Marshall-Olkin Extended Burr III Distribution

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Abstract

In this paper, a new extended Burr III distribution using Marshall and Olkin method is introduced. Some characteristics and properties of the new distribution are studied. Maximum likelihood and Bayesian methods are used to estimate model parameters. Also, the asymptotic distributions of the maximum likelihood is used to obtain confidence intervals for the parameters. Monte Carlo simulations are carried out to investigate the performance of the new distribution. Finally, a real data set is analyzed to illustrate the results.

Keywords: Marshall-Olkin extended Burr III distribution, Maximum likelihood method, Bayesian method

1 Introduction

Marshall and Olkin [14] proposed a family of survival functions by adding a parameter to obtain a family of distributions. The new resulting distributions are more flexible than the original distributions.
The method starts with the survival function (SF) of any selected distribution, \( \hat{F}(x) = 1 - F(x) \), where, \( F(x) \) is the cumulative distribution function (CDF). That is, the SF of the resulting distribution can be written as

\[
G(x; \alpha) = \frac{\alpha \hat{F}(x)}{1 - (1 - \alpha) \hat{F}(x)} ; -\infty < x < \infty, 0 < \alpha < \infty
\] (1)

Here, \( \alpha \) is the new added parameter and if \( \alpha = 1 \), \( G = F \). The probability density function (PDF) and the CDF for the new distribution are, respectively, given by

\[
g(x; \alpha) = \frac{\alpha f(x)}{(1 - \alpha \hat{F}(x))^2} \] (2)

\[
G(x; \alpha) = \frac{F(x)}{1 - \alpha \hat{F}(x)} = 1 - \frac{\alpha \hat{F}(x)}{1 - \alpha \hat{F}(x)} \] (3)

Burr [3] introduced a family of twelve distributions. The Burr III distribution include a widespread region in the skewness and kurtosis plane. This covers many well-known families of distributions such as the gamma, Weibull, and lognormal families [20]. Lindsay et al. [13] compared between the Burr III distribution with four parameters and the Weibull distribution using the diameter data. Shao [17] considered the constrained maximum likelihood method to estimate the Burr III parameters using toxicity data. Shao [18] suggested an extension of the 3-parameters Burr III distribution using re-parameterization method in low-flow frequency analysis. They used the moments, probability-weighted moments, and maximum likelihood methods to estimate the distribution parameters. Wang and Lee [21] estimated the parameters of the extended three-parameter Burr III using M-estimators for complete data with outliers and compared it with ML and least squares methods. Moreover, Wang and Lee [22] used M-estimators and ML and AM-estimators when the data is asymmetric to estimate the parameters of Burr III for complete data with outliers and compared them. Several studies have been conducted to propose several distribution extensions and discussed their properties and inference. These includes the Marshall-Olkin extended (MOE) Weibull proposed by Ghitany et al. [9], MOE Pareto proposed by Ghitany [7], MOE gamma proposed by Ristic et al. [15], MOE Lomax using censored data proposed by Ghitany et al. [8], MOE exponential distribution proposed by Srivastava et al. [19], MOE Uniform distribution proposed by Jose and Krishna [12], both MOE power log-normal and MOE log-logistic distributions proposed by Gui [10], [11], and MOE Burr type XII distribution proposed by Al-Saiari et al. [1]. In addition, Santos-Neto et al. [16] introduced the MOE Weibull family of distributions and studied some of its mathematical properties. Barreto-Souza et al. [2] and Cordeiro and
Lemonte [5] provided general results of the Marshall and Olkin’s family of distributions. This article introduces the Marshall-Olkin extended Burr III (MOEBIII) distribution and study its performance and flexibility using a simulation study and real data. The rest of this paper is organized as follows: In Section 2, the Marshall-Olkin Extended Burr III distribution is introduced with a study of some of its properties. Sections 3 discusses maximum likelihood estimation method for the unknown parameters and their asymptotic properties. Bayesian estimation of the MOEBIII parameters. A simulation study is conducted in Section 5. The introduced distribution is fitted to a real data set to show its performance in Section 6. Finally, concluding remarks are given in Section 7.

2 Marshall-Olkin extended Burr III

If X is a random variable from MOEBIII distribution, then the SF of MOEBIII is given by

\[ G(x; \alpha, c, k) = \frac{\alpha[1-(1+x^{-c})^{-k}]}{[1-(1-\alpha)[1-(1+x^{-c})^{-k}]]} \]

The corresponding CDF and PDF for the MOEBIII distribution are obtained from (2) and (3) as:

\[ G(x; \alpha, c, k) = \frac{(1+x^{-c})^{-k}}{[1-(1-\alpha)[1-(1+x^{-c})^{-k}]]}, \quad x, \alpha, c, k > 0. \]

\[ g(x; \alpha, c, k) = \frac{\alpha k c x^{-(c+1)(1+x^{-c})^{-(k+1)}}}{[1-(1-\alpha)[1-(1+x^{-c})^{-k}]]^2} \]

It is clear that if \( \alpha=1 \), we obtain the original distribution, Burr III, with two parameters.

