A Class of Complete Lie Algebras II

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Abstract

In this paper, by using a matrix method, we explicitly determine the maximal torus of a class of quasifiliform Lie algebras $C_n$ and prove that this class of Lie algebras are complete and has a left-symmetric algebra structure.

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1 Introduction

A Lie algebra is called complete if its center is zero, and all its derivations are inner. The definition of complete Lie algebras was given by N. Jacobson in 1962 [2]. The first important result of complete Lie algebras first appeared in 1951, in the context of Schenkman’s theory of subinvariant Lie algebras [10]. In recent years, different authors have concentrated on classifications and structural properties of complete Lie algebras [3, 6, 7, 12]. The complete nilpotent Lie algebra is one of the interesting results have been obtained [12]. But which nilpotent Lie algebra is complete is still an open problem.

Left-symmetric algebras first have been studied in theory of affine manifolds, affine structures on Lie groups and convex homogeneous cones [4],[5],[11]. It is well-known that under the commutator $[x, y] = xy - yx$, A left-symmetric

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algebra becomes a Lie algebra. But not all Lie algebras have left-symmetric algebra structures.

In [1], J. R. Gómez, A. Jiménez-Merchán and J. Reyes have given the classification of quasi-filiform Lie algebras of maximal length. In this paper, we explicitly determine the maximal torus of a class of quasifiliform Lie algebras $C_n$ and prove that they are complete and have left-symmetric algebra structures.

2 Preliminary

**Definition 2.1** Let $N$ be a Lie algebra. A maximal torus on $N$ is a maximal abelian subalgebra of $\text{Der} N$, which consists of semisimple linear transformations.

**Lemma 2.2** [9] Let $H_1$ and $H_2$ be two maximal torus on $N$, then there exist $\theta \in \text{Aut} N$, such that $H_2 = \theta H_1 \theta^{-1}$.

As all maximal tori on $N$ are mutually conjugated, so the dimension of a maximal torus on $N$ is an invariant of $N$ called the rank of $N$ (denoted by $\text{rank}(N)$). A nilpotent Lie algebra is called maximal rank nilpotent Lie algebra if $\text{rank}(N)=\text{dim} N/[[N,N]]$.

If $H$ is a maximal torus on a nilpotent Lie algebra $N$, define the bracket in $H^+N$, by $[h_1 + n_1, h_2 + n_2] = h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in H, n_i \in N, i = 1, 2$, then $H^+N$ is a solvable Lie algebra.

**Definition 2.3** [12] Let $H$ be a maximal torus on a nilpotent Lie algebra $N$, if a solvable Lie algebra $H^+N$ is complete, then $N$ is called a complete nilpotent Lie algebra.

**Lemma 2.4** [8] Let $H$ be a maximal torus on a nilpotent Lie algebra $N$ and the decomposition of $N$ with respect to $H$ be

$$N = \sum_{\alpha \in \Delta} N_\alpha,$$

where $\Delta = \{ \alpha \in H^* | N_\alpha \neq 0 \}$. Defined the bracket in $g = H^+N$, by $[h_1 + n_1, h_2 + n_2] = h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in H, n_i \in N, i = 1, 2$, then $H^+N$ is a solvable Lie algebra. If $0 \notin \Delta$ and $C(g) = 0$, then

$$\text{Der} g = \text{ad} g + D_0$$

where $D_0 = \{ \phi \in \text{Der} g | \phi(H) = 0 \}$. 
Lemma 2.5 [1] Let \( g \) be an \( n \)-dimensional non-split quasifiliform Lie algebra of maximal length \( l(g) = n \geq 13 \). Then the algebra \( g \) is isomorphic to \( A_n \) (\( n \) odd), or to \( B_n \) or to \( C_n \), where \( A_n, B_n, C_n \) are defined in a basis \( X = \{x_0, x_1, \cdots, x_{n-2}, y\} \) as follows:

\[
\begin{align*}
A_n : & \quad \begin{cases}
[x_0, x_i] = x_{i+1}, & 1 \leq i \leq n-3, \\
[x_i, x_{n-2-i}] = (-1)^{i-1}y, & 1 \leq i \leq \frac{n-3}{2}.
\end{cases} \\
B_n : & \quad \begin{cases}
[x_0, x_i] = x_{i+1}, & 1 \leq i \leq n-3, \\
[x_i, y] = x_{i+2}, & 1 \leq i \leq n-4.
\end{cases} \\
C_n : & \quad \begin{cases}
[x_0, x_i] = x_{i+1}, & 1 \leq i \leq n-3, \\
[x_i, y] = x_{i+2}, & 1 \leq i \leq n-4, \\
[x_1, x_i] = x_{i+3}, & 2 \leq i \leq n-5.
\end{cases}
\end{align*}
\]

