The Solution of the Sinh-Gordon Wave Equation
Using Lattice-Boltzmann and
the Tanh-Coth Method

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Abstract

In this paper we solved the temporal nonlinear one-dimensional Sinh-Gordon equation using lattice-Boltzmann using a $d1q3$ velocity scheme, and the Tanh-Coth solitary wave method. We find several families of solutions.

Keywords: 1D Sinh-Gordon, lattice-Boltzmann. tanh-coth method

1 Introduction

The Sinh-Gordon equation (ShGEq), is a very important analytical tool that helps to explain physical phenomena as diverse as integrable Ising-Ising models, conformal field theory till fluid dynamics, [1]- [3]. Important work has been done in order to find solutions to ShGEq. For instance, a remarkable analysis using the Painlevé property and the Bäcklund transformation, [4], is applied to ShGEq in ref. [5]. In the same way, in ref.[6], solitary wave solutions using the Painlevé property and the tanh method, are achieved. Also, the $(G'/G)$-expansion method is applied to the double sinh-Gordon equation to get solutions [7]. Equally, a special transformations join with Jacobian elliptic functions are used, in [8], in order to accomplished solutions to ShGEq.
In addition, another striking technique called, lattice Boltzmann (LB), has proven to be versatile in diverse research fields, such as fluid dynamics [9], till relativistic quantum mechanics [10].

The manuscript is organized as follows. Section (2), presents the lattice-Boltzmann model model applied to Sinh-Gordon equation. In section (3), the definition of the moments of the particle distribution are given and we get the ShGEq. Section (4), we get the equilibrium distribution function. In Section (5), we apply the Tanh-Coth method, [11], in order to obtain solitary wave solutions of the ShGEq. At last, in section (6), we present results and conclusions.

2 The lattice Boltzmann model

The lattice-Boltzmann equation is:

$$f_i(x + v_x \delta t, t + \delta t) - f_i(x, t) = \Omega_i(x, t) + \Xi_i(x, t)$$

(1)

Where $\Omega(\vec{x}, t)$ is the collision term, and it is given by:

$$\Omega_i(x, t) = \sum_j A_{i,j} [f_j(x, t) - f_{eq}^i(x, v_x)]$$

(2)

Using the B.G.K approximation for the collision term, [12]:

$$\Omega_i(x, t) = -\frac{1}{\tau} [f_i(x, t) - f_{eq}^i(x, v_x)]$$

(3)

Expanding the one-particle distribution function in a Taylor series, we have:

$$\Delta f_i = f(x + \delta x, t + \delta t) - f_i(x, t) = (\delta t \frac{\partial}{\partial t} + \delta x \frac{\partial}{\partial x}) f_i + (\frac{\delta t^2}{2} \frac{\partial^2}{\partial t^2}) f_i$$

(4)

$$+ \frac{\delta x^2}{2} \frac{\partial^2}{\partial x^2} + \delta t \delta x \frac{\partial^2}{\partial x \partial t} f_i$$

Considering $\delta x = v_x \delta t$, $\delta x = \delta tv_x$, and replacing in eq.(4)

$$f(x + v_x \delta t, t + \delta t) - f_i(x, t) = \delta t \left[ \frac{\partial}{\partial t} + v_{x,i} \frac{\partial}{\partial x} \right] f_i + \frac{\delta t^2}{2} \left[ \frac{\partial}{\partial t} + v_{x,i} \frac{\partial}{\partial x} \right]^2 f_i$$

(5)

Doing a perturbative expansion at first order in the spatial derivatives, second order to the time derivative, and second order to the particle distribution function, we have:
The left-hand side of eq. (1) is:

$$\Delta f_i = \delta t \left[ \frac{\partial}{\partial t} + v_{x,i} \frac{\partial}{\partial x} \right] f_i + \frac{\delta t^2}{2} \left[ \frac{\partial}{\partial t} + v_{x,i} \frac{\partial}{\partial x} \right]^2 f_i$$  \hspace{1cm} (7)

