A Geometric Approach
to Hexachordal Combinatoriality

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Abstract

A geometric approach is developed for the synthesis of all hexachords possessing any type of combinatoriality.

1 Introduction

Combinatoriality as a device for musical composition made its first appearance implicitly in the music of Arnold Schoenberg and his Second Viennese School in the 1920s and 1930s [1, 11, 13, 15] while its first explicit statement awaited the writings of Ernst Krenek in 1940 [4]. This line of musical development reached its apex in the work of Milton Babbitt on hexachordal combinatoriality in the 1950s and 1960s [2, 9].

Since that time, the utility of geometric considerations in the analysis of combinatorial properties of hexachords has been widely appreciated [5, 10]. Alternatively, the present paper purports to study the geometric synthesis of hexachordal combinatoriality.

Specifically, the geometric properties [6] of rotational symmetry, rotational antisymmetry, reflectional symmetry and reflectional antisymmetry will be employed to systematically generate all hexachords possessing R-combinatoriality, P-combinatoriality, RI-combinatoriality and I-combinatoriality, respectively. The reader is directed to [14, pp. 222-230] for relevant definitions and musical context.
2 R-Combinatoriality \((T_n : H \rightarrow H)\)

A hexachord \(H\) is R-combinatorial if and only if it maps onto itself under some \(T_n\), i.e. if and only if it is rotationally symmetric. If \(T_n\) is the identity transformation, this R-combinatoriality is trivial. Thus, only nontrivial R-combinatoriality, where \(n \neq 0\), will be considered henceforth.

Any such \(T_n\) produces a permutation of the vertices of \(H\) so that the group of such symmetries of \(H\) must be isomorphic to a subgroup of the symmetric group \(S_6\) [3, p. 300]. Thus, by Lagrange’s Theorem [3, p. 290], the order of this subgroup must be a divisor of 6, i.e. it must be equal to 1, 2, 3 or 6.

The group of rotational symmetries of the hexachord \(H\) must also be a subgroup of the group of rotations of the regular dodecagon, i.e. the cyclic group \(C_{12}\) [3, p. 283]. The only subgroups of \(C_{12}\) of order 1, 2, 3, 6 are, respectively, \(\{T_0\}\), \(\{T_0, T_6\}\), \(\{T_0, T_4, T_8\}\), \(\{T_0, T_2, T_4, T_6, T_8, T_{10}\}\). The first corresponds to the trivial R-combinatoriality possessed by all hexachords.

The second and fourth subgroups both contain \(T_6\). By the Common Tones Theorem for Transposition [6], any hexachord invariant under a half-turn must contain three tritones (diameters). The only three ways to combine three diameters to form a hexagon in prime form are shown in Figure 1. From left to right, we will denote these hexachords using their opening intervals as \(H_{114}\) (containing three adjacent diameters), \(H_{123}\) (containing a pair of adjacent diameters) and \(H_2\) (containing no adjacent diameters). The first two are R-combinatorial under \(\{T_0, T_6\}\) while the last is R-combinatorial under \(\{T_0, T_2, T_4, T_6, T_8, T_{10}\}\).

The third subgroup contains \(T_4\). By the Common Tones Theorem for Transposition [6], any hexachord invariant under a 120° rotation must contain six major thirds (M3). The twelve M3s of the musical clock form four disjoint equilateral triangles and any hexachord possessing 120° rotational symmetry must be the union of a pair of them. The only ways to form the union of two such equilateral triangles with the resulting hexachord in prime form are shown.
in Figure 2. The first, $H_{13}$ (containing a pair of adjacent equilateral triangles), is R-combinatorial under $\{T_0, T_4, T_8\}$ while the second is a reappearance of $H_2$ (containing a pair of nonadjacent equilateral triangles).

By the above geometric reasoning, this exhausts all possible instances of nontrivial R-combinatoriality. The resultant four hexachords possessing nontrivial R-combinatoriality are collected together in Figure 3. The first three are the all-combinatorial hexachords at multiple transpositional levels (6, 3 and 2, respectively) while the fourth is the only hexachord possessing only R- and I- combinatoriality at precisely two transpositional levels [8].

### 3 P-Combinatoriality ($T_n : H \rightarrow \overline{H}$)

A hexachord $H$ is P-combinatorial if and only if it maps onto its complement under some $T_n$, i.e. if and only if it is rotationally antisymmetric. By the Common Tones Theorem for Transposition [6], any hexachord mapping onto its complement under $T_n$ must exclude interval class $n$ from its interval vector. Thus, all P-combinatorial hexachords may be generated by successively excluding each interval class.

