Harnack Inequalities and Applications for Stochastic Functional Differential Equations Driven by Sub-fBm

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Abstract
In this paper, Harnack inequalities are established for stochastic functional differential equations driven by sub-fractional Brownian motion (fBm) with Hurst parameter \(0 < H < 1\). As applications, strong Feller property and log-Harnack inequality are given. We also get derivative formula and give the corresponding Harnack inequality.

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1 Introduction
Under a curvature condition, Wang [16] established the following type dimension-free Harnack inequality for diffusion semigroups on a Riemannian manifold \(M\):

\[(P_t f)^{\alpha}(y) \leq (P_t f^{\alpha})(x)e^{c(t)\rho(x,y)^2}, \quad f \geq 0, t > 0, \alpha > 1, x, y \in M,\]

where \(c(t) > 0\) is explicitly determined by \(\alpha\) and the curvature lower bound. This type of inequality has been studied extensively. One can refer to the

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litteratures [8, 9, 10, 18, 19]. The Harnack inequality has various applications, see for instance, [13, 14, 17] for strong Feller property and contractivity properties; [1, 3] for short times behaviors of infinite dimensional diffusions; [4, 14] for heat kernel estimates and entropy-cost inequalities.

Recently, by the approach of coupling and Girsanov transformations Fan [7] established the Harnack inequality for stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. On the other hand, as an extension of a Brownian motion, Bojdecki et al. [5] have introduced and studied a rather general class of self-similar Gaussian processes. This process is called the sub-fractional Brownian motion (sub-fBm in short) which has properties analogous to those of fBm (self-similarity, long-rang dependence, Hölder paths), but it do not have stationary increments.

Our paper has been influenced by [7]. Following [7], in this paper we obtain the Harnack inequalities for a class of stochastic functional differential equations driven by sub-fractional Brownian motion with Hurst parameter $0 < H < 1$ using the approach of coupling and Girsanov transformations. As application, strong Feller property and log-Harnack inequality are derived. We also get derivative estimate and give the corresponding Harnack inequality.

The paper is organized as follows. In section 2, we give some preliminaries on sub-fractional Brownian motion. In section 3, we establish the Harnack inequality by using the approach of coupling and Girsanov transformations, and present some applications. In section 4, we are devoted to obtain derivative estimate and give the corresponding Harnack inequality.

2 Preliminaries

Let $S^H = \{S^H(t), t \in [0, T]\}$ be a sub-fractional Brownian motion with Hurst parameter $H \in (0, 1)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $S^H$ is a centered Gauss process with covariance function

$$\mathbb{E}(S^H(t)S^H(s)) = s^{2H} + t^{2H} - \frac{1}{2}[(t + s)^{2H} + |t - s|^{2H}].$$

In particular, if $H = \frac{1}{2}$, $S^H$ is coincides with the standard Brownian motion. $S^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $S^H$.

Now, we consider the kernels $n_H$ and $\Psi_H$ introduced in [6] and [15]

$$n_H(t, s) = \frac{\sqrt{\pi}}{2H \Gamma(H + \frac{1}{2})} s^{\frac{3}{2} - H} \frac{d}{ds} \left( \int_s^t (x^2 - s^2)^{H - \frac{1}{2}} ds \right) I(0, t)(s)$$

$$= \frac{\sqrt{\pi}}{2H \Gamma(H + \frac{1}{2})} s^{\frac{3}{2} - H} \left( \frac{(t^2 - s^2)^{H - \frac{1}{2}}}{t} + \int_s^t \frac{(x^2 - s^2)^{H - \frac{1}{2}}}{x^2} dx \right) I(0, t)(s)$$

$$\Psi_H(t, s) = \frac{\sqrt{\pi}}{2H \Gamma(H + \frac{1}{2})} s^{\frac{3}{2} - H} \left( \frac{(t^2 - s^2)^{H - \frac{1}{2}}}{t} + \int_s^t \frac{(x^2 - s^2)^{H - \frac{1}{2}}}{x^2} dx \right) I(0, t)(s)$$
and
\[
\Psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2} - H)} \left[ t^{H-\frac{3}{2}} (t^2 - s^2)^{\frac{1}{2} - H} - \left( H - \frac{3}{2} \right) \int_s^t (x^2 - s^2)^{\frac{1}{2} - H} x^{H-\frac{3}{2}} dx \right] I_{(a,t)}(s).
\] (2.1)