The PDF of the MOEBIII is plotted in Figure 1, which displays the new parameter \( \alpha \) with different values of \( c \) and \( k \).
The hazard rate of the MOEBIII \( r(x; \alpha, c, k) \) and the hazard rate of the original distribution \( h(x) \) have the following relation

\[
    r(x; \alpha, c, k) = \frac{h(x)}{1 - a F(x)} = \frac{kc x^{-(c+1)(1+x^{-c})^{-1}}} {1 - \alpha [1 - (1 + x^{-c})^{-k}][1 - (1 + x^{-c})^{-k}]}, \tag{7}
\]

From (7), it can be verified that for different values of \( \alpha \):

i. For \( c > \frac{1}{k} \), \( r(x; \alpha, c, k) \) is an inverted U-shaped function of time (unimodal).

ii. For \( c \leq \frac{1}{k} \), \( r(x; \alpha, c, k) \) is L-shaped (see Figure 2).

The relationship between the \( r(x; \alpha, c, k) \) and \( \bar{G}(x; \alpha, c, k) \) of the extended distributions with the original distributions depends on the new parameter \( \alpha > 1 \) or \( 0 < \alpha < 1 \), see [6], [14].

That is, for \( 0 < \alpha < 1 \), it can be shown that for \( x > 0 \)

\[
    \frac{k cx^{-(c+1)(1+x^{-c})^{-1}}} {1 - (1 + x^{-c})^{-k}} \leq r(x; \alpha, c, k) \leq \frac{k cx^{-(c+1)(1+x^{-c})^{-1}}} {\alpha [1 - (1 + x^{-c})^{-k}]}, \tag{8}
\]

\[
    [1 - (1 + x^{-c})^{-k}]^{1/\alpha} \leq \bar{G}(x; \alpha, c, k) \leq 1 - (1 + x^{-c})^{-k}. \tag{9}
\]

For \( \alpha > 1 \),

\[
    \frac{k cx^{-(c+1)(1+x^{-c})^{-1}}} {\alpha [1 - (1 + x^{-c})^{-k}]} \leq r(x; \alpha, c, k) \leq \frac{k cx^{-(c+1)(1+x^{-c})^{-1}}} {1 - (1 + x^{-c})^{-k}}, \tag{10}
\]
1 - (1 + x^{-c})^{-k} \leq \tilde{G}(x; \alpha, c, k) \leq [1 - (1 + x^{-c})^{-k}]^{1/\alpha}. \quad (11)

Figure 2: The hazard rate of the MOEBIII: (I) \( \alpha = 0.8, k=0.1, c=2; \) (II) \( \alpha = 3, k=0.1, c=2; \) (III) \( \alpha = 0.8, k = 5, c = 2; \) (IV) \( \alpha = 3, k = 5, c = 2 \)

The quantile of the MOEBIII

\[ x_q = \left[ \left( \frac{1 - q}{\alpha q} \right) + 1 \right]^{1/k} - 1 \] , \quad 0 \leq q \leq 1. \quad (12)

The median of the MOEBIII

\[ \text{median}(x) = \left[ 1 + \frac{1}{\alpha} \right]^{1/k} - 1. \quad (13) \]

The mode of the MOEBIII is obtained by solving \( \frac{\partial \log g(x; \alpha, c, k)}{\partial x} = 0 \)

\[
\frac{d}{dx} \log g(x; \alpha, c, k) = \left( c + 1 \right) \frac{-1}{x} + c(k + 1) \frac{x^{-(c+1)}}{(1+x^{-c})}, \\
-2(1 - \alpha) kc \frac{(1+x^{-c})^{-k+1}x^{-(c+1)}}{[1-(1-\alpha)(1-(1+x^{-c})-k)]} = 0. \quad (14)
\]

For this selected parameters values \( k=1.5, \alpha=0.8, c=2, \) the mode is 1.7745 as shown in Figure 1 (III), which indicates that the PDF is unimodal for the selected parameter values.