Replacing eqs. (6) in eq. (7)

$$-\frac{1}{\tau} (f_i^0 + \epsilon f_i^1 + \epsilon^2 f_i^2 - f^{eq}) = \delta t((\epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}) + v_{x,i}[\epsilon \frac{\partial}{\partial x_1}])f_i^0$$  \hspace{1cm} (8)

$$+\epsilon f_i^1 + \epsilon^2 f_i^2 + \frac{\delta t^2}{2}[(\epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}) + v_{x,i}[\epsilon \frac{\partial}{\partial x_1}]]^2(f_i^0 + \epsilon f_i^1 + \epsilon^2 f_i^2) + \epsilon^2 \phi_i$$

The source term in LB eq. (1), [13]-[14], is defined as:

$$\Xi_i(x, t) = \epsilon^2 \phi_i$$  \hspace{1cm} (9)

The terms at order $\epsilon$ in eq. (8) and assuming $f_i^0 = f^{eq}$, we have:

$$-\frac{1}{\tau} \epsilon f_i^1 = \epsilon \delta t \left[ \frac{\partial}{\partial t_1} + v_{x,i} \frac{\partial}{\partial x_1} \right] f_i^0$$  \hspace{1cm} (10)

The terms at order $\epsilon^2$ in eq (8), we get:

$$-\frac{1}{\tau} [\epsilon^2 f_i^2] = \epsilon^2 \delta t \frac{\partial}{\partial t_2} (f_i^0) + \epsilon^2 \delta t \frac{\partial}{\partial t_1} + v_{x,i} \frac{\partial}{\partial x_1} f_i^0$$  \hspace{1cm} (11)

Replacing eq. (10) in eq. (11)

$$-\frac{1}{\tau} [\epsilon^2 f_i^2] = \epsilon^2 (f_i^0) + \epsilon^2 \delta t \left[ \frac{\partial}{\partial t_1} + v_{x,i} \frac{\partial}{\partial x_1} \right] f_i^1 (1 - \frac{1}{2\tau})$$  \hspace{1cm} (12)

Therefore

$$-\frac{1}{\tau} f_i^2 = \delta t \frac{\partial}{\partial t_2} f_i^0 + \delta t \left[ \frac{\partial}{\partial t_1} + v_{x,i} \frac{\partial}{\partial x_1} \right] f_i^1 (1 - \frac{1}{2\tau})$$  \hspace{1cm} (13)

Summing eq. (10) to eq. (13), we have

$$-\frac{1}{\tau \delta t} [\epsilon f_i^1 + \epsilon^2 f_i^2] = [\frac{\partial}{\partial t_1} + v_{x,i} \frac{\partial}{\partial x_1}](f_i^0) + \epsilon^2 f_i^1 (1 - \frac{1}{2\tau}) + \epsilon^2 \frac{\partial}{\partial t_2} f_i^0$$  \hspace{1cm} (14)
3 Moments of the distribution function

We suppose, by convenience, the next set of moments of the distributions

\[
\frac{\partial \rho}{\partial t} = \sum_i \left( f^0_i \right) \tag{15}
\]

\[- \frac{\partial \rho}{\partial x} = \sum_i \left( v_{x,i} f^0_i \right) \tag{16}\]

\[
\Pi^0_{\alpha,\beta} = \sum_i v_{i,\alpha} v_{i,\beta} f^0_i = \delta_{\alpha,\beta} \lambda \rho \tag{17}\]

\[
\sum_i \left( f^k_i \right) = 0, \ k > 0 \tag{18}\]

\[
\sum_i \left( v_{x,i} f^k_i \right) = 0, \ k > 0 \tag{19}\]

Summing on \(i\) in eq. (15) and using eq. (16)

\[
0 = \frac{\partial^2}{\partial t^2} (\rho) + \sum_i v_{x,i} \frac{\partial}{\partial x} \left( f^0_i \right) + \epsilon \sum_i \phi_i \tag{19}\]

Now we assume

\[
\frac{\partial}{\partial x} \left( v_{x,i} f^0_i \right) = v_{x,i} \frac{\partial}{\partial x} \left( f^0_i \right) + \left( \frac{\partial}{\partial x} v_{x,i} \right) f^0_i \tag{20}\]

\[
v_{x,i} \frac{\partial}{\partial x} \left( f^0_i \right) = - \left( \frac{\partial}{\partial x} v_{x,i} \right) f^0_i + \frac{\partial}{\partial x} \left( v_{x,i} f^0_i \right) \tag{21}\]