According to Dzhaparidze and Van Zantenet [6] and Tudor [15], we have the following relations between the sub-fractional Brownian motion and the Brownian motion.

**Lemma 2.1.** The process
\[
W(t) = \int_0^t \Psi_H(t,s) dS^H(s)
\] (2.2)
is the unique Brownian motion such that
\[
S^H(s) = c(H) \int_0^t n_H(t,s) dW(s)
\] (2.3)
where
\[
c^2 (H) = \frac{\Gamma(1 + 2H) \sin \pi H}{\pi}.
\]
Moreover $S^H$ and $W$ generate the same filtration.

The following lemma due to Mendy [11] gives an estimate of $\Psi$ which we use throughout this paper.

**Lemma 2.2.** (i) For every $H < \frac{1}{2}$ and $T > 0$,
\[
\Psi(t,s) \leq C(H) s^{2H-2}, \quad 0 \leq s < t \leq T,
\] (2.4)

(ii) For every $H > \frac{1}{2}$ and $T > 0$,
\[
\Psi(t,s) \leq C(H) s^{H-3/2} (t-s)^{1/2-H} + C'(H) s^{H-3/2}, \quad 0 \leq s < t \leq T,
\] (2.5)

where $C(H)$ and $C'(H)$ are two generic positive constants depending only on $H$.

In what follows we shall denote by $n_H$ also the operator on $L^2([0,T])$ induced by the kernel
\[
n_H(f)(t) := \int_0^t n_H(t,s) f(s) ds,
\]
and similarly for $\Psi_H$. Note that the operator $\Psi_H$ is indeed inverse of the operator $n_H$. 

\[\text{Harnack inequalities and applications ...}\]
Let $r > 0$ be fixed, and let $\mathcal{L} = C([-r,0]; \mathbb{R})$ be equipped with the uniform norm $\| \cdot \|_\infty$. We consider the following stochastic functional differential equation driven by sub-fractional Brownian motion on $\mathbb{R}$,

$$
\begin{cases}
  dX(t) = b(t, X(t))dt + F(t, X_t)dt + dS^H(t), \\
  X_0 = \xi,
\end{cases}
$$

(2.6)

where $\xi \in \mathcal{L}$, for each $t \geq 0$, $X_t \in \mathcal{L}$ is fixed as $X_t(u) = X(t + u), u \in [-r,0]$.

The aim of the paper is to consider the Harnack inequality for the equation (2.1) for $0 < H < 1$. We define

$$
P_T f(\xi) := \mathbb{E} f(X^\xi_T), \quad t \in [0,T], \quad f \in \mathcal{B}_b(\mathcal{L}),
$$

where $X^\xi_T$ is the solution to the equation (2.1) and $\mathcal{B}_b(\mathcal{L})$ denotes the set of all bounded measurable functions on $\mathcal{L}$.

### 3 Harnack inequality and their applications

Let us start with the following hypothesis (H1):

(i) $|b(t, x) - b(t, y)| \leq K_1 |x - y|, \forall x, y \in \mathbb{R}, t \in [0,T]$, where $K_1 > 0$ is a constant;

(ii) The mapping $t \mapsto b(t, 0)$ is bounded on $[0,T]$;

(iii) The map $F$ is globally Lipschitz on $\mathcal{L}$, i.e. for some $K_2 > 0$,

$$
|F(t, x) - F(t, y)| \leq K_2 \|x - y\|_\infty \quad \forall x, y \in \mathcal{L}, \ t \in [0,T].
$$

It is clear that under (H1), due to Mendy [11], the equation (2.6) has a unique solution.