The fact that the product of two rotations is also a rotation ($T_m T_n = T_{m+n}$) reveals an intimate connection between P- and R-combinatoriality. Since $T_m : H \leftrightarrow \overline{H}$ and $T_n : H \leftrightarrow \overline{H}$ implies that $T_{m+n} : H \rightarrow H$, the collection of transpositions carrying $H$ onto its complement is the product of one of the previously four considered subgroups of rotations with the square root of its
corresponding group generator. Specifically,

\[ \{T_0\} \circ T_{12}^{1/2} = \{T_0\} \circ T_6 = \{T_6\} \]

\[ \{T_0, T_6\} \circ T_6^{1/2} = \{T_0, T_6\} \circ T_3 = \{T_3, T_9\} \]

\[ \{T_0, T_4, T_8\} \circ T_4^{1/2} = \{T_0, T_4, T_8\} \circ T_2 = \{T_2, T_6, T_{10}\} \]

\[ \{T_0, T_2, T_4, T_6, T_8, T_{10}\} \circ T_2^{1/2} = \{T_0, T_2, T_4, T_6, T_8, T_{10}\} \circ T_1 = \{T_1, T_3, T_5, T_7, T_9, T_{11}\} \]

are the only collections of transpositions associated with P-combinatoriality.

Invoking the Common Tones Theorem for Transpositions [7], the first collection of transpositions excludes the tritone (π), the second excludes minor thirds (m3), the third excludes major seconds (M2) and π while the last, by necessity, excludes minor seconds (m2), m3 and perfect fourths (P4). Thus, the interval vector of the latter must be \(<0, 6, 0, 6, 0, 3>\) since its entries must sum to 15. Thereby containing six M2 and no m2, it must be the whole-tone hexachord, \(H_2\), shown in the left frame of Figure 4.
Hence, the remainder of the P-combinatorial hexachords may be found by systematically excluding M2, m3 and π. The computational burden in so listing the P-combinatorial hexachords is dramatically reduced by invoking the following fundamental result.

**Theorem 1 (Hexachordal Prime Forms)** All hexachords in prime form must open with one of the following five triadic patterns:

- \([0, 1, 2, \cdot, \cdot, \cdot]\)
- \([0, 1, 3, \cdot, \cdot, \cdot]\)
- \([0, 1, 4, \cdot, \cdot, \cdot]\)
- \([0, 2, 3, \cdot, \cdot, \cdot]\)
- \([0, 2, 4, \cdot, \cdot, \cdot]\)

**Proof:** See [7] for a straightforward geometric demonstration.

Hence, we next consider the construction of hexachords in prime form that exclude each of the distinguished interval classes \(\{M2, m3, \pi\}\), in turn, and, for each excluded interval class, inspect the five (or fewer!) permissible opening triadic patterns. N.B.: Prime form precludes \(E\) as a final pitch-class.

**Figure 4: Excluding Minor and Major Seconds**

- **ic-2:** The only opening triadic pattern of a hexachord in prime form which excludes ic-2 is \([0, 1, 4, \cdot, \cdot, \cdot]\). This automatically excludes pitch-classes 2, 3, 6, \(T, E\). The final pitch-class, reading clockwise from 0, must be 9 since choosing 8 would also force the inclusion of edge 5 – 7 which forms
ic-2 which must be excluded, while choosing an interval class smaller than 8 would not allow the accommodation of a hexachord. Thus, we are ultimately led to the only possible such hexachord, shown in the right frame of Figure 4, which is \([0, 1, 4, 5, 8, 9]\).

\[\begin{array}{c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
E & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}\]

Figure 5: Excluding Minor Thirds

- **ic-3**: The only opening triadic patterns of a hexachord in prime form which exclude ic-3 are \([0, 1, 2, \cdot, \cdot, \cdot]\) and \([0, 2, 4, \cdot, \cdot, \cdot]\).

  - \([0, 1, 2, \cdot, \cdot, \cdot]\): This automatically excludes pitch-classes 3, 4, 5, 9, \(T, E\) leaving only pitch-classes 6, 7, 8 to complete the hexachord. Thus, we are immediately led to the only possible such hexachord, shown in the left frame of Figure 5, which is \([0, 1, 2, 6, 7, 8]\).