3 Maximum likelihood estimation

Suppose that \( X = (X_1, X_2, ..., X_n) \) is a random sample of size \( n \) from MOEBIII distribution, then the log-likelihood function is given by
\[ l(\alpha, c, k) = n(\log \alpha + \log c + \log k) - (c + 1) \sum_{i=1}^{n} \log(x_i) - (k + 1) \sum_{i=1}^{n} \log(\varphi(x_i; c)) - 2 \sum_{i=1}^{n} \log(\omega(x_i; \alpha, c, k)), i = 1, \ldots, n \quad (15) \]

where
\[ \varphi(x_i; c) = (1 + x_i^{-c}) \text{ and } \omega(x_i; \alpha, c, k) = 1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}] \] . \quad (16)

To obtain ML estimates of the parameters, equation (15) is maximized by equating the partial derivative with respect to \( \alpha, c, \) and \( k \) to 0. Thus, the following three nonlinear equations should be solved iteratively
\[
\frac{\partial l(\hat{\alpha}, \hat{c}, \hat{k})}{\partial \hat{\alpha}} = \frac{n}{\hat{\alpha}} - 2 \sum_{i=1}^{n} \frac{[\varphi_2(x_i; \hat{c})]^{-k}}{\omega_2(x_i; \hat{\alpha}, \hat{c}, \hat{k})} = 0. \quad (17)
\]
\[
\frac{\partial l(\hat{\alpha}, \hat{c}, \hat{k})}{\partial \hat{c}} = \frac{n}{\hat{c}} - \sum_{i=1}^{n} \log(x_i) + (\hat{k} + 1) \sum_{i=1}^{n} x_i^{-\hat{c}} \log(x_i) \frac{x_i^{-\hat{c}} \log(x_i)}{\omega_2(x_i; \hat{\alpha}, \hat{c}, \hat{k})} = 0. \quad (18)
\]
\[
\frac{\partial l(\hat{\alpha}, \hat{c}, \hat{k})}{\partial \hat{k}} = \frac{n}{\hat{k}} - \sum_{i=1}^{n} \log(\varphi_2(x_i; \hat{c})) + 2(1 - \hat{\alpha}) \sum_{i=1}^{n} \frac{[\varphi_2(x_i; \hat{c})]^{-k} \log(\varphi_2(x_i; \hat{c})^{-k+1})}{\omega_2(x_i; \hat{\alpha}, \hat{c}, \hat{k})} = 0. \quad (19)
\]

It is clear that a numerical method is needed to obtain the solution for the nonlinear equations (17-19). This is can be solved using Newton-Raphson method. The invariance property of ML estimation can be used to obtain the ML estimators for \( \hat{\theta}(.) \) and \( r(.) \).

Moreover, approximate confidence interval of the parameters is obtained using the asymptotic distribution of the ML estimates of \( \hat{\theta} \). The distribution of the ML estimate for large sample under some regularity conditions have approximately multivariate normal distribution with mean \( \theta = (\alpha, c, k) \) and asymptotic variance-covariance matrix equivalent to the inverse of Fisher information matrix given in the Appendix. Therefore, the \( 100 \left( 1 - \frac{y}{2} \right) \% \) approximate confidence interval of the parameters \( \theta = (\alpha, c, k) \) are given by
\[ \hat{\alpha} \pm Z_{\gamma/2} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{c} \pm Z_{\gamma/2} \sqrt{\text{var}(\hat{c})}, \quad \hat{k} \pm Z_{\gamma/2} \sqrt{\text{var}(\hat{k})}. \quad (20) \]

The likelihood ratio test (LRT) is conducted to test if the MOEBIII model is statistically better than the original Burr III distribution, where under \( H_0 \), the LRT statistic \( \Lambda = -2[L(1, \hat{c}, \hat{k}) - L(\hat{\alpha}, \hat{c}, \hat{k})] \) has approximately a chi-square distribution with 1 degree of freedom.
4 Bayesian estimation

Suppose we assume that the unknown vector of parameters \( \theta = (\alpha, c, k) \) have the following independent prior distributions
\[
\alpha \sim \text{uniform}(a_1, b_1), c \sim \text{uniform}(a_2, b_2), k \sim \text{uniform}(a_3, b_3).
\]
Then, the joint posterior probability can be written as:
\[
\pi^*(\alpha, c, k|x) = B\alpha^n c^n k^n \frac{\prod_{i=1}^{n} (x_i)^{-c(1+c)}}{\prod_{i=1}^{n} \omega(x_i; \alpha, c, k)^2} \times \pi(\alpha, c, k), \tag{21}
\]
where \( \pi(\alpha, c, k) \) is the joint prior distribution of the parameters, \( \varphi(x_i; c) \) and \( \omega(x_i; \alpha, c, k) \) are defined in (16), and \( B \) is the normalizing constant.
Bayesian and credible interval estimates of the parameters are obtained numerically using Markov chain Monte Carlo (MCMC) techniques. That is, samples are simulated for the joint posterior distribution in (21) using the Metropolis-Hasting algorithm to obtained the mean posterior estimates of the parameters \( \theta \) by MCMC, see [4].