**Theorem 3.1.** If (H1) holds, then for any nonnegative $f \in \mathcal{B}_b(\mathcal{L})$, $T > 0$, $p > 1$, and $x, y \in \mathcal{L}$,

$$(P_T f(y))^p \leq P_T f^p(x) \exp\left[\frac{p}{p-1} C(T, H) \rho^2(T, r, K_1, K_2, \xi, \eta)\right],$$

where $B(\cdot, \cdot)$ is the standard Beta functions,

$$(p - 1) C(T, H) = \begin{cases}
  \frac{C^2(H)T^{4H + 1}}{8H^2(4H + 1)^2}, & H < \frac{1}{2}, \\
  \frac{C^2(H)T^{2H + 2}}{2(4H + 1)^2}, & H > \frac{1}{2},
\end{cases}$$

$$
\rho(T, r, K_1, K_2, \xi, \eta) = \inf_{r \leq s \leq T} \rho(T, r, s, K_1, K_2, \xi, \eta)
$$

and

$$
\rho(T, r, s, K_1, K_2, \xi, \eta) = K_2 \|\xi - \eta\|_\infty + K_2 e^{K_1 s} |\xi(0) - \eta(0)| + \frac{K_1 |\xi(0) - \eta(0)|}{(1 - e^{-K_1 (s-r)})}.
$$
Proof. The proof will be divided into three steps.  
Step 1. As in [2] we shall employ a coupling argument. Let 
\[ G(x) = \begin{cases} \frac{x}{|x|^\varepsilon}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \]

The function \( G \) is continuous and the gradient of the convex function \( \frac{1}{1+\varepsilon}|x|^{1+\varepsilon} \) on \( \mathbb{R} \), hence it is also monotone, i.e.
\[ \langle G(x) - G(y), x - y \rangle \geq 0 \quad \forall x, y \in \mathbb{R}, \]
where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R} \). On the other hand, for \( \forall m \geq 1, x \in \mathbb{R}^+, \varepsilon \in (0,1) \), we have
\[ x^\varepsilon \leq mx + \frac{1}{m^\varepsilon}. \]

The above inequality implies that \( G \) is linear growth. Thus, for fixed \( \gamma > 0 \), by [11], there exists a unique process \( (Y(t))_{t \geq 0} \) solving
\[ \begin{cases} dY(t) = b(t, Y(t))dt + F(t, X_t)dt + \gamma \cdot G(X(t) - Y(t))dt + dS^H(t) \\ Y(0) = \eta(0), \end{cases} \quad (3.1) \]
and which we extend by \( Y(t) = \eta(t) \) for \( t \in [-r, 0) \).

Note that \( d(X(t) - Y(t)) = (b(t, X(t)) - b(t, Y(t)))dt - \gamma \cdot G(X(t) - Y(t))dt \), thus applying the Tanaka formula to \( |X(t) - Y(t)| \), we have
\[ d|X(t) - Y(t)| = \text{sgn}(X(t) - Y(t))d(X(t) - Y(t)) = \text{sgn}(X(t) - Y(t))(b(t, X(t)) - b(t, Y(t)))dt - \gamma \cdot |X(t) - Y(t)|^\varepsilon dt. \]

In view of (H1), for all \( t \geq 0 \) we get
\[ d|X(t) - Y(t)| \leq K_1 |X(t) - Y(t)| dt - \gamma \cdot |X(t) - Y(t)|^\varepsilon dt. \]
This implies that
\[ d\left( e^{-K_1 t} |X(t) - Y(t)| \right) \leq -\gamma e^{-K_1 t} \cdot |X(t) - Y(t)|^\varepsilon dt. \]