  - \([0, 2, 4, \cdot, \cdot, \cdot]\): This automatically excludes pitch-classes 1, 3, 5, 7, 9, \(E\) leaving only pitch-classes 6, 8, \(T\) to complete the hexachord. Thus, we are immediately led to the only possible such hexachord, shown in the right frame of Figure 5, which is the whole-tone hexachord \([0, 2, 4, 6, 8, T]\).

- **ic-6**: See Figure 6.

  - \([0, 1, 2, \cdot, \cdot, \cdot]\): This automatically excludes pitch-classes 6, 7, 8. Choosing 9, \(T\) or \(E\) as the last pitch-class would require the inclusion of at least ic-4 in order to skip 6, 7, 8. However, prime form would then require at least ic-4 at the end of the chord which would preclude the inclusion of 9, \(T\) or \(E\) which is a contradiction! Hence, the last pitch-class must be \(\leq 5\) with 5 being the only choice leading
to a hexachord. Thus, we are ultimately led to the only possible such hexachord, shown in the top frame of Figure 6, which is the chromatic hexachord $[0, 1, 2, 3, 4, 5]$. 

- $[0, 1, 3, \cdot, \cdot, \cdot]$: This automatically excludes pitch-classes 2, 6, 7, 9. A choice of $T$ or $E$ as the last pitch-class would require the inclusion of at least ic-3 in order to skip 6, 7. However, prime form would then require at least ic-3 at the end of the chord which would preclude the inclusion of $T$ or $E$ which is a contradiction! Hence, the last pitch-class must be $\leq 8$ with 8 being the only choice leading to a hexachord. Thus, we are ultimately led to the only possible such hexachord, shown in the middle-left frame of Figure 6, which is $[0, 1, 3, 4, 5, 8]$.

- $[0, 1, 4, \cdot, \cdot, \cdot]$: This automatically excludes pitch-classes 2, 3, 6, 7, $T$. The presence of edge 1 – 4 implies that, in order for the chord
to be in prime form, the final interval class must be at least ic-3. But, a final interval class larger than ic-3 could not accommodate a hexachord so that the last pitch-class must be 9. Thus, we are immediately led to the only possible such hexachord, shown in the middle-center frame of Figure 6, which is [0, 1, 4, 5, 8, 9].

Figure 7: P-Combinatorial Hexachords

- [0, 2, 3, ···, ···]: This automatically excludes pitch-classes 1, 6, 8, 9. In order to be in prime form, the final interval class must be at least ic-2. But a choice of T as the last pitch-class would require a penultimate interval class of at least ic-3 in order to skip over 8, 9 and this would, in turn, violate the conditions for prime form. The only remaining choice for final pitch-class which accommodates a hexachord is 7. Thus, we are immediately led to the only possible such hexachord, shown in the middle-right frame of Figure 6, which is [0, 2, 3, 4, 5, 7].
The automatic exclusion of pitch-classes 1, 3, 6, 8, T. In order to be in prime form, the final interval class must be at least ic-2. But, since T has been excluded, this final interval class must be at least ic-3. The only remaining choice for final pitch-class which accommodates a hexachord is 9. Thus, we are immediately led to the only possible such hexachord, shown in the bottom frame of Figure 6, which is the diatonic hexachord [0, 2, 4, 5, 7, 9]

By the above geometric reasoning, this exhausts all possible instances of P-combinatoriality. The resultant seven hexachords possessing P-combinatoriality are collected together in Figure 7. Observe that the first six entries form the complete collection of all-combinatorial hexachords [8] while the seventh entry is thereby the only hexachord in prime form possessing only P-combinatoriality at a single transpositional level.

Comparing Figures 3 and 7, we note the general paucity of hexachords possessing (nontrivial) transpositional combinatoriality: there are only nine of them since H2, H13 and H114 recur. As we now turn our attention to inversional combinatoriality, we will observe a concomitant upsurge in the number of qualifying hexachords.

4 RI-Combinatoriality \((T_nI : H \rightarrow H)\)

A hexachord \(H\) is RI-combinatorial if and only if it maps onto itself under some \(T_nI\), i.e. if and only if it is reflectionally symmetric. The collection of rotations and reflections mapping \(H\) onto itself must form a subgroup of the symmetry group of the regular dodecagon, i.e. the dihedral group \(D_{24}\) [3, p. 446].

