Some Inequalities Involving
Geometric and Harmonic Means

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Abstract
Using a forward-backward induction method, we proved that \( \frac{n}{1+\beta} \) is a lower bound of the series \( \sum_{i=1}^{n} \frac{1}{1+a_i} \), where \( a_i \) is a positive integer greater than 1, and \( \beta \) is the geometric mean from \( a_1 \) to \( a_n \). We also proved that \( \frac{n}{1+\gamma} \) is an upper bound of the series, where \( \gamma \) is the harmonic mean from \( a_1 \) to \( a_n \).

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1 Introduction
Upper and lower bounds are important topics in many mathematical areas. Even though calculus might be the most popular tool to study the upper and
lower bounds, one may sometimes find that the knowledge of inequalities comes in very handy. In this paper, we study the series \( \sum_{i=1}^{n} \frac{1}{1+a_i} \) for positive \( a_i \)'s, and develop several inequalities to find the upper and lower bounds of the series.

We first introduce the notations and some inequalities we will be using in this paper. For sequence \( \{a_i\}_{i=1}^{n} \) with positive \( a_i \)'s, we use \( \alpha_{(m,n)} \) to indicate the arithmetic mean from \( a_m \) to \( a_n \). Without any confusion, let \( \alpha = \alpha_{(1,n)} \).

Similarly, let \( \beta_{(m,n)} \) be the geometric mean from \( a_m \) to \( a_n \), and let \( \beta = \beta_{(1,n)} \).

For harmonic mean, let \( \gamma_{(m,n)} \) be the harmonic mean from \( a_m \) to \( a_n \), and let \( \gamma = \gamma_{(1,n)} \). We already know the famous AM-GM-HM inequality, which states: \( \alpha \geq \beta \geq \gamma \), with equality when all \( a_i \)'s are equal.

In our paper we also apply Chrystal’s Inequality. This inequality is provided as an exercise in Chrystal’s own book [3], and is mentioned as a property in [5] without proof. Here we provide our own proof.

**Theorem 1.1. (Chrystal)** Let \( a_i \) be a positive real number for all \( i \), let \( n \) be a natural number, and let \( \beta \) be the geometric mean from \( a_1 \) to \( a_n \), then

\[
\prod_{i=1}^{n} (1 + a_i) \geq (1 + \beta)^n.
\]

**Proof.** Since \( a_i > 0 \) for all \( i \), and \( \prod_{i=1}^{n} a_i = \beta^n \), we invoke Theorem 40 in [5, p.39] to prove the claim.

The Theorem 40 states: for non-negative sequences \( \{x_i\}_{i=1}^{n} \) and \( \{y_i\}_{i=1}^{n} \), if \( m_1 + m_2 + \cdots + m_n = 1 \), then

\[
(x_1 + y_1)^{m_1} (x_2 + y_2)^{m_2} \cdots (x_n + y_n)^{m_n} \geq x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} + y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n},
\]

and the equality holds when \( x_i = k y_i \) for all \( i \), where \( k \) is a fixed number.

Let \( \{x_i\}_{i=1}^{n} = \{1\}_{i=1}^{n}, \{y_i\}_{i=1}^{n} = \{a_i\}_{i=1}^{n} \), and \( m_i = \frac{1}{n} \) for all \( i \). We have

\[
\prod_{i=1}^{n} (1 + a_i)^{\frac{1}{n}} \geq \left( \prod_{i=1}^{n} 1 + \prod_{i=1}^{n} (a_i)^{\frac{1}{n}} \right) = (1 + \beta).
\]

After we exponentiate both sides by \( n \), we have the claimed inequality. And apparently the equality holds when all \( a_i \)'s are equal. \( \square \)

There is another famous inequality involving a sum of fractions, the Radon’s Inequality, which we also apply in our paper.

**Theorem 1.2. (Radon)** If \( n \in \mathbb{N}, x_i \geq 0, y_i > 0, i \in \{1, 2, \cdots, n\} \), and \( m \geq 0 \), then

\[
\frac{x_1^{m+1}}{y_1^m} + \frac{x_2^{m+1}}{y_2^m} + \cdots + \frac{x_n^{m+1}}{y_n^m} \geq \frac{(x_1 + x_2 + \cdots + x_n)^{m+1}}{(y_1 + y_2 + \cdots + y_n)^m},
\]

with equality if and only if \( \frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n} \).
For the proof of Radon’s Inequality, see [6, p.1351]. Interested readers may also see [1] for its generalizations and other comments.