Then, by virtue of the above inequality and the chain rule, we have that
\[ \begin{aligned} d \left( e^{-K_1 t} |X(t) - Y(t)| \right)^{1-\varepsilon} &= (1 - \varepsilon) \left( e^{-K_1 t} |X(t) - Y(t)| \right)^{-\varepsilon} d \left( e^{-K_1 t} |X(t) - Y(t)| \right) \\ &\leq -\gamma (1 - \varepsilon) e^{-K_1 t(1-\varepsilon)} dt \end{aligned} \]
and
\[ \left( e^{-K_1 t} |X(t) - Y(t)| \right)^{1-\varepsilon} \leq \left( |X(0) - Y(0)|^{1-\varepsilon} - \frac{\gamma}{K_1} (1 - e^{-K_1 t(1-\varepsilon)}) \right)_+. \]
Thus,

\[ |X(t) - Y(t)| \leq e^{K_1 t} \left( |X(0) - Y(0)|^{1-\varepsilon} - \frac{\gamma}{K_1 (1 - e^{-K_1 (1-\varepsilon)})} \right)^{1/(1-\varepsilon)} \]  

(3.2)

Hence, for \( s \in (r, T] \), choosing

\[ \gamma = \gamma_s = \frac{K_1 |\xi(0) - \eta(0)|^{1-\varepsilon}}{(1 - e^{-K_1 (1-\varepsilon)(s-r)})} > 0. \]

This implies \( X(t) = Y(t) \) for all \( t \geq s - r \) and \( X_t = Y_t \) in \( C \) for all \( t \geq s \).

Step 2. Let

\[ u_t = F(t, X_t) - F(t, Y_t) - \gamma \cdot G(X(t) - Y(t)), \quad \widetilde{S}^H(t) = \int_0^t u_s ds + S^H(t), \quad t \in [0, T]. \]

Note that

\[ |u_t| \leq K_2 \|X_t - Y_t\|_\infty + \gamma |X(t) - Y(t)|^\varepsilon. \]

Thus, we have

\[ |u_t| \leq K_2 \|X_t - Y_t\|_\infty + K_2 e^{K_1 s} |\xi(0) - \eta(0)| + \frac{K_1 |\xi(0) - \eta(0)|}{(1 - e^{-K_1 (s-r)})} =: \rho(T, r, s, K_1, K_2, \xi, \eta). \]  

(3.3)

Now, due to the above inequality it holds that

\[ \int_0^T |u_t|^2 dt \leq T \rho^2(T, r, s, K_1, K_2, \xi, \eta). \]

Hence,

\[ \int_0^T u_s dv \in L^{H+\frac{1}{2}}_0([0, T]). \]

According to integral representation of sub-fractional Brownian motion and the definition of the operator \( n_H \), we deduce

\[ \widetilde{S}^H(t) = \int_0^t u_s dv + \int_0^t n_H(t, v) dW(v) \int_0^t n_H(t, v) \left[ (\Psi_H \int_0^s u_z dz)(v) dv + dW(v) \right] \]

\[ =: \int_0^t n_H(t, v) d\widetilde{W}(v). \]
Now, let
\[ R_T = \exp \left[ -\int_0^T \left( \Psi_H \int_0^z u_z \, dz \right) (v) \, dW(v) - \frac{1}{2} \int_0^T \left( \Psi_H \int_0^z u_z \, dz \right)^2 (v) \, dv \right]. \]