Since the product of two reflections is a rotation \((T_mI \circ T_nI = T_{m-n}I)\) and the product of a rotation with a reflection is a reflection \((T_mI \circ T_nI = T_{m+n}I; T_mI \circ T_n = T_{m-n}I)\), these subgroups of \(D_{24}\) must be extensions of the four previously encountered subgroups of \(C_{12}\). Specifically, they must assume one of the following four forms:

\[
\{T_0, T_kI\},
\{T_0, T_6, T_kI, T_{k+6}I\},
\{T_0, T_4, T_8, T_kI, T_{k+4}I, T_{k+8}I\},
\{T_0, T_2, T_4, T_6, T_8, T_{10}, T_kI, T_{k+2}I, T_{k+4}I, T_{k+6}I, T_{k+8}I, T_{k+10}I\}.
\]

The permissible values of \(k\) are determined by the constraint that the hexachord \(H\) appear in prime form.

We now construct all RI-combinatorial hexachords by considering, in turn, the image of each of the five permissible opening triadic patterns under \(T_nI\) (0 ≤
$n \leq 11$). Some simple observations will suffice to reduce somewhat the computational burden. Firstly, if $n$ is even then the pitch-classes lying on the axis of symmetry must either both be in $H$ or both be in $\overline{H}$ (since $T_n I : H \rightarrow \overline{H}$ as well as $T_n I : H \rightarrow H$). Also, since $T_{11} I : 0 \rightarrow 11$, a $m2$ appears at the end of the chord thereby precluding prime form so that this reflection need not be considered. Finally, since the only hexachord in prime form ending in a $M2$ is $H_2$, $T_0 I$ and $T_{10} I$ need only be considered in conjunction with the opening triadic pattern $[0, 2, 4, \cdot, \cdot, \cdot]$.

![Diagram of hexachords](image)

Figure 8: RI-Combinatorial Hexachords: $[0, 1, 2, \cdot, \cdot, \cdot]$

- $[0, 1, 2, \cdot, \cdot, \cdot]$: (See Figure 8)
  - $T_0 I$: One hexachord not in prime form.
  - $T_{11} I$: Four hexachords not in prime form.
  - $T_{22} I$: Two hexachords not in prime form. Two hexachords in prime form: $[0, 1, 2, 5, 7, 9]$ and $[0, 1, 2, 6, 7, 8]$. 
– $T_3I$: Two hexachords not in prime form. Two hexachords in prime form: [0, 1, 2, 3, 6, 9] and [0, 1, 2, 3, 7, 8].
– $T_4I$: One hexachord in prime form: [0, 1, 2, 3, 4, 8].
– $T_5I$: One hexachord in prime form: [0, 1, 2, 3, 4, 5].
– $T_6I$: One hexachord in prime form: [0, 1, 2, 3, 5, 6].
– $T_7I$: One hexachord in prime form: [0, 1, 2, 3, 5, 7].
– $T_8I$: One hexachord in prime form: [0, 1, 2, 3, 6, 7] (repeat).
– $T_9I$: One hexachord not in prime form.
– $T_{10}I$: One hexachord not in prime form.
– $T_{11}I$: One hexachord not in prime form.

Figure 9: RI-Combinatorial Hexachords: [0, 1, 3, ·, ·, ·]

- [0, 1, 3, ·, ·, ·]: (See Figure 9)
  - $T_9I$: One hexachord not in prime form.
- \( T_1I \): Three hexachords not in prime form.
- \( T_2I \): Conflict: 0 → 2.
- \( T_3I \): Conflict: 1 → 2.
- \( T_4I \): Two hexachords not in prime form. One hexachord in prime form: \([0, 1, 3, 4, 7, 9]\).
- \( T_5I \): Conflict: 3 → 2.
- \( T_6I \): One hexachord in prime form: \([0, 1, 3, 5, 6, 9]\).
- \( T_7I \): One hexachord in prime form: \([0, 1, 3, 4, 6, 7]\).
- \( T_8I \): One hexachord in prime form: \([0, 1, 3, 5, 7, 8]\).
- \( T_9I \): One hexachord not in prime form.
- \( T_{10}I \): One hexachord not in prime form.
- \( T_{11}I \): One hexachord not in prime form.

Figure 10: RI-Combinatorial Hexachords: \([0, 1, 4, \cdot, \cdot, \cdot]\)
Hexachordal combinatoriality

- \([0, 1, 4, \cdot, \cdot, \cdot]\): (See Figure 10)
  - \(T_0I\): One hexachord not in prime form.
  - \(T_1I\): Two hexachords in prime form: \([0, 1, 4, 5, 8, 9]\) and \([0, 1, 4, 6, 7, 9]\).
  - \(T_2I\): Conflict: \(0 \rightarrow 2\).
  - \(T_3I\): Conflict: \(1 \rightarrow 2\).
  - \(T_4I\): Conflict: \(1 \rightarrow 3\).
  - \(T_5I\): One hexachord in prime form: \([0, 1, 4, 5, 8, 9]\) (repeat).
  - \(T_6I\): Conflict: \(4 \rightarrow 2\).
  - \(T_7I\): Conflict: \(4 \rightarrow 3\).
  - \(T_8I\): One hexachord not in prime form.
  - \(T_9I\): One hexachord not in prime form.
  - \(T_{10}I\): One hexachord not in prime form.
  - \(T_{11}I\): One hexachord not in prime form.

