Another Lambert Series Result for Fibonacci Reciprocals

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Abstract

In this short note we derive another Lambert series expression for a series containing a product of Fibonacci reciprocals.

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1 Introduction

The Fibonacci numbers $F_n$ are defined for $n \geq 0$ as $F_{n+2} = F_{n+1} + F_n$ with initial conditions $F_0 = 0, F_1 = 1$. They can be expressed compactly via the Binet form given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0,$$

(1)

where $\alpha$ and $\beta$ are roots of the quadratic equation $x^2 - x - 1 = 0$, that is

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

(2)

Clearly, we have $\alpha > 1, -1 < \beta < 0$ and $\alpha \beta = -1$. We are interested in studying a special infinite sum containing a product of Fibonacci reciprocals.

1Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.
These sums are known to be challenging and some open problems still exist ([5]). In some cases the sums allow expressions in terms of other functions. Almkvist [1] derives a list of expressions in terms of theta functions. Still other reciprocal Fibonacci sums may be expressed in terms of the Lambert series. This series is defined by

\[ L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}, \quad |x| < 1. \] (3)

In [4] it is shown that

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n}} = (\alpha - \beta)[L(\beta^2) - L(\beta^4)], \] (4)

whereas the author of [2] proofs that

\[ \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1}} = 2(\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)] + \beta. \] (5)

The purpose of this short note is to fill the gap between the two results. We establish the following expression:

**Theorem 1.1**

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+1}} = (\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)] + \beta. \] (6)

**Corollary 1.2**

\[ \sum_{n=1}^{\infty} \frac{F_{2n+1} - 1}{F_{2n} F_{2n+1}} = (\alpha - \beta)[L(\beta^4) - 2L(\beta^8)] - \beta, \] (7)

\[ \sum_{n=1}^{\infty} \frac{F_{2n+1} + 1}{F_{2n} F_{2n+1}} = (\alpha - \beta)[2L(\beta^2) - 3L(\beta^4) + 2L(\beta^8)] + \beta, \] (8)

and

\[ \sum_{n=1}^{\infty} \frac{1}{F_{2n-1} F_{2n}} = (\alpha - \beta)[L(\beta^2) - 2L(\beta^4) + 2L(\beta^8)]. \] (9)
2 Proof of the Theorem

Our proof starts with the following identity for $k \geq 0$ (see Eq. (3.4) in [3]):

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}} = \frac{1}{\alpha^{k+1}F_{k+1}}. \quad (10)$$

Hence,

$$\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}} = \sum_{k=1}^{\infty} \frac{1}{\alpha^{k}F_{k}}. \quad (11)$$

Now,

$$\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}} = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}}. \quad (12)$$

Combining the last two equations and using Eq. (3.1) of [2] gives

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}} = -\frac{1}{2\alpha} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{F_{k}F_{k+1}}. \quad (13)$$

The statement follows from Eq. (5) and the observation that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{F_{n+k}F_{n+k+1}} = \sum_{k=1}^{\infty} \frac{1}{F_{2k}F_{2k+1}}. \quad (14)$$

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References


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