Geometric Proofs of the Complementary Chords Theorems

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Abstract

The Common Tones Theorem for Transposition [14, p. 108] is used to provide a simple geometric proof of the Complementary Chords Theorem [12, p. 74] of Mathematical Music Theory. Then, the Common Tones Theorem for Inversion [17, pp. 83-85] is used to prove a new Complementary Chords Theorem for Inversion.

1 Introduction

A geometric approach to Mathematical Music Theory was sketched in [5] and then subsequently applied to some salient features of the octatonic [6] and diatonic [7, 8] scales. The present paper adapts this approach in order to show that the Complementary Chords Theorem for Transposition [12, p. 74] is an immediate consequence of the Common Tones Theorem for Transposition [14, p. 108]. This observation yields the simplest available proof of this foundational result of Mathematical Music Theory.

After stating and proving the Complementary Chords Theorem for Transposition we state its immediate corollary, the Hexachord Theorem for Transposition. Then, the Common Tones Theorem for Transposition, the Complementary Chords Theorem for Transposition and the Hexachord Theorem for Transposition are generalized to $n$-tone equal temperament. Subsequently, the Patterson Theorems of Discrete Geometry are reviewed and related to these three generalized theorems.
Then, after stating and proving the new Complementary Chords Theorem for Inversion we state its immediate corollary, the new Hexachord Theorem for Inversion. Finally, the Common Tones Theorem for Inversion, the Complementary Chords Theorem for Inversion and the Hexachord Theorem for Inversion are generalized to \(n\)-tone equal temperament.

The following elementary (and geometrically self-evident) result will be of particular utility in the subsequent development.

**Lemma 1** Let \(P\) be an arbitrary chord and \(\overline{P}\) its complement.

1. \(\overline{T_kP} = T_k\overline{P}\)
2. \(\overline{T_kIP} = T_kI\overline{P}\)

**Proof:** Denote the aggregate by \(U\).

1. \(T_kP \cup \overline{T_kP} = U \land T_kP \cap \overline{T_kP} = \emptyset\)
   \(\therefore \overline{T_kP} = T_k\overline{P}\)
2. \(T_kIP \cup \overline{T_kIP} = U \land T_kIP \cap \overline{T_kIP} = \emptyset\)
   \(\therefore \overline{T_kIP} = T_kI\overline{P}\)

The interested reader is referred to my *Prelude to Musical Geometry* [5] for all mathemusical terminology. In addition, the following results proven geometrically in [9] will be indispensable below.

**Theorem 1 (Common Tones Theorem for Transposition)** If the interval vector for a chord \(P\) is \(\langle n_1, n_2, \ldots, n_6 \rangle\), then, for \(k = 1, 2, \ldots, 5\), the number of pitches in \(P\) that are invariant under transposition by \(k\) semitones up or down (tones common to the chords \(P\) and \(T_\pm kP\)) is \(n_k\). However, the number of pitches that are invariant under transposition by a tritone \((k = 6)\) is \(2n_6\).

**Theorem 2 (Common Tones Theorem for Inversion)** If the index vector for a chord \(P\) is \(\langle n_0, n_1, \ldots, n_{11} \rangle\), then, for \(k = 0, 1, \ldots, 11\), the number of pitches in \(P\) that are invariant under inversion at index \(k\) (tones common to the chords \(P\) and \(T_kIP\)) is \(n_k\).

**Corollary 1** The index vector of \(T_kP\) is the cyclic permutation by \(2k\) places to the right of the index vector of \(P\).

**Corollary 2** The index vector of \(T_kIP\) is the cyclic permutation by \(2k + 1\) places to the right of the retrograde of the index vector of \(P\).
2 Complementary Chords Theorem for Transposition

Figure 1: Diatonic Scale

Figure 1 displays the diatonic collection $\mathcal{D}$ (comprised of the white keys on the piano) together with its axis of symmetry while Figure 2 displays its intervallic structure. The interval vector of $\mathcal{D}$ is easily computed to be $\langle 2, 5, 4, 3, 6, 1 \rangle$.

Figure 2: Intervallic Structure of Diatonic Scale
Figure 3 superimposes its complement $\overline{D}$, the pentatonic collection (comprised of the black keys on the piano), whose interval vector may likewise be computed to be $(0, 3, 2, 1, 4, 0)$.