3 Main Results

Let \( \sigma \) be a linear transformation of a Lie algebra \( g \), then \( \sigma \in \text{Der}(g) \) if and only if with respect to a basis \( \{x_1, x_2, \cdots, x_n\} \) of \( g \), \( \sigma \) satisfies for any \( 1 \leq i, j \leq n \),

\[ \sigma[x_i, x_j] = [\sigma x_i, x_j] + [x_i, \sigma x_j]. \]

Lemma 3.1 Let \( \sigma \) be a linear transformation of a Lie algebra \( g \) and \( A = \text{diag}(a_1, a_2, \cdots, a_n) \) be the matrix of \( \sigma \) with respect to a basis \( \{x_1, x_2, \cdots, x_n\} \) of \( g \). Suppose that

\[ [x_i, x_j] = \sum_{k=1}^{n} b^k_{ij}x_k. \]

Then \( \sigma \in \text{Der}(g) \) if and only if for any \( 1 \leq i, j \leq n \),

\[ (a_i + a_j)b^k_{ij} = a_kb^k_{ij}, \quad 1 \leq k \leq n. \]

Lemma 3.2 Let \( C_n \) be the \( n \)-dimensional non-split quasifiliform Lie algebra of maximal length \( l(g) = n \geq 13 \) defined in Lemma 2.5. Suppose that \( \phi \) is a semisimple linear transformation of \( C_n \) with respect to the given basis \( X \). Then \( \phi \) is a derivation of \( C_n \) if and only if \( \phi \) satisfies

\[
\begin{align*}
\phi(x_0) &= a_0x_0, \quad \phi(x_1) = 3a_0x_1, \quad \phi(x_2) = 4a_0x_2, \quad \phi(x_3) = 5a_0x_3, \cdots, \\
\phi(x_{n-2}) &= na_0x_{n-2}, \quad \phi(y) = 2a_0y,
\end{align*}
\]

where \( a_0 \in \mathbb{F} \).
Proof. Let \( \phi \) be a semisimple linear transformation of \( C_n \), we may assume that
\[
\phi(x_0) = a_0x_0, \phi(x_1) = a_1x_1, \ldots, \phi(x_{n-2}) = a_{n-2}x_{n-2}, \phi(y) = by.
\]
Then by lemma 3.1, we have that \( \phi \in \text{Der}(C_n) \) if and only if
\[
\begin{cases}
a_0 + a_i = a_{i+1}, & 1 \leq i \leq n - 3; \\a_i + b = a_{i+2}, & 1 \leq i \leq n - 4; \\a_1 + a_i = a_{i+3}, & 2 \leq i \leq n - 5.
\end{cases}
\]
Solving the system of linear equations, we have \( \phi \in \text{Der}(C_n) \) if and only if
\[
\begin{cases}
a_1 = 3a_0, \\
a_2 = 4a_0, \\
\vdots \\
a_{n-2} = na_0, \\
b = 2a_0.
\end{cases}
\]
Hence \( \phi \in \text{Der}(C_n) \) if and only if
\[
\phi(x_0) = a_0x_0, \phi(x_1) = 3a_0x_1, \phi(x_2) = 4a_0x_2, \phi(x_3) = 5a_0x_3, \ldots, \\
\phi(x_{n-2}) = na_0x_{n-2}, \phi(y) = 2a_0y,
\]
Hence the conclusion holds. \( \square \)

Theorem 3.3 Let \( C_n \) be the \( n \)-dimensional non-split quasifiliform Lie algebra of maximal length \( l(g) = n \geq 13 \) defined in Lemma 2.5. Then the linear transformation
\[
\phi(x_0) = x_0, \phi(x_1) = 3x_1, \phi(x_2) = 4x_2, \phi(x_3) = 5x_3, \ldots, \\
\phi(x_{n-2}) = nx_{n-2}, \phi(y) = 2y
\]
generates a maximal torus on \( C_n \).

