat{\theta}(-1)) = x_3(r)$, $MSE(\hat{\theta}(1)) = x_4(r)$ and $MSE(\hat{\theta}(2)) = x_5(r)$ which are obtain from (3.1) and (3.2).

Lertprapai [5] proposed that the $i$th estimator is better than the $j$th estimator if

$$\int_{-1}^{1} x_i(r)w(r)dr < \int_{-1}^{1} x_j(r)w(r)dr.$$ 

Since $w(r) = 1$ and consider $-1 < r < 1$, we must then have

$$\int_{-1}^{1} x_4(r)dr < \int_{-1}^{1} x_2(r)dr.$$ 

For $n \geq 4$ we could show that

$$\int_{-1}^{1} x_4(r)dr < \int_{-1}^{1} x_2(r)dr < \int_{-1}^{1} x_3(r)dr < \int_{-1}^{1} x_1(r)dr < \int_{-1}^{1} x_5(r)dr.$$ 

Comparing each pair of inequality, thus we show in 4 cases as follow:

**Case 1:** $\int_{-1}^{1} x_4(r)dr < \int_{-1}^{1} x_2(r)dr$

so we get

$$\int_{-1}^{1} \left(1 - \alpha(-1)^2 \right)dr < \int_{-1}^{1} \left(1 - \alpha(-2)^2 \right)dr,$$

$$\frac{2(3n^2 - 14n + 23)}{3(n-1)^3} < \frac{2(3n^2 + 10n - 1)}{3(n+2)/n+1}$$

which we may rewrite as

$$3n^4 - 4n^3 + 2n^2 - 100n - 45 > 0.$$
By mathematical induction, assume this inequality is true for \( n \geq 4 \).

For \( n = 4 \),
\[
3(4)^4 - 4(4)^3 + 2(4)^2 - 100(4) - 45 = 99 > 0.
\]

This is true. Assume the truth of the statement \( 3n^4 - 4n^3 + 2n^2 - 100n - 45 > 0 \) for some \( n \), now
\[
3(n+1)^4 - 4(n+1)^3 + 2(n+1)^2 - 100(n+1) - 45 = 3n^4 + 8n^3 + 8n^2 - 96n - 144
= (3n^4 - 4n^3 + 2n^2 - 100n - 45) + 12n^3 + 6n^2 + 4n - 99
\]
which is statement for \( n+1 \).

Since \( 3n^4 - 4n^3 + 2n^2 - 100n - 45 > 0 \) and \( 12n^3 + 6n^2 + 4n - 99 \) is greater than zero for \( n \geq 4 \). So the statement is true for \( n \geq 4 \) and its truth for \( n \) implies its truth for \( n+1 \). Therefore it is true for all \( n \).

**Case 2:** \( \int_{-1}^{1} x_2(r) \, dr < \int_{-1}^{1} x_3(r) \, dr \)

so we get
\[
\int_{-1}^{1} \left(1 - \alpha(-2)\right)^2 r^2 + \frac{\alpha^2(-2)}{n-1} \, dr < \int_{-1}^{1} \left(1 - \alpha(-1)\right)^2 r^2 + \frac{\alpha^2(-1)}{n-1} \, dr ,
\]
\[
\frac{2(3n^2 - 10n - 1)}{3(n+2)(n+1)^2} < \frac{2(3n-2)}{3n^2} ,
\]
which we may rewrite as
\[
8n^2 - 4n - 4 > 0 ,
\]
\[
4(2n+1)(n-1) > 0 \text{ or } (n-1) > 0 \text{ that obvious for } n \geq 4 .
\]

**Case 3:** \( \int_{-1}^{1} x_3(r) \, dr < \int_{-1}^{1} x_4(r) \, dr \)

so we get
\[
\int_{-1}^{1} \left(1 - \alpha(-1)\right)^2 r^2 + \frac{\alpha^2(-1)^2}{n-1} \, dr < \int_{-1}^{1} \frac{1}{n} \, dr ,
\]
\[
\frac{2(3n-2)}{3n^2} < \frac{2}{n} ,
\]
or \( 2 > 0 \) which is true.

**Case 4:** \( \int_{-1}^{1} x_4(r) \, dr < \int_{-1}^{1} x_5(r) \, dr \)

so we get
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\[
\int_{-1}^{1} \frac{1}{n} dr < \int_{-1}^{1} \left( (1-\alpha(2))^2 r^2 + \frac{\alpha^2(2)}{n-1} \right) dr ,
\]

\[
\frac{2}{n} < \frac{2(3n^4 - 5n^3 + 48n^2 - 453n + 839)}{3(n-1)^3},
\]

which we may rewrite as

\[
10n^4 + 18n^3 - 423n^2 + 824n + 3 > 0 .
\]

By mathematical induction, assume this inequality is true for \( n \geq 4 \).

For \( n = 4 \),

\[
10(4)^4 + 18(4)^3 - 423(4)^2 + 824(4) + 3 = 243 > 0 .
\]

This is true.

Assume the truth of the statement \( 10n^4 + 18n^3 - 423n^2 + 824n + 3 > 0 \) for some \( n \), now

\[
10(n+1)^4 + 18(n+1)^3 - 423(n+1)^2 + 824(n+1) + 3 = 10n^4 + 58n^3 - 309n^2 + 72n + 432
\]

\[
= (10n^4 + 18n^3 - 423n^2 + 824n + 3) + 40n^3 + 114n^2 - 752n + 429
\]

\[
= (10n^4 + 18n^3 - 423n^2 + 824n + 3) + 2n(20n^2 + 57n - 376) + 429
\]

which is statement for \( n+1 \).

Since \( 10n^4 + 18n^3 - 423n^2 + 824n + 3 > 0 \) and \( 20n^2 + 57n - 376 \) is greater than zero for \( n \geq 4 \). So the statement is true for \( n \geq 4 \) and its truth for \( n \) implies its truth for \( n+1 \). Therefore, it is true for all \( n \).

These complete the proof in four cases.

References


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