Subtracting these two interval vectors yields $(2, 2, 2, 2, 2, 1)$. Denoting the cardinality of the original chord by $m = 7$, the cardinality of its complement is $12 - m = 5$ and these cardinalities differ by $d = m - (12 - m) = 2m - 12 = 2$. Observe that the difference of the interval vector of the original chord with its complement is given by $(d, d, d, d, d, d/2)$. Thus, the interval vector for $\overline{D}$ may be obtained from that for $D$ by simply subtracting $d$ from its first five components and $d/2$ from its last component. That this is not an accidental occurrence for the diatonic/pentatonic collections but rather applies to all pairs of chords and their complements is the gist of the Complementary Chords Theorem for Transposition.

This important theorem, as well as its many proofs, has a rich history which is recounted in [2] and [16]. The simplest proof currently appearing in the literature is due to Blau [3] and rests upon an analysis of the effect on the interval vector of interchanging two neighboring pitches between a chord and its complement. After precisely stating this theorem, the present paper details an even simpler proof employing the Common Tones Theorem for Transposition.
Theorem 3 (Complementary Chords Theorem for Transposition)

Let $\mathcal{P}$ be a pc-set of cardinality $m$ with interval vector $\langle n_1, n_2, \ldots, n_6 \rangle$, then $\overline{\mathcal{P}}$ has cardinality $12 - m$ and interval vector $\langle n_1 - d, n_2 - d, \ldots, n_6 - d/2 \rangle$, where $d = 2m - 12$ is the difference in cardinalities of $\mathcal{P}$ and $\overline{\mathcal{P}}$.

Proof:

• For $k = 1, \ldots, 5$, the Common Tones Theorem for Transposition implies that $\mathcal{P}$ and $T_{\pm k}\mathcal{P}$ share $n_k$ pitches. Since the cardinality of $\mathcal{P}$ is $m$, this implies that $\mathcal{P}$ and $T_{\pm k}\overline{\mathcal{P}} = T_{\pm k}\overline{\mathcal{P}}$ (equality by Lemma 1, Part 1) share $m - n_k$ pitches. Since the cardinality of $\overline{\mathcal{P}}$ is $12 - m$, this implies that $\mathcal{P}$ and $T_{\pm k}\overline{\mathcal{P}}$ share $(12 - m) - (m - n_k) = n_k - d$ pitches.

• For $k = 6$, $\mathcal{P}$ and $T_{\pm k}\mathcal{P}$ share $2n_k$ pitches, $\mathcal{P}$ and $T_{\pm k}\overline{\mathcal{P}}$ share $m - 2n_k$ pitches, and $\overline{\mathcal{P}}$ and $T_{\pm k}\overline{\mathcal{P}}$ share $(12 - m) - (m - 2n_k) = 2n_k - d/2$ pitches.

• By the Common Tones Theorem for Transposition, the interval vector of $\overline{\mathcal{P}}$ is $\langle n_1 - d, n_2 - d, \ldots, n_6 - d/2 \rangle$.

The inherently geometric nature of the Complementary Chords Theorem for Transposition can be gleaned from Figure 4 whose left-hand column portrays the case of interval class $k = 4$ (major third) while the right-hand column does likewise for interval class $k = 6$ (tritone). The first row displays the pitch polygon for the diatonic scale, $\mathcal{D}$ ($m = 7$), where its pitch classes are shown as solid dots together with its three major thirds on the left and its single tritone on the right. The pitch classes of its complement $\overline{\mathcal{D}}$ ($12 - m = 5$), the pentatonic scale, are shown as asterisks.

The second row displays the effect on $\mathcal{D}$ of applying the transposition $T_k$. As guaranteed by the Common Tones Theorem for Transposition, $T_4\mathcal{D}$ shares $n_4 = 3$ pitches with $\mathcal{D}$ while $T_6\mathcal{D}$ shares $2n_6 = 2$ pitches with $\mathcal{D}$. These shared pitches have been circled. Thus, $\mathcal{D}$ shares $m - n_4 = 4$ pitches with $T_4\overline{\mathcal{D}} = T_4\overline{\mathcal{D}}$ and $m - 2n_6 = 5$ pitches with $T_6\overline{\mathcal{D}} = T_6\overline{\mathcal{D}}$. These shared pitches correspond to solid dots lying outside $T_k\mathcal{D}$ and have been enclosed by diamonds. Hence, $T_4\overline{\mathcal{D}}$ shares $n_4 = (12 - m) - (m - n_4) = 5 - 4 = 1$ pitch with $\overline{\mathcal{D}}$. This pitch corresponds to the lone asterisk lying outside $T_4\mathcal{D}$ and has been enclosed in a square on the left of the second row. Likewise, $T_6\overline{\mathcal{D}}$ shares $2n_6 = (12 - m) - (m - 2n_6) = 5 - 5 = 0$ pitches with $\overline{\mathcal{D}}$, which is why there is no asterisk lying outside $T_6\mathcal{D}$ on the right of the second row.