The term is assumed \(\frac{\partial}{\partial x} v_{x,i} = 0\), which means an irrotational fluid. Then, we have:

\[
v_{x,i} \frac{\partial f^0_i}{\partial x} = \frac{\partial (v_{x,i} f^0_i)}{\partial x} \tag{22}\]

We consider \(\epsilon \sum_i \phi_i = \lambda \Phi, [13]-[14]\). Therefore, in eq. (19):

\[
0 = \frac{\partial^2}{\partial t^2} (\rho) + \frac{\partial}{\partial x} \left( \sum_i v_{x,i} f^0_i \right) + \lambda \Phi \tag{23}\]

And using eq. (17), we have the nonlinear wave equation:

\[
\frac{\partial^2}{\partial t^2} (\rho) = \frac{\partial}{\partial x} \left( \frac{\partial \rho}{\partial x} \right) - \lambda \Phi \tag{24}\]
If we consider

\[ \lambda \Phi = - \sinh (\rho) \]  \hspace{1cm} (25)

Therefore, eq. (24) is

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho}{\partial x^2} + \sinh (\rho) \]  \hspace{1cm} (26)

### 4 Distribution function d1q3

Defining the distribution function \( f_i^{(eq)} = f_i^0 \), and using the d1q3 velocity one-dimensional scheme with \( e_\alpha = \{0, c, -c\} \), as:

\[
 f_i^{(eq)} \left( e_\alpha, e_\beta \right) = \begin{cases} 
 -\frac{\partial \rho}{\partial t} - \frac{\lambda \rho}{2c^2} & \rightarrow \quad i = 0 \\
 \frac{\lambda \rho}{2c^2} - \frac{1}{2} \frac{\partial \rho}{\partial x} & \rightarrow \quad i = 1 \\
 \frac{\lambda \rho}{2c^2} + \frac{1}{2} \frac{\partial \rho}{\partial x} & \rightarrow \quad i = 2 
\end{cases}
\]  \hspace{1cm} (27)

The temporal and spatial derivative of \( \rho(x, t) \) is applied with the backward difference discretization scheme

\[ \frac{\partial \rho(x, t)}{\partial t} \rightarrow \frac{\rho(x, t) - \rho(x, t - \Delta t)}{\Delta t} \]  \hspace{1cm} (28)

and

\[ \frac{\partial \rho(x, t)}{\partial x} \rightarrow \frac{\rho(x, t) - \rho(x - \Delta x, t)}{\Delta x} \]  \hspace{1cm} (29)

### 5 The tanh-Coth method

We start using the next trasformation:
\[ u = x - kt \]  

\[ \frac{\partial}{\partial t} = -k \frac{d}{du}; \quad \frac{\partial}{\partial x} = \frac{d}{du}; \quad \frac{\partial^2}{\partial x^2} = \frac{d^2}{du^2}; \quad \frac{\partial^2}{\partial t^2} = k^2 \frac{d^2}{du^2} \]

Then, eq. (26)

\[ \frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 \rho}{\partial x^2} = \sinh(\rho) \]  

\[ k^2 \frac{d^2 \rho}{du^2} - \frac{d^2 \rho}{du^2} = \sinh(\rho) \]  

\[ (k^2 - 1) \frac{d^2 \rho}{du^2} = \sinh(\rho) \]  

Also, we do the next transformation, [8]:

\[ \rho = 2 \tanh^{-1}(v) \]

Then

\[ \frac{d \rho}{du} = \frac{d \rho}{dv} \frac{dv}{du} = \frac{2}{1 - v^2} \frac{dv}{du} \]

And

\[ \frac{d^2 \rho}{du^2} = \frac{4v}{(1 - v^2)^2} \left( \frac{dv}{du} \right)^2 + \frac{2}{1 - v^2} \frac{d^2 v}{du^2} \]