## 2 Main Results

We consider the series \( \sum_{i=1}^{n} \frac{1}{1+a_i} \) for all positive \( a_i \)’s. Since all \( a_i \)’s are positive, hence all \( \frac{1}{1+a_i} \), after applying the AM-GM-HM inequality we find that

\[
\sum_{i=1}^{n} \frac{1}{1+a_i} \geq \frac{n}{\left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n}} \geq \frac{n^2}{n+(a_1+\cdots+a_n)} = \frac{n}{1+\alpha}.
\]

We already know that \( \alpha \geq \beta \geq \gamma \), so \( \frac{n}{1+\gamma} \geq \frac{n}{1+\alpha} \). Naturally, we would like to compare \( \frac{n}{\left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n}} \) and \( \sum_{i=1}^{n} \frac{1}{1+a_i} \) with \( \frac{n}{1+\beta} \) and \( \frac{n}{1+\gamma} \). Here is our first result when considering \( \frac{n}{\left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n}} \).

**Theorem 2.1.** Let \( a_i \) be a positive real number for all \( i \), let \( n \) be a natural number, and let \( \beta \) be the geometric mean from \( a_1 \) to \( a_n \), then

\[
\frac{n}{1+\beta} \geq \frac{n}{\left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n}}.
\]

**Proof.** This inequality is equivalent to \( \left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n} \geq (1+\beta) \), the Chrystal’s Inequality. \( \square \)

Since \( \beta \geq \gamma \), we easily get that \( \frac{n}{1+\gamma} \geq \sqrt[n]{\frac{n}{\left[(1+a_1)\cdots(1+a_n)\right]^\frac{1}{n}}} \) as well. We now move on to the series \( \sum_{i=1}^{n} \frac{1}{1+a_i} \).

**Theorem 2.2.** Let \( a_i \) be a positive real number for all \( i \), let \( n \) be a natural number, and let \( \beta \) be the geometric mean from \( a_1 \) to \( a_n \), we have

\begin{align*}
(i) \quad & \sum_{i=1}^{n} \frac{1}{1+a_i} \geq \frac{n}{1+\beta}, \quad \text{if} \quad a_i \geq 1; \\
(ii) \quad & \sum_{i=1}^{n} \frac{1}{1+a_i} \leq \frac{n}{1+\beta}, \quad \text{if} \quad 0 < a_i < 1.
\end{align*}

**Proof.** We use a forward-backward induction method introduced by Cauchy (see [2]) to prove (i) first.

If \( n = 1 \), this inequality is trivial.

We then use the regular mathematical induction to prove the special case when \( n = 2^m \). If \( m = 1 \), that is \( n = 2 \), we need to prove that

\[
\frac{1}{1+a_1} + \frac{1}{1+a_2} \geq \frac{2}{1+\sqrt{a_1a_2}}.
\]
Rearranging both sides, we can simplify the above inequality to
\[ \sqrt{a_1a_2} (a_1 + a_2 + 2) \geq (a_1 + a_2 + 2a_1a_2). \]

We notice that
\[ \sqrt{a_1a_2} (a_1 + a_2 + 2) - (a_1 + a_2 + 2a_1a_2) = (\sqrt{a_1a_2} - 1) (a_1 + a_2 - 2\sqrt{a_1a_2}). \]

Since \( a_i \geq 1 \), \( \sqrt{a_1a_2} - 1 \geq 0 \). Also, \( (a_1 + a_2 - 2\sqrt{a_1a_2}) = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0 \). The claimed inequality is true.

For an arbitrary but fix integer \( k > 1 \), assume that the inequality is true for \( n = 2^k \). That is
\[ \sum_{i=1}^{2^k} \frac{1}{1 + a_i} \geq 2^k \frac{1 + \beta(1,2^k)}{1 + \beta(2^k+1,2^{k+1})}. \]

according to our inductive hypothesis. Since we already proved the case when \( n = 2 \),
\[ \frac{2^k}{1 + \beta(1,2^k)} + \frac{2^k}{1 + \beta(2^k+1,2^{k+1})} \geq \frac{2^{k+1}}{1 + \sqrt{\beta(1,2^k)\beta(2^k+1,2^{k+1})}} = \frac{2^{k+1}}{1 + \beta(1,2^{k+1})}. \]

By the principle of mathematical induction, the inequality is true when \( n = 2^m \) for any positive integer \( m \).

If \( n \) is not a power of 2, there exists an integer \( k \) such that \( 2^{k-1} < n < 2^k \). Let \( a_{n+1} = a_{n+2} = \cdots = a_{2k} = \beta(1,n) = \beta \). So \( a_1 \cdots a_{2k} = \beta^{2^k} \), equivalently \( \beta(1,2^k) = \beta \). We already know that
\[ \sum_{i=1}^{2^k} \frac{1}{1 + a_i} \geq \frac{2^k}{1 + \beta(1,2^k)} = \frac{2^k}{1 + \beta}; \]

which is equivalent to
\[ \sum_{i=1}^{n} \frac{1}{1 + a_i} + \frac{2^k - n}{1 + \beta} \geq \frac{2^k}{1 + \beta}. \]

Therefore,
\[ \sum_{i=1}^{n} \frac{1}{1 + a_i} \geq \frac{n}{1 + \beta}. \]

To prove (ii) we only need to consider the special case when \( n = 2 \). Since \( a_i < 1 \), \( \sqrt{a_1a_2} - 1 < 0 \). Therefore
\[ \sqrt{a_1a_2} (a_1 + a_2 + 2) - (a_1 + a_2 + 2a_1a_2) = (\sqrt{a_1a_2} - 1) (\sqrt{a_1} - \sqrt{a_2})^2 \leq 0. \]
Some inequalities involving geometric and harmonic means

Equivalently,
\[ \frac{1}{1 + a_1} + \frac{1}{1 + a_2} \leq \frac{2}{1 + \sqrt{a_1 a_2}}. \]

Part (ii), hence the theorem, is then proved. \( \square \)

If we replace the geometric mean with the harmonic mean, we then have the upper bound of the series.