The designations of points on the circumference of each pitch circle in the third row are identical to those in the second row, but now $\overline{\mathcal{D}}$ is drawn as a solid polygon while $T_k\overline{\mathcal{D}}$ is drawn as a dashed polygon thereby graphically illustrating the shared pitch on the left and the lack of shared pitches on the
right. By the Common Tones Theorem for Transposition, \( \pi_4 = 1 = n_4 - d \) and \( \pi_6 = 0/2 = 0 = n_6 - d/2 \) \((d = 7 - 5 = 2)\) in accordance with the Complementary Chords Theorem for Transposition.

Regener [15, pp. 203-204] considers a more general result pertaining to the relationship between an arbitrary pair of chords and their complements. The Complementary Chords Theorem for Transposition is then obtained by specializing to the case where the pair of chords are transpositions of one another. Not only does this combinatorial approach entail a sacrifice in simplicity, it also obscures the essentially geometric flavor of the Complementary Chords Theorem for Transposition.

Setting \( m = 6 \) in the Complementary Chords Theorem for Transposition, we obtain the celebrated Hexachord Theorem for Transposition [1, p. 93] as
Theorem 4 (Hexachord Theorem for Transposition) Complementary hexachords share the same interval vector.

3  $n$-Tone Equal Temperament: Transposition

The above theorems, as well as their proofs, generalize immediately to the case of $n$-tone equal temperament [4] (simply replace 12 by $n$).

Theorem 5 (Generalized Common Tones Theorem for Transposition) Let $n$ be the number of pitch-classes in equal temperament.

- **$n$ even:** If the interval vector for a chord $\mathcal{P}$ is $\langle \nu_1, \nu_2, \ldots, \nu_{n/2} \rangle$, then, for $k = 1, 2, \ldots, n/2$, the number of pitches in $\mathcal{P}$ that are invariant under transposition by $k$ semitones up or down (tones common to the chords $\mathcal{P}$ and $T_{\pm k}\mathcal{P}$) is $\nu_k$. However, the number of pitches that are invariant under transposition by a diameter ($k = n/2$) is $2\nu_{n/2}$.

- **$n$ odd:** If the interval vector for a chord $\mathcal{P}$ is $\langle \nu_1, \nu_2, \ldots, \nu_{\lfloor n/2 \rfloor} \rangle$, then, for $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, the number of pitches in $\mathcal{P}$ that are invariant under transposition by $k$ semitones up or down (tones common to the chords $\mathcal{P}$ and $T_{\pm k}\mathcal{P}$) is $\nu_k$.

Theorem 6 (Generalized Complementary Chords Theorem for Transposition) Let $n$ be the number of pitch-classes in equal temperament.

- **$n$ even:** Let $\mathcal{P}$ be a pc-set of cardinality $m$ with interval vector $\langle \nu_1, \nu_2, \ldots, \nu_{n/2} \rangle$, then $\overline{\mathcal{P}}$ has cardinality $n - m$ and interval vector $\langle \nu_1 - d, \nu_2 - d, \ldots, \nu_{n/2} - d/2 \rangle$, where $d = 2m - n$ is the difference in cardinalities of $\mathcal{P}$ and $\overline{\mathcal{P}}$.

- **$n$ odd:** Let $\mathcal{P}$ be a pc-set of cardinality $m$ with interval vector $\langle \nu_1, \nu_2, \ldots, \nu_{\lfloor n/2 \rfloor} \rangle$, then $\overline{\mathcal{P}}$ has cardinality $n - m$ and interval vector $\langle \nu_1 - d, \nu_2 - d, \ldots, \nu_{\lfloor n/2 \rfloor} - d \rangle$, where $d = 2m - n$ is the difference in cardinalities of $\mathcal{P}$ and $\overline{\mathcal{P}}$.