Next we want to show \((\tilde{S}^H(t))_{0 \leq t \leq T}\) is an \(\mathcal{F}^{S_H}_t\)-sub-fractional Brownian motion with Hurst parameter \(H\) under the new probability \(R_T P\). Due to [12], it suffices to show that \(\mathbb{E} R_T = 1\). By the definition of \(\Psi_H\) and Lemma 2.2, we have for \(H < \frac{1}{2}\)
\[
\left| (\Psi_H \int_0^z u_z \, dz)(v) \right| = \int_0^v \Psi(v, z) \cdot (\int_0^z u_r \, dr) \, dz \leq \rho(T, r, s, K_1, K_2, \xi, \eta) \int_0^v \Psi(v, z) \, zdz
\]
\[
\leq \rho(T, r, s, K_1, K_2, \xi, \eta) \cdot \frac{C(H)v^{2H}}{2H}
\]
and for \(H > \frac{1}{2}\)
\[
\left| (\Psi_H \int_0^z u_z \, dz)(v) \right| = \int_0^v \Psi(v, z) \cdot (\int_0^z u_r \, dr) \, dz \leq \rho(T, r, s, K_1, K_2, \xi, \eta) \int_0^v \Psi(v, z) \, zdz
\]
\[
\leq \rho(T, r, s, K_1, K_2, \xi, \eta) C(H) B(H + \frac{1}{2} - H) v + \rho(T, r, s, K_1, K_2, \xi, \eta) \frac{C'(H)}{H} v^{H + \frac{1}{2}}.
\]
As a consequence, we get
\[
\mathbb{E} \exp \frac{1}{2} \int_0^T (\Psi_H \int_0^z u_r \, dr)^2 (s) \, ds \leq \exp[\rho^2(T, r, s, K_1, K_2, \xi, \eta) C(T, H)].
\]

Using the Novikov criterion, we have \(\mathbb{E} R_T = 1\).

Step 3. From step 2, we can rewrite (3.1) in the following form
\[
\begin{cases}
  dY(t) = b(t, Y(t)) \, dt + F(t, Y_t) \, dt + d\tilde{S}^H(t), \\
  Y_0 = \eta,
\end{cases}
\]
where \((\tilde{S}^H(t))_{0 \leq t \leq T}\) is an \(\mathcal{F}^{S_H}_t\)-fractional Brownian motion with Hurst parameter \(H\) under the new probability \(R_T P\). By the uniqueness of the solution and \(X_T = Y_T, \text{a.s.}\), we have
\[
P_T f(y) = \mathbb{E} f(X_T^y) = \mathbb{E} R_T f(Y_T^y) = \mathbb{E} R_T f(X_T^y)
\]
Applying the Hölder inequality to (3.5), we obtain
\[
(P_T f(y))^p \leq P_T f^p(x) \cdot (\mathbb{E} R_T^p)^{p-1}.
\]
Now we will estimate moments of $R_T$.

Denote $\alpha = \frac{p}{p-1}$ and $M_T = -\int_0^T (\Psi_H \int_0^t u_z \, dz)^2 \, dW(v)$. Since $(R_t)_{0 \leq t \leq T}$ is a $\mathbb{P}$ martingale, by (3.4) we have

\[
E R_T^{\alpha} = E \exp[\alpha M_T - \frac{1}{2} \alpha \langle M \rangle_T] = E \exp[\alpha M_T - \frac{1}{2} \alpha^2 \langle M \rangle_T + \frac{1}{2} \alpha (\alpha - 1) \langle M \rangle_T]
\]

\[
\leq \exp[\alpha (\alpha - 1) C(T, H) \rho^2(T, r, K_1, K_2, \xi, \eta)].
\]

(3.6)

Substituting (3.7) into (3.6), we get the desired result.

As a direct application of the Harnack type inequalities derived above, by Proposition 3.1 of [18] we get the strong Feller property on $P_T$.

**Proposition 3.1.** Assume (H1). Then $P_T$ is strong Feller and the following estimate holds

\[
|P_T f(\xi) - P_T f(\eta)| \leq \|f\|_{\infty} [2C(T, H)]^{\frac{1}{2}} \rho(T, r, K_1, K_2, \xi, \eta) \exp[C(T, H) \rho^2(T, r, K_1, K_2, \xi, \eta)],
\]

for every $T > 0$, $\xi, \eta \in \mathcal{L}$ and $f \in \mathcal{B}_b(\mathcal{L})$.