**Theorem 2.3.** Let \( a_i \) be a real number for all \( i \), let \( n \) be a natural number, and let \( \gamma \) be the harmonic mean from \( a_1 \) to \( a_n \), we have
\[ \sum_{i=1}^{n} \frac{1}{1 + a_i} \leq \frac{n}{1 + \gamma}. \]

**Proof.** We use the same technique to prove this inequality. Since \( n = 1 \) is a trivial case, we start with \( n = 2 \). We want to prove that
\[ \frac{1}{1 + a_1} + \frac{1}{1 + a_2} \leq \frac{2}{1 + \frac{2a_1 a_2}{a_1 + a_2}}. \]

After simplification, we find that this inequality is equivalent to \( 2a_1 a_2 \leq (a_1^2 + a_2^2) \), which is obviously true due to AM-GM inequality.

For an arbitrary but fix positive integer \( k \), assume that the inequality is true for \( n = 2^k \). That is,
\[ \sum_{i=1}^{2^k} \frac{1}{1 + a_i} \leq \frac{2^k}{1 + \gamma(1,2^k)}. \]

Therefore,
\[ \sum_{i=1}^{2^{k+1}} \frac{1}{1 + a_i} = \sum_{i=1}^{2^k} \frac{1}{1 + a_i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{1 + a_i} \leq \frac{2^k}{1 + \gamma(1,2^k)} + \frac{2^k}{1 + \gamma(2^k+1,2^{k+1})}. \]

Since
\[ \frac{2 \gamma(1,2^k) \gamma(2^k+1,2^{k+1})}{\gamma(1,2^k) + \gamma(2^k+1,2^{k+1})} = \frac{2 \cdot \left( \frac{2^k}{a_1 + \cdots + a_{2^k}} \right) \cdot \left( \frac{2^k}{a_{2^k+1} + \cdots + a_{2^{k+1}}} \right)}{\frac{2^k}{a_1 + \cdots + a_{2^k}} + \frac{2^k}{a_{2^k+1} + \cdots + a_{2^{k+1}}} = \frac{2^{k+1}}{a_1 + \cdots + a_{2^{k+1}}} = \gamma(1,2^{k+1}),} \]
applying special case of \( n = 2 \), we have
\[ \frac{2^k}{1 + \gamma(1,2^k)} + \frac{2^k}{1 + \gamma(2^k+1,2^{k+1})} \leq \frac{2^{k+1}}{1 + \frac{2 \gamma(1,2^k) \gamma(2^k+1,2^{k+1})}{\gamma(1,2^k) + \gamma(2^k+1,2^{k+1})}} = \frac{2^{k+1}}{1 + \gamma(1,2^{k+1})}. \]
By the principle of mathematical induction, the inequality is true when \( n = 2^m \)
for any positive integer \( m \).

If \( n \) is not a power of 2, let \( k \) be the smallest integer that \( n < 2^k \), and let \( a_{n+1} = a_{n+2} = \cdots = a_{2^k} = \gamma(1,n) = \gamma \). Since
\[
\sum_{i=1}^{2^k} \frac{1}{1 + a_i} \leq \frac{2^k}{1 + \gamma(1,2^k)},
\]
and
\[
\gamma(1,2^k) = \frac{2^k}{\frac{1}{a_1} + \cdots + \frac{1}{a_n} + \frac{2^k - n}{\gamma}} = \frac{\gamma \cdot 2^k}{n + (2^k - n)} = \gamma,
\]
we have
\[
\sum_{i=1}^{n} \frac{1}{1 + a_i} + \frac{2^k - n}{1 + \gamma} \leq \frac{2^k}{1 + \gamma}.
\]
Equivalently,
\[
\sum_{i=1}^{n} \frac{1}{1 + a_i} \leq \frac{n}{1 + \gamma}.
\]

**Remark.** As mentioned in the Introduction, the Radon’s Inequality can also be applied to the series \( \sum_{i=1}^{n} \frac{1}{1+a_i} \). However, after applying the special case when \( m = 1 \), we find that
\[
\sum_{i=1}^{n} \frac{1}{1 + a_i} \geq \frac{n^2}{n + (a_1 + a_2 + \cdots + a_n)} = \frac{n}{1 + \alpha},
\]
which is the original result when applying AM-HM inequality. It unfortunately does not improve our lower bound.

**References**


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