Theorem 7 (Generalized Hexachord Theorem for Transposition) Let $n$ be the number of pitch-classes in equal temperament. For $n$ even, complementary demichords (i.e. chords with $n/2$ pitches) share the same interval vector.
4 Patterson’s Theorems

In the context of Discrete Geometry, two sets sharing the same interval vector are termed homometric [18]. In the crystallography literature, such sets are known to be governed by the Patterson Theorems [13].

Theorem 8 (Patterson’s First Theorem) If two subsets of a regular \( m \)-gon are homometric then their complements are homometric.

Theorem 9 (Patterson’s Second Theorem) Every \( m \)-point subset of a regular \( 2m \)-gon is homometric to its complement.

Patterson’s First Theorem is an immediate corollary to the Generalized Complement Theorem for Transposition while Patterson’s Second Theorem is equivalent to the Generalized Hexachord Theorem for Transposition.

5 Complementary Chords Theorem for Inversion

Returning now to the diatonic collection \( \mathcal{D} \), its index vector is easily computed to be \( ⟨3, 4, 5, 2, 7, 2, 5, 4, 3, 6, 2, 6⟩ \) while that of its complement \( \overline{\mathcal{D}} \), the pentatonic collection, is likewise computed to be \( ⟨1, 2, 3, 0, 5, 0, 3, 2, 1, 4, 0, 4⟩ \).

Subtracting these two interval vectors yields \( ⟨2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2⟩ \). Denoting the cardinality of the original chord by \( m = 7 \), the cardinality of its complement is \( 12 - m = 5 \) and these cardinalities differ by \( d = m - (12 - m) = 2m - 12 = 2 \). Observe that the difference of the index vector of the original chord with its complement is given by \( ⟨d, d, d, d, d, d, d, d, d, d, d, d⟩ \). Thus, the index vector for \( \overline{\mathcal{D}} \) may be obtained from that for \( \mathcal{D} \) by simply subtracting \( d \) from its components. That this is not an accidental occurrence for the diatonic/pentatonic collections but rather applies to all pairs of chords and their complements is the gist of the new Complementary Chords Theorem for Inversion.

Theorem 10 (Complementary Chords Theorem for Inversion) Let \( \mathcal{P} \) be a pc-set of cardinality \( m \) with index vector \( ⟨n_0, n_1, \ldots, n_{11}⟩ \), then \( \overline{\mathcal{P}} \) has cardinality \( 12 - m \) and index vector \( ⟨n_0 - d, n_1 - d, \ldots, n_{11} - d⟩ \), where \( d = 2m - 12 \) is the difference in cardinalities of \( \mathcal{P} \) and \( \overline{\mathcal{P}} \).

Proof:

- For \( k = 0, \ldots, 11 \), the Common Tones Theorem for Inversion implies that \( \mathcal{P} \) and \( T_k I \mathcal{P} \) share \( n_k \) pitches. Since the cardinality of \( \mathcal{P} \) is \( m \), this implies that \( \mathcal{P} \) and \( T_k I \overline{\mathcal{P}} = T_k I \overline{\mathcal{P}} \) (equality by Lemma 1, Part 2) share \( m - n_k \) pitches. Since the cardinality of \( \overline{\mathcal{P}} \) is \( 12 - m \), this implies that \( \overline{\mathcal{P}} \) and \( T_k I \overline{\mathcal{P}} \) share \( (12 - m) - (m - n_k) = n_k - d \) pitches.
• By the Common Tones Theorem for Inversion, the index vector of $\overline{\mathcal{P}}$ is $(n_0 - d, n_1 - d, \ldots, n_{11} - d)$.

Figure 5: Geometric Proof of Complementary Chords Theorem for Inversion

The inherently geometric nature of the Complementary Chords Theorem for Inversion can be gleaned from Figure 5 which explores the effect on the index vector of applying $T_0I$ to $\mathcal{D}$. The first row displays the pitch polygon for the diatonic scale, $\mathcal{D}$ ($m = 7$), where its pitch classes are shown as solid dots. The pitch classes of its complement $\overline{\mathcal{D}}$ ($12 - m = 5$), the pentatonic scale, are shown as asterisks.

The second row displays the effect on $\mathcal{D}$ of applying the inversion $T_0I$. As guaranteed by the Common Tones Theorem for Inversion, $T_0I\mathcal{D}$ shares $n_0 = 3$ pitches with $\mathcal{D}$. These shared pitches have been circled. Thus, $\mathcal{D}$ shares
$m - n_0 = 4$ pitches with $\overline{T_0 I D} = T_0 I \overline{D}$. These shared pitches correspond to solid dots lying outside $T_0 I D$ and have been enclosed by diamonds. Hence, $T_0 I \overline{D}$ shares $\pi_0 = (12 - m) - (m - n_0) = 5 - 4 = 1$ pitch with $\overline{D}$. This pitch corresponds to the lone asterisk lying outside $T_0 I D$ and has been enclosed in a square on the left of the second row.