- \([0, 2, 3, \cdot, \cdot, \cdot]\): (See Figure 11)
  - \(T_0I\): One hexachord not in prime form.
  - \(T_1I\): Conflict: \(0 \rightarrow 1\).
  - \(T_2I\): Three hexachords not in prime form.
  - \(T_3I\): Conflict: \(2 \rightarrow 1\).
  - \(T_4I\): Conflict: \(3 \rightarrow 1\).
  - \(T_5I\): Three hexachords not in prime form.
  - \(T_6I\): One hexachord in prime form: \([0, 2, 3, 4, 6, 9]\).
  - \(T_7I\): One hexachord in prime form: \([0, 2, 3, 4, 5, 7]\).
  - \(T_8I\): One hexachord in prime form: \([0, 2, 3, 5, 6, 8]\).
  - \(T_9I\): One hexachord in prime form: \([0, 2, 3, 6, 7, 9]\).
  - \(T_{10}I\): One hexachord not in prime form.
  - \(T_{11}I\): One hexachord not in prime form.

- \([0, 2, 4, \cdot, \cdot, \cdot]\): (See Figure 12)
  - \(T_0I\): One hexachord in prime form: \([0, 2, 4, 6, 8, T]\).
  - \(T_1I\): Conflict: \(0 \rightarrow 1\).
  - \(T_2I\): One hexachord not in prime form. One hexachord in prime form: \([0, 2, 4, 6, 8, T]\) (repeat).
Figure 11: RI-Combinatorial Hexachords: $[0, 2, 3, \ldots, 9]$  

- $T_3I$: Conflict: $2 \rightarrow 1$.
- $T_4I$: Two hexachords not in prime form. One hexachord in prime form: $[0, 2, 4, 6, 8, T]$ (repeat).
- $T_5I$: Conflict: $2 \rightarrow 3$.
- $T_6I$: One hexachord in prime form: $[0, 2, 4, 6, 8, T]$ (repeat).
- $T_7I$: Conflict: $4 \rightarrow 3$.
- $T_8I$: One hexachord in prime form: $[0, 2, 4, 6, 8, T]$ (repeat).
- $T_9I$: One hexachord in prime form: $[0, 2, 4, 5, 7, 9]$.
- $T_{10}I$: One hexachord in prime form: $[0, 2, 4, 6, 8, T]$ (repeat).
- $T_{11}I$: One hexachord not in prime form.

By the above geometric reasoning, this exhausts all possible instances of RI-combinatoriality. The resultant twenty hexachords with RI-combinatoriality are collected together in Figure 13. Discounting the previously encountered
six all-combinatorial hexachords [8], this yields precisely fourteen hexachords possessing only RI-combinatoriality at a single transpositional level.

5 I-Combinatoriality \((T_n I : H \rightarrow \overline{H})\)

A hexachord \(H\) is I-combinatorial if and only if it maps onto its complement under some \(T_n I\), i.e. if and only if it is reflectionally antisymmetric. Now, if \(T_n I : H \rightarrow \overline{H}\) then it can have no fixed points. This implies that \(n\) is odd, since if it were even then the two pitch-classes at the endpoints of the axis of reflection would be fixed points of the associated inversion.

Furthermore, if \(T_m I : H \leftrightarrow \overline{H}\) and \(T_n I : H \leftrightarrow \overline{H}\) then \(T_m I \circ T_n I = T_{m-n} : H \rightarrow H\). Hence, the collection of inversions carrying \(H\) onto its complement is the product of one of the previously four considered subgroups of rotations with an inversion of odd suffix. Specifically,

\[
\{T_0\} \circ T_{2k+1} I = \{T_{2k+1} I\}
\]
Figure 13: RI-Combinatorial Hexachords

\[
\begin{align*}
\{T_0, T_6\} \circ T_{2k+1}I &= \{T_{2k+1}I, T_{2k+7}I\} \\
\{T_0, T_4, T_8\} \circ T_{2k+1}I &= \{T_{2k+1}I, T_{2k+5}I, T_{2k+9}I\} \\
\{T_0, T_2, T_4, T_6, T_8, T_{10}\} \circ T_{2k+1}I &= \{T_1I, T_3I, T_5I, T_7I, T_9I, T_{11}I\}
\end{align*}
\]

are the only collections of transpositions associated with I-combinatoriality. The permissible values of \(k\) are determined by the constraint that the hexachord \(H\) appear in prime form.