Proof. By Lemma 3.2, we know that the transformation
\[
\phi(x_0) = x_0, \phi(x_1) = 3x_1, \phi(x_2) = 4x_2, \phi(x_3) = 5x_3, \ldots, \\
\phi(x_{n-2}) = nx_{n-2}, \phi(y) = 2y
\]
is a derivation of \( C_n \). Let \( H \) be a maximal torus on \( C_n \) such that \( \phi \in H \). \( \forall h \in H \) and suppose the matrix of \( h \) with respect to the given basis is \( M_h = (h_{ij})_{n \times n} \). Since any maximal torus is abelian, we have
\[
[\phi, h] = 0,
\]
which means \([M_h, \text{diag}(1, 3, 4, \cdots, n, 2)] = 0\). Since the diagonal entries of the matrix \(\text{diag}(1, 3, 4, \cdots, n, 2)\) are different to each others, we have \(M_h\) is a diagonal matrix. Similarly as the proof of Lemma 3.2, we have \(h \in \text{Der}(C_n)\) if and only if
\[
\begin{align*}
    h(x_0) &= a_0 x_0, h(x_1) = 3a_0 x_1, h(x_2) = 4a_0 x_2, h(x_3) = 5a_0 x_3, \cdots, \\
    h(x_{n-2}) &= na_0 x_{n-2}, h(y) = 2a_0 y,
\end{align*}
\]
where \(a_0 \in \mathbb{F}\). Therefore, \(h = a_0 \phi\). Hence \(H\) is generated by \(\phi\). \(\square\)

**Theorem 3.4** Let \(C_n\) be the \(n\)-dimensional non-split quasifiliform Lie algebra of maximal length \(l(g) = n \geq 13\) defined in Lemma 2.5. Then there exists a left-symmetric algebra structure on \(C_n\).

**Proof.** From the proof of Lemma 3.2, we know that the transformation
\[
\phi(x_0) = x_0, \phi(x_1) = 3x_1, \cdots, \phi(x_{n-2}) = nx_{n-2}, \phi(y) = 2y
\]
is a derivation of \(C_n\) and obviously nonsingular. It is well-known that a Lie algebra with a nonsingular transformation \(D\) has a left-symmetric algebra structure \(xy = D^{-1}[x, Dy]\). Hence the conclusion holds. \(\square\)

**Theorem 3.5** Let \(C_n\) be the \(n\)-dimensional non-split quasifiliform Lie algebra of maximal length \(l(g) = n \geq 13\) defined in Lemma 2.5. Then \(C_n\) is a complete nilpotent Lie algebra.

**Proof.** From Theorem 3.3, there exists a maximal torus \(H\) such that the transformation
\[
\phi(x_0) = x_0, \phi(x_1) = 3x_1, \phi(x_2) = 4x_1, \cdots, \phi(x_{n-2}) = nx_{n-2}, \phi(y) = 2y
\]
is in \(H\). So there isn’t zero root space in the decomposition of
\[
C_n = \sum_{\alpha \in \Delta \subseteq H^*} (C_n)_{\alpha}
\]
with respect to \(H\), and \(C(H + C_n) \neq 0\). Hence by Lemma 2.4, \(\text{Der}g = \text{ad}g + D_0\), where \(g = H + C_n\).
\[
\forall \varphi \in D_0, \text{for any } 0 \neq x_\beta \in (C_n)_\beta, \text{ set }
\]
\[
\varphi(x_\beta) = h_1 + \sum_{\alpha \in \Delta} x_\alpha,
\]
where \(h_1 \in H, x_\alpha \in (C_n)_\alpha\). \(\forall h \in H, \) from
\[
\varphi[h, x_\beta] = [\varphi(h), x_\beta] + [h, \varphi(x_\beta)] = [h, \varphi(x_\beta)],
\]
we have
\[
\beta(h)h_1 + \sum_{\alpha \in \Delta} \beta(h)x_\alpha = \left[ h, h_1 + \sum_{\alpha \in \Delta} x_\alpha \right] = \sum_{\alpha \in \Delta} \alpha(h)x_\alpha.
\]

By \(0 \notin \Delta, \beta \neq 0\), we have \(h_1 = 0\) and if \(\beta \neq \alpha, x_\alpha = 0\). Thus \(\varphi((C_n)_\beta) \subseteq (C_n)_\beta\).

From \(\phi \in H\), we obtain that \(\dim(C_n)_\beta = 1\). So \(\varphi(x_\beta) = k_i x_\beta, k_i \in \mathbb{C}\). It means that \(\varphi|_{C_n}\) is a semisimple derivation of \(C_n\). So \(C \varphi|_{C_n}\) is a torus on \(C_n\) and \([\varphi|_{C_n}, H]\) = 0. Because \(H\) is a maximal torus on \(C_n\), we have \(\varphi|_{C_n} \in H\). It means that there exists an \(h' \in H\) such that \(\varphi|_{C_n} = \text{ad} h'\). Noting that \(\varphi(H) = 0\), we immediately have
\[
\varphi = \text{ad} h'.
\]

Hence \(D_0 \subseteq \text{ad} g\) and \(\text{Der} g = \text{ad} g\). Therefore, \(g = H + C_n\) is a complete Lie algebra and \(C_n\) is a complete nilpotent Lie algebra. \(\square\)

References


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