5 Simulation study

Simulation study has been performed to investigate the performance of the ML and Bayesian estimates. This simulation was conducted for different sample sizes and different parameter values to study the effect of the new parameter \( \alpha \) in two cases, the case of \( \alpha < 1 \) and the case of \( \alpha > 1 \). Table 1 reports the ML and Bayesian estimates of the unknown parameters along with the relative mean square error (RMSE). It is clear from the results that the RMSE for the Bayesian estimates of \( \alpha, c, \) and \( k \) are smaller than their corresponding RMSE of ML estimates. Therefore, it can be concluded that the Bayesian method based on non-informative priors has provided better estimates of the parameters compared to the ML method.

<table>
<thead>
<tr>
<th>n</th>
<th>Parameters</th>
<th>ML</th>
<th></th>
<th>Bayesian</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha )</td>
<td>( k )</td>
<td>( c )</td>
<td>( \hat{\alpha} )</td>
</tr>
<tr>
<td>50</td>
<td>0.8</td>
<td>1.5</td>
<td>2</td>
<td>1.6222 (0.2344)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8</td>
<td></td>
<td>0.5690 (0.1615)</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>2</td>
<td></td>
<td>1.7883 (0.2364)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8</td>
<td></td>
<td>0.6495 (0.2145)</td>
</tr>
</tbody>
</table>
Table 1: (Continued): ML and Bayesian estimates with their (RMSE) under MOEBIII distribution

<table>
<thead>
<tr>
<th></th>
<th>1.5</th>
<th>2</th>
<th>1.6070 (0.1798)</th>
<th>0.8831 (0.2929)</th>
<th>2.0062 (0.0461)</th>
<th>1.5123 (0.0048)</th>
<th>0.7827 (0.0029)</th>
<th>2.0084 (0.0086)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.5</td>
<td>0.8</td>
<td>0.5340 (0.0678)</td>
<td>0.8311 (0.1475)</td>
<td>0.8129 (0.0217)</td>
<td>0.5409 (0.0107)</td>
<td>0.7485 (0.0073)</td>
<td>0.7894 (0.0064)</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>2</td>
<td>1.7125 (0.1712)</td>
<td>2.8072 (0.4077)</td>
<td>1.9714 (0.0196)</td>
<td>1.4735 (0.0084)</td>
<td>3.0352 (0.0022)</td>
<td>2.0141 (0.0075)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8</td>
<td>0.6013 (0.1256)</td>
<td>2.7657 (0.3762)</td>
<td>0.7884 (0.0097)</td>
<td>0.4964 (0.0076)</td>
<td>3.1255 (0.0241)</td>
<td>0.8124 (0.0046)</td>
</tr>
</tbody>
</table>

6 Application to real data

The data set consists of 36 observations on chromium in marine water data. This data are the result of the study of the influence of the proportion of toxic in chromium in marine waters. The study is conducted by specialists in biology to preserve the environment. For more details see [17].

The validity of the fitted model is checked by applying the Kolmogorov-Smirnov (K-S) statistics on the fitted CDF and the empirical CDF. This is conducted by replacing the parameters with their ML and Bayesian estimates as seen in Figure 3. That is, the K-S test statistics is $D=0.1985$ using ML estimates with the corresponding $p$-value = 0.1171, and the K-S test statistics is $D=0.139$ using Bayesian estimates with the corresponding $p$-value = 0.4903. The K-S test values indicate that the MOEBIII distribution provides best fit for this data set.

![Figure 3: The plot for the fitted and empirical CDF of the MOEBIII distribution using ML and Bayesian estimates](image)

Model fitting

The flexibility of the MOEBIII distribution is measured by comparing the fitted model with the original distribution Burr III by calculating the Akaike information criterion (AIC) and Bayesian information criterion (BIC) as presented in Table 2. It can be seen that the Burr III distribution has higher AIC and BIC values compared to the MOEBIII distribution, which confirms that MOEBIII is more suitable for this
data set. In addition, the LRT statistics and its corresponding p-value are Λ=21.4237, LRT < 0.05, which rejects the assumption that the Burr III distribution is better compared to the MOEBIII distribution. Therefore, we can conclude that the MOEBIII distribution is more suitable for the given data.