We next construct all I-combinatorial hexachords by considering, in turn, the image under \(T_{2k+1}I\) (\(0 \leq k \leq 5\)) of each of the five permissible opening triadic patterns. Some simple observations will suffice to reduce somewhat the computational burden. Firstly, a pitch-class may be included in \(H\) if and only if its mirror image is excluded. Also, a hexachord in prime form may not end with a m2. Finally, the only hexachord in prime form ending in T is the whole-tone hexachord, \([0, 2, 4, 6, 8, T]\), and the only hexachord in prime form ending in 5 is the chromatic hexachord, \([0, 1, 2, 3, 4, 5]\). Thus, we need only consider other hexachords ending in 9, 8, 7 or 6.
$T_1 I$: (See Figure 14)

- $[0, 1, 2, \cdot, \cdot, \cdot]:$ Conflict: $0 \leftrightarrow 1$.
- $[0, 1, 3, \cdot, \cdot, \cdot]:$ Conflict: $0 \leftrightarrow 1$.
- $[0, 1, 4, \cdot, \cdot, \cdot]:$ Conflict: $0 \leftrightarrow 1$.
- $[0, 2, 3, \cdot, \cdot, \cdot]:$ $E$ and $T$ are automatically excluded. If we end in 9 then we must exclude 4 while prime form then requires the inclusion of 7 thereby leading directly to $[0, 2, 3, 5, 7, 9]$. If we end in 8 then we must exclude 5 and include 4 while prime form then requires the inclusion of 6 thereby leading directly to $[0, 2, 3, 4, 6, 8]$. If we end in 7 then we must exclude 6 and include 4 and 5 thereby leading directly to $[0, 2, 3, 4, 5, 7]$. If we end in 6 then we must also include 5 and 4 thereby leading directly to $[0, 2, 3, 4, 5, 6]$ which is not in prime form.
– $[0, 2, 4, \ldots, \cdot, \cdot, \cdot]$; E and 9 are automatically excluded while T and 8 are automatically included thereby leading directly to $[0, 2, 4, 6, 8, T]$. 

Figure 15: I-Combinatorial Hexachords: $T_3I$

- $T_3I$: (See Figure 15)

– $[0, 1, 2, \ldots, \cdot, \cdot, \cdot]$: Conflict: $1 \leftrightarrow 2$.

– $[0, 1, 3, \ldots, \cdot, \cdot, \cdot]$: Conflict: $0 \leftrightarrow 3$.

– $[0, 1, 4, \ldots, \cdot, \cdot, \cdot]$; If we end in 9 then we must exclude 6 and include 5 while prime form then requires the inclusion of either 8 or 7 thereby leading directly to either $[0, 1, 4, 5, 8, 9]$ and $[0, 1, 4, 5, 7, 9]$. If we end in 8 then we must exclude 7 and include 5 and 6 thereby leading directly to $[0, 1, 4, 5, 6, 8]$. If we end in 7 then we must include 5 and 6 thereby leading directly to $[0, 1, 4, 5, 6, 7]$ which is not in prime form.

– $[0, 2, 3, \ldots, \cdot, \cdot, \cdot]$: Conflict: $0 \leftrightarrow 3.$
– $[0, 2, 4, \cdot, \cdot, \cdot]$: Ending in $T$ leads directly to $[0, 2, 4, 6, 8, T]$. If we end in 9 then we must exclude 6 and include 5 while prime form then requires the inclusion of 7 thereby leading directly to $[0, 2, 4, 5, 7, 9]$. If we end in 8 then we must exclude 7 and include 5 and 6 thereby leading directly to $[0, 2, 4, 5, 6, 8]$ which is not in prime form. If we end in 7 then we must include 5 and 6 thereby leading directly to $[0, 2, 4, 5, 6, 7]$ which is not in prime form.