Replacing in eq. (33)

\[ \frac{2(k^2 - 1) \frac{d^2 v}{du^2}}{1 - v^2} + \frac{4(k^2 - 1)v \left( \frac{dv}{du} \right)^2}{(1 - v^2)^2} = \sinh(2 \tanh^{-1}(v)) \]

And

\[ \sinh(2 \tanh^{-1}(v)) = 2 \sinh(\rho) \cosh(\rho) = 2 \frac{v}{\sqrt{1 - v^2}} \frac{1}{\sqrt{1 - v^2}} \]

Then

\[ (k^2 - 1)(1 - v^2) \frac{d^2 v}{du^2} + 2(k^2 - 1)v \left( \frac{dv}{du} \right)^2 - v + v^3 = 0 \]  

Now, we introduce a new independent variable:
Solution of the sinh-Gordon wave equation

\[ Y(x, t) = \tanh(u) \]  

Then, the derivatives of \( u \), are:

\[ \frac{d}{du} = (1 - Y^2) \frac{d}{dY} \]

\[ \frac{d^2}{du^2} = -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2} \]

The solutions are postulated as:

\[ v(\xi) = \sum_{i=1}^{q} a_i Y^i + \sum_{i=1}^{q} a_{i+q} Y^{-i} \]

Then, replacing eqs. (40)

\[ -2(k^2 - 1)(1 - v^2)Y(1 - Y^2) \frac{dv}{dY} + (k^2 - 1)(1 - v^2)(1 - Y^2)^2 \frac{d^2v}{dY^2} \]

\[ -2(k^2 - 1)v(1 - Y^2)^2 \left( \frac{dv}{dY} \right)^2 - v + v^3 = 0 \]

Now, we balance the highest-order linear derivative with the highest order nonlinear terms in eq. (44). We get:

\[ Y^4 \frac{d^2v}{dY^2} \rightarrow v^3 \rightarrow m + 2 = 3m \rightarrow m = 1 \]

\[ v = a_0 + a_1 Y + a_2 Y^{-1}, \rightarrow \frac{dv}{dY} = a_1 - a_2 Y^{-2}, \rightarrow \frac{d^2v}{dY^2} = -2a_2 Y^{-3} \]

If \( k_1 = k^2 - 1 \), we have in eq. (46):

\[ -2k_1 a_1 Y + 2k_1 a_1 Y^3 + 2k_1 a_2 Y^{-1} - 2k_1 a_2 Y \]

\[ -2k_1 a_1^2 Y^5 - 4k_1 a_0 a_1 Y^4 - 2k_1 a_1^2 a_2 Y^3 + 2k_1 a_1 a_2 Y^3 - 2k_1 a_0^2 a_1 Y^3 \]

\[ -2k_1 a_2^3 Y^3 + 4k_1 a_0 a_2 a_1 Y^2 Y^2 - 4k_1 a_0 a_2^2 Y^2 Y^2 - 2k_1 a_1 a_2 Y^2 + 2k_1 a_1 a_2 Y + 2k_1 a_2 a_1 Y \]

\[ +2k_1 a_0 a_2 Y + 2k_1 a_0 a_1 Y + 2k_1 a_0 a_2 Y^{-1} - 2k_1 a_1 a_0 Y^{-1} - 2k_1 a_2 a_0 Y^{-1} \]

\[ +4k_1 a_1 a_2 Y - 2k_1 a_2 Y^{-3} - 4k_1 a_2 Y^{-1} - 2k_1 a_2 Y \]

\[ +(2k_1 a_2^3 Y^{-5} + 4k_1 a_0 a_2^2 Y^{-4} + 2k_1 a_1 a_2 a_1 Y^{-3} - 4k_1 a_2^2 Y^{-3} + 4k_1 a_1 a_2 Y^{-3} \]
The spatiotemporal, LB, evolution of \( \rho(x, t) \) using a \( d1q3 \) lattice velocity, for two \( \tanh - \text{sech}(x) \) eq. (46) initial profiles. The system size is \( L = 100 \).