Table 2: ML, Bayesian estimates with their confidence and credible intervals along with the corresponding AIC, BIC of the Burr III and MOEB III distributions

<table>
<thead>
<tr>
<th>Model</th>
<th>ML Estimate</th>
<th>95% Confidence Interval</th>
<th>AIC</th>
<th>BIC</th>
<th>Bayesian Estimate</th>
<th>95% Credible Interval</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Burr III</td>
<td>k</td>
<td>9.7</td>
<td>(5.33,14.09)</td>
<td>586.6</td>
<td>583.4</td>
<td>8.95</td>
<td>(7.87, 9.85)</td>
<td>594.9</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>0.45</td>
<td>(0.36,0.54)</td>
<td></td>
<td></td>
<td>0.45</td>
<td>(0.40,0.52)</td>
<td></td>
</tr>
<tr>
<td>MOEB III</td>
<td>k</td>
<td>8.4</td>
<td>(5.66,11.14)</td>
<td>438.9</td>
<td>434.2</td>
<td>8.1</td>
<td>(7.81, 8.39)</td>
<td>456.8</td>
</tr>
<tr>
<td></td>
<td>a</td>
<td>5.1</td>
<td>(3.43,6.77)</td>
<td></td>
<td></td>
<td>5.1</td>
<td>(4.82, 5.39)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>0.6</td>
<td>(0.40,0.795)</td>
<td></td>
<td></td>
<td>0.65</td>
<td>(0.58, 0.74)</td>
<td></td>
</tr>
</tbody>
</table>

7 Concluding remarks

In this paper, Marshall-Olkin extended Burr III distribution is proposed. Some of the statistical properties of the MOEBIII distribution are studied. ML and Bayesian methods are used to estimate the parameters of the proposed distribution. A simulation study real data set are conducted and the results show that the MOEBIII distribution is flexible and provides a suitable fit for the data.

Appendix

The asymptotic variance-covariance matrix $I^{-1}(\theta)$ is given by

$$I^{-1}(\theta) = \begin{pmatrix}
-\frac{\partial^2 l(\theta)}{\partial \alpha^2} & -\frac{\partial^2 l(\theta)}{\partial \alpha \partial c} & -\frac{\partial^2 l(\theta)}{\partial \alpha \partial k} \\
-\frac{\partial^2 l(\theta)}{\partial c \partial \alpha} & -\frac{\partial^2 l(\theta)}{\partial c^2} & -\frac{\partial^2 l(\theta)}{\partial c \partial k} \\
-\frac{\partial^2 l(\theta)}{\partial k \partial c} & -\frac{\partial^2 l(\theta)}{\partial k \partial c} & -\frac{\partial^2 l(\theta)}{\partial k^2}
\end{pmatrix},$$

where $I(\theta)$ is the asymptotic Fisher information matrix, and the second partial derivatives are
\[
\frac{\partial^2 l}{\partial \alpha^2} = \frac{-n}{\alpha^2} + 2 \sum_{i=1}^{n} \frac{[1 - (1 + x_i^{-c})^{-k}]^2}{[1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}]]^2}
\]

\[
\frac{\partial^2 l}{\partial \alpha \partial c} = 2k \sum_{i=1}^{n} \frac{x_i^{-c} \log(x_i) (1 + x_i^{-c})^{-(k+1)}}{[1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}]]^2}
\]

\[
\frac{\partial^2 l}{\partial \alpha \partial k} = -2 \sum_{i=1}^{n} \frac{(1 + x_i^{-c})^{-k} \log(1 + x_i^{-c})}{[1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}]]^2}
\]

\[
\frac{\partial^2 l}{\partial k^2} = \frac{-n}{k^2} - 2\alpha(1 - \alpha) \sum_{i=1}^{n} \frac{(1 + x_i^{-c})^{-k}(\log(1 + x_i^{-c}))^2}{[1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}]]^2}
\]

\[
\frac{\partial^2 l}{\partial c \partial k} = \sum_{i=1}^{n} \frac{x_i^{-c} \log(x_i)}{(1 + x_i^{-c})} - 2(1 - \alpha) \sum_{i=1}^{n} \frac{x_i^{-c} \log(x_i) (1 + x_i^{-c})^{-(k+1)}}{[1 - (1 - \alpha)[1 - (1 + x_i^{-c})^{-k}]]^2}
\]

\[
+ 2\alpha(1 - \alpha)k \sum_{i=1}^{n} \frac{x_i^{-c(1+x_i^{-c})^{-(k+1)}} \log(x_i) \log(1+x_i^{-c})}{[1-(1-\alpha)[1+(1+x_i^{-c})^{-k}]]^2}
\]

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