![Diagram of I-Combinatorial Hexachords: $T_5I$](image)

- $T_5I$: (See Figure 16)

  - $[0, 1, 2, \cdot, \cdot, \cdot]$: 3, 4, and 5 are automatically excluded. If we end in 9 then we must exclude 8 thereby leading directly to $[0, 1, 2, 6, 7, 9]$ which is not in prime form. If we end in 8 then this leads directly to $[0, 1, 2, 6, 7, 8]$. We cannot end in 7 or 6 since that would not leave sufficiently many pitches with which to form a hexachord.
– $[0, 1, 3, \cdot, \cdot, \cdot, \cdot, \cdot]$: 4 and 5 are automatically excluded. If we end in 9 then we must exclude 8 thereby leading directly to $[0, 1, 3, 6, 7, 9]$ which is not in prime form. If we end in 8 then this leads directly to $[0, 1, 3, 6, 7, 8]$. We cannot end in 7 or 6 since that would not leave sufficiently many pitches with which to form a hexachord.

– $[0, 1, 4, \cdot, \cdot, \cdot, \cdot, \cdot]$: Conflict: $1 \leftrightarrow 4$.

– $[0, 2, 3, \cdot, \cdot, \cdot, \cdot, \cdot]$: Conflict: $2 \leftrightarrow 3$.

– $[0, 2, 4, \cdot, \cdot, \cdot, \cdot, \cdot]$: 5 is automatically excluded. If we end in $T$ then we are directly led to $[0, 2, 4, 6, 8, T]$. If we end in 9 then we must exclude 8 thereby leading directly to $[0, 2, 4, 6, 7, 9]$ which is not in prime form. If we end in 8 then this leads directly to $[0, 2, 4, 6, 7, 8]$ which is not in prime form. We cannot end in 7 or 6 since that would not leave sufficiently many pitches with which to form a hexachord.

Figure 17: I-Combinatorial Hexachords: $T_7I$
• $T_7I$: (See Figure 17)

- $[0, 1, 2, \cdot, \cdot, \cdot]$:
  5, 6 and 7 are automatically excluded. If we end in 9 then prime form requires that we include 8 thereby leading directly to $[0, 1, 2, 4, 8, 9]$ and $[0, 1, 2, 3, 8, 9]$, neither of which is in prime form.

- $[0, 1, 3, \cdot, \cdot, \cdot]$:
  4, 6 and 7 are automatically excluded while 5 is automatically included. In turn, this requires that we include 8 and 9 to complete the hexachord thereby leading directly to $[0, 1, 3, 5, 8, 9]$ which is not in prime form.

- $[0, 1, 4, \cdot, \cdot, \cdot]$:
  6 and 7 are automatically excluded while 5 is automatically included. In turn, this requires that we include 8 and 9 to complete the hexachord thereby leading directly to $[0, 1, 4, 5, 8, 9]$.

- $[0, 2, 3, \cdot, \cdot, \cdot]$:
  4, 5 and 7 are automatically excluded while 6 is automatically included. In turn, this requires that we include 8 and 9 to complete the hexachord thereby leading directly to $[0, 2, 3, 6, 8, 9]$ which is not in prime form.

- $[0, 2, 4, \cdot, \cdot, \cdot]$:
  5 and 7 are automatically excluded while 6 is automatically included thereby leading directly to $[0, 2, 4, 6, 8, 9]$ which is not in prime form.

• $T_9I$: (See Figure 18)

- $[0, 1, 2, \cdot, \cdot, \cdot]$:
  7, 8 and 9 are automatically excluded which leaves only 3, 4, 5 and 6 from which to select three pitch-classes to complete a hexachord. But 3–6 and 4–5 form mirror images so that only one of each pair may be included which thereby precludes the completion of a hexachord.

- $[0, 1, 3, \cdot, \cdot, \cdot]$:
  6, 8 and 9 are automatically excluded while 7 is automatically included but, since 4–5 are mirror images, only one of them can be included thereby precluding the completion of a hexachord.

- $[0, 1, 4, \cdot, \cdot, \cdot]$:
  5, 8 and 9 are automatically excluded while 6 and 7 are automatically included leaving no sixth pitch-class with which to complete a hexachord.

- $[0, 2, 3, \cdot, \cdot, \cdot]$:
  6, 7 and 9 are automatically excluded while 8 is automatically included but, since 4–5 are mirror images, only one of them can be included thereby precluding the completion of a hexachord.