As an immediate application of Theorem 3.1, by Corollary 1.2 of [14], we may also establish the following result on log-Harnack inequality.

**Corollary 3.1.** Let (H1) hold, then the log-Harnack inequality

\[
P_T(\log f)(\eta) \leq \log P_T f(\xi) + C(T, H) \rho^2(T, r, K_1, K_2, \xi, \eta),
\]

holds for $\forall \xi, \eta \in \mathcal{L}, \, t > 0, \, f \in \mathcal{B}_b(\mathcal{L})$ and $f \geq 1$.

## 4 Derivative formula

In this part, we begin with the following hypothesis (H3):

(i) We have $|\partial_2 b(t, x)| \leq K$, $\forall x \in \mathbb{R}$, $t \in [0, T]$, where $K > 0$ is a constant, $\partial_2 b(t, x)$ denotes the derivative for the second variable;

(ii) The mapping $t \mapsto b(t, 0)$ is bounded on $[0, T]$;

(iii) The map $F$ is globally Lipschitz on $\mathcal{L}$, i.e. for some $K_2 > 0$,

\[
|F(t, x) - F(t, y)| \leq K_2 \|x - y\|_{\infty} \quad \forall x, y \in \mathcal{L}, \, t \in [0, T].
\]

The aim is to establish a Bismut type derivative formula for $P_T$ which will imply the Harnack inequality. For $f \in \mathcal{B}_b(\mathcal{L})$, $\xi, \eta \in \mathcal{L}$, $T > r$, we will consider

\[
D_\eta P_T f(\xi) := \lim_{\varepsilon \to 0} \frac{P_T f(\xi + \varepsilon \eta) - P_T f(\xi)}{\varepsilon}.
\]
Theorem 4.1. (Derivative formula) Assume Hypothesis (H3). Then, for each $T > r$, nonnegative $f \in B_b(L)$, and $\xi, \eta \in L$, $D_\eta P_T f(\xi)$ exists and satisfies

$$|D_\eta P_T f(\xi)| \leq \mathbb{E}f(X_T^\xi)N_T,$$

where

$$N_T = \left[[\overline{K} + K_2 + \frac{1}{T-r})|\eta(0)| + K_2\|\eta\|_\infty\right] \int_0^T C'(s, H) dW_s$$

and

$$C'(s, H) = \begin{cases} \frac{C(H)s^{2H}}{H}, & H < \frac{1}{2}, \\ \frac{C'(H)s^{H+\frac{1}{2}} + C(H)B(H+\frac{1}{2}, \frac{3}{2} - H)s}{H+\frac{1}{2}}, & H > \frac{1}{2}. \end{cases}$$

**Proof.** The proof is similar to Theorem 4.1 of [9], so we omit it here. □

As an application of the derivative estimate above, by Corollary 3.1 of [7] we have the following Harnack inequality.

**Corollary 4.1.** If (H3) holds, then for any nonnegative $f \in B_b(L)$, $\xi, \eta \in L$, $T > 0$

$$(P_T f(\eta))^p \leq P_T f^p(\xi) \exp\left[\frac{p}{p-1}C'(T, H)\varrho(\overline{K}, K_2, T, r, \xi, \eta)\right],$$

where

$$\varrho(\overline{K}, K_2, T, r, \xi, \eta) = \left[[\overline{K} + K_2 + \frac{1}{T-r})|\eta(0) - \xi(0)| + K_2\|\eta - \xi\|_\infty\right]^2$$

and

$$C'(T, H) = \begin{cases} \frac{C^2(H)T_4^{4H+1}}{4H^2(4H+1)}, & H < \frac{1}{2}, \\ \frac{C^2(H)T_4^{4H+2}}{(H+\frac{1}{2})^2(H+1)} + \frac{2C^2(H)B^2(H+\frac{1}{2}, \frac{3}{2} - H)T^3}{3}, & H > \frac{1}{2}. \end{cases}$$

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