\[
\begin{align*}
+2k_1a_0^2a_2Y^{-3} &+ 4k_1a_0a_1a_2Y^2 - 8k_1a_0a_2^2Y^{-2} + 4k_1a_0a_1a_2Y^{-2} \\
+4k_1a_1a_2Y - 4k_1a_1^2a_2Y + 2k_1a_0^2a_2Y + 2k_1a_1a_2Y^{-1} - 8k_1a_1a_2Y^{-1} \\
+2k_1a_0^2a_2Y^{-1} &- 4k_1a_0a_2^2Y^{-1} + 4k_1a_0a_2^2 - 8k_1a_0a_1a_2 \\
-a_0 &- a_1Y - a_2Y^{-1} + a_1^3Y^3 + a_2^3Y^{-3} + 3a_0a_1a_2Y^{-2} \\
+3a_1^2a_2Y &+ 3a_0a_1Y + 3a_1a_2Y^{-1} + 3a_0a_2Y^{-1} + 6a_0a_1a_2 + a_0^3 \\
-2a_1^2k_1Y^5 &- 2a_0^2k_1Y^5 - 2a_0a_2^2k_1Y^4 - 2a_0a_2^2k_1Y^{-4} + 2a_1^2a_2k_1Y^{-3} \\
+4a_1^2k_1Y^3 &+ 4a_0^2k_1Y^3 + 2a_1a_2^2k_1Y^{-3} + 4a_0a_1a_2k_1Y^2 + 4a_0a_2^2k_1Y^2 \\
+4a_0a_2^2k_1Y^{-2} &+ 4a_0a_1a_2k_1Y^{-2} + 2a_1a_2^2k_1Y - 4a_1^2a_2k_1Y - 2a_1^3k_1Y \hspace{1cm} \\
-2a_2^2k_1Y^{-1} &- 4a_1a_2^2k_1Y^{-1} + 2a_2^2a_1k_1Y^{-1} - 2a_0a_2^2k_1 \\
-8a_0a_1a_2k_1 &- 2a_0a_2^2k_1 = 0
\end{align*}
\]

We suppose \( a_1 \neq 0 \) and \( a_2 \neq 0 \). Doing some algebra, we get:

\[
a_2 = \frac{8k_1a_1}{(8k_1 - 3)} \tag{48}
\]

\[
a_{1,2} = \pm \sqrt{\frac{1 + 2k_1 + 4k_1 \frac{8k_1}{(8k_1 - 3)} \left( \frac{8k_1}{(8k_1 - 3)} \right)^2 - 6k_1 \frac{8k_1}{(8k_1 - 3)} + \frac{24k_1}{(8k_1 - 3)} - 2k_1}{(8k_1 - 3)^2}} \tag{49}
\]

\[
k_1 = \pm \sqrt{\frac{3}{32}} \tag{50}
\]
Solution of the sinh-Gordon wave equation

\[ a_{13,4} = \pm i \sqrt{\frac{2k_1}{(2k_1 \frac{8k_1}{(8k_1-3)} + 1 + 6k_1)}} \]  

(51)

\[ a_{15,6} = \pm \sqrt{\frac{2k_1}{(-2k_1(\frac{8k_1}{(8k_1-3)})^2 - 4k_1(\frac{8k_1}{(8k_1-3)})^2 + 4k_1 \frac{8k_1}{(8k_1-3)} + (\frac{8k_1}{(8k_1-3)})^2)}} \] 

(52)

\[ a_{17,8} = \pm \sqrt{\frac{1 - 6k_1}{(4k_1(\frac{8k_1}{(8k_1-3)})^2 - 16k_1 \frac{8k_1}{(8k_1-3)} + 3 \frac{8k_1}{(8k_1-3)} - 2k_1(\frac{8k_1}{(8k_1-3)})^2 + 4k_1)}} \] 

(53)

Then, we find eight families of solutions.

6 Conclusions

This paper presents the LB and Tanh-Coth methods applied to find solutions of the one-dimensional Sinh-gordon equation. The general structure of the solutions are:

\[ \rho_i = 2 \tanh^{-1} (a_{1,i} \tanh (x - kt) + a_{2,i} \coth (x - kt)) \]  

(54)

We find 8 families of solutions for the field \( \rho \). The extension to two or three dimensions and to the Sin-Gordon equation is straightforward.

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References


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