- $[0, 2, 4, \cdot, \cdot, \cdot]$:
  5, 7 and 9 are automatically excluded while 6 and 8 are automatically included leading directly to $[0, 2, 4, 6, 8, 9]$.
Figure 18: I-Combinatorial Hexachords: $T_9 I$

- $T_{11} I$: (See Figure 19)
  - $[0, 1, 2, \cdot, \cdot, \cdot]:$ 9, $T$ and $E$ are automatically excluded. If we end in 8 (excluding 3) and include 7 (excluding 4) then we are led directly to $[0, 1, 2, 6, 7, 8]$ and $[0, 1, 2, 5, 7, 8]$. If we end in 8 (excluding 3) and exclude 7 (including 4) then we are led directly to $[0, 1, 2, 4, 6, 8]$ and $[0, 1, 2, 4, 5, 8]$. If we end in 7 (excluding 4 and including 3) then we are led directly to $[0, 1, 2, 3, 6, 7]$ and $[0, 1, 2, 3, 5, 7]$. If we end in 6 then we must exclude 5 and include 3 and 4 thereby leading directly to $[0, 1, 2, 3, 4, 6]$. If we end in 5 then we must include 3 and 4 thereby leading directly to $[0, 1, 2, 3, 4, 5]$.

- $[0, 1, 3, \cdot, \cdot, \cdot]:$ 8, $T$ and $E$ are automatically excluded while 9 is automatically included thereby leading directly to $[0, 1, 3, 5, 7, 9]$, $[0, 1, 3, 4, 6, 9]$ and $[0, 1, 3, 4, 6, 9]$.

- $[0, 1, 4, \cdot, \cdot, \cdot]:$ 7, $T$ and $E$ are automatically excluded while 8 and 9 are automatically included thereby leading directly to $[0, 1, 4, 6, 8, 9]$, which is not in prime form, and $[0, 1, 4, 5, 8, 9]$. 
Figure 19: I-Combinatorial Hexachords: $T_{11}I$

- $[0, 2, 3, \cdot, \cdot, \cdot]: 8, 9$ and $E$ are automatically excluded while $T$ is automatically included thereby precluding the formation of a hexachord in prime form.

- $[0, 2, 4, \cdot, \cdot, \cdot]: 7, 9$ and $E$ are excluded while $8$ and $T$ are automatically included thereby leading directly to $[0, 2, 4, 6, 8, T]$.

By the above geometric reasoning, this exhausts all possible instances of I-combinatoriality. The resultant nineteen hexachords possessing I-combinatoriality are collected together in Figure 20. Discounting the previously encountered six all-combinatorial hexachords [8], this yields thirteen hexachords possessing only I-combinatoriality, twelve of them at a single inversional level and $[0, 1, 3, 6, 7, 9]$, which we have already seen to be R-combinatorial at two transpositional levels, at two inversional levels.
Figure 20: I-Combinatorial Hexachords

6 Conclusion

Let us summarize the results of the preceding geometric syntheses. Taken together, Figures 3 and 7 contain the 6 all-combinatorial hexachords, the 1 hexachord possessing only R- and I-combinatoriality at precisely two levels and the 1 hexachord possessing only R- and P-combinatoriality at a single level. Excluding these hexachords from Figure 13 leaves the 14 hexachords possessing only R- and RI-combinatoriality at one level. Likewise, excluding these same hexachords from Figure 20 leaves the 12 hexachords possessing only R- and I-combinatoriality at one level. This accounts for \(6 + 1 + 1 + 14 + 12 = 34\) of the 50 possible hexachords in prime form [14, p. 264].

The remaining 16 hexachords possess only the (trivial) R-combinatoriality provided by the identity transformation and they may be listed via the following geometric procedure. First of all, since the only hexachord in prime form ending in a \(T\) must be the whole-tone hexachord while the only hexachord in prime form ending in a 5 must be the chromatic hexachord, and
these two hexachords are all-combinatorial, we may restrict our attention to those hexachords opening with one of the five permissible opening triadic patterns and ending on one of the pitch-classes 9, 8, 7, 6. Thus, the remaining two pitch-classes required to round out the hexachord are chosen to respect prime form while avoiding any of the previously constructed 34 hexachords possessing nontrivial hexachordal combinatoriality. The end result of this process is on display in Figure 21.

![Figure 21: (Trivially) R-Combinatorial Hexachords](image)

The present paper has treated the occurrence of mixed/multiple combinatoriality as a residual benefit of our geometric synthesis of all types of hexachordal combinatoriality. Yet, it is possible to generate all instances of mixed/multiple combinatoriality without the need to first delineate all types of hexachordal combinatoriality. Such a direct geometric construction for hexachords possessing mixed/multiple combinatoriality will be presented in [8].
References


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