The designations of points on the circumference of each pitch circle in the third row are identical to those in the second row, but now $\overline{D}$ is drawn as a solid polygon while $T_0 I \overline{D}$ is drawn as a dashed polygon thereby graphically illustrating the shared pitch. By the Common Tones Theorem for Inversion, $\pi_0 = 1 = n_4 - d$ ($d = 7 - 5 = 2$) in accordance with the Complementary Chords Theorem for Inversion.

Figure 6: Prime Forms of Diatonic (Left) and Pentatonic (Right) Collections

Of course, $D$ and $\overline{D}$ are not presently in prime form. As displayed in Figure 6 (Left), the prime form of the diatonic collection is given by $T_1 D$. Thus, by Corollary 1, its index vector is obtained by a cyclic shift of 2 to the right of the previously computed index vector of $D$: $\langle 2, 6, 3, 4, 5, 2, 7, 2, 5, 4, 3, 6 \rangle$. Likewise, as displayed in Figure 6 (Right), the prime form of the pentatonic collection is given by $T_6 \overline{D}$. Thus, by Corollary 1, its index vector is obtained by a cyclic shift of 12 to the right of the index vector of $\overline{D}$. However, as this is the identity transformation, the index vector of the prime form of the pentatonic collection coincides with that previously computed for $\overline{D}$: $\langle 1, 2, 3, 0, 5, 0, 3, 2, 1, 4, 0, 4 \rangle$.

Setting $m = 6$ in the Complementary Chords Theorem for Inversion, we obtain the new Hexachord Theorem for Inversion as an immediate corollary.

**Theorem 11 (Hexachord Theorem for Inversion)** Complementary hexachords share the same index vector.
6 \(n\)-Tone Equal Temperament: Inversion

The above theorems, as well as their proofs, generalize immediately to the case of \(n\)-tone equal temperament \([4]\) (simply replace 12 by \(n\)).

**Theorem 12** (Generalized Common Tones Theorem for Inversion)
Let \(n\) be the number of pitch-classes in equal temperament. If the index vector for a chord \(P\) is \(\langle \nu_0, \nu_1, \ldots, \nu_{n-1} \rangle\), then, for \(k = 0, 1, \ldots, n - 1\), the number of pitches in \(P\) that are invariant under inversion at index \(k\) (tones common to the chords \(P\) and \(T_k I P\)) is \(\nu_k\).

Note that Corollaries 1 and 2 still apply without modification.

**Theorem 13** (Generalized Complementary Chords Theorem for Inversion)
Let \(n\) be the number of pitch-classes in equal temperament. Let \(P\) be a pc-set of cardinality \(m\) with index vector \(\langle \nu_0, \nu_1, \ldots, \nu_{n-1} \rangle\), then \(\overline{P}\) has cardinality \(n - m\) and index vector \(\langle \nu_0 - d, \nu_1 - d, \ldots, \nu_{n-1} - d \rangle\), where \(d = 2m - n\) is the difference in cardinalities of \(P\) and \(\overline{P}\).

**Theorem 14** (Generalized Hexachord Theorem for Inversion) Let \(n\) be the number of pitch-classes in equal temperament. For \(n\) even, complementary demichords (i.e. chords with \(n/2\) pitches) share the same index vector.

7 Conclusion

As should be abundantly clear from the preceding sections, simple geometric arguments are of considerable utility in the development of Mathematical Music Theory. Not only does this provide the most insightful entrée to well established results such as the Common Tones Theorems, the Complementary Chords Theorem for Transposition and the Hexachord Theorem for Transposition, but it also leads to natural generalizations of these foundational results such as the Complementary Chords Theorem for Inversion and the Hexachord Theorem for Inversion.

As a direct consequence of these Complementary Chords Theorems, complementary hexachords share both their interval and index vectors. As such, they have identical combinatorial properties \([10, 11]\). Consequently, \(Z\)-related hexachords are inextricably linked by their common combinatorial properties. An exhaustive treatment of hexachordal combinatoriality from a geometric perspective will appear in \([10, 11]\).

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References


Complementary chords theorems


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