The r-Polynomials in Hecke Algebras with Unequal Parameters

Dongcheng Li and Xigou Zhang

College of Mathematics & Information Science
JXNU, Nanchang, P.R. China, 330022

Abstract

This paper is concerned with some important properties of r-polynomials in Hecke algebras with unequal parameters. In chapter one, we briefly introduce some conceptions in Coxeter group $\mathcal{W}$ and some notations in Hecke algebra $\mathcal{H}$. In chapter two, we give some important properties of some r-polynomials which satisfy $r_{y,w} = \prod_{i=1}^{l(w)-l(y)}(v_{s_i} - v_{s_i}^{-1})$ when $y \leq w$ in $\mathcal{W}$ and $w = s_1s_2\cdots s_q$ is any reduced expression.

Keywords: Coxeter group; Coxeter elements; Hecke algebra; r-polynomials

1. Introduction

Let $(\mathcal{W}, S)$ be a Coxeter system (see [1]), For the generators set $S$, we say the group $W$ is of rank $|S|$, and call any order product of some elements in $S$ is a Coxeter element (see [2]) of $W$.

Definition 1.1.1 (see [1]) Let $w = s_1s_2\cdots s_r \in W$ be any expression with $s_i \in S$ for $1 \leq i \leq r$. Define the length function $l(w)$ to be the smallest $r$ for which such a expression exists, and we call the expression reduced. Note that the reduced expression of $w$ is not necessary uniquely determined.
Let “≤” be the Bruhat ordering in W, equivalently \( y \leq w \) for \( y, w \in W \) if and only if \( y \) is a subexpression of any reduced expression of \( w \), which is clearly a partial order in \( W \). In particular, let \( W \) be the dihedral group \( D_m \) for any \( y, w \in W \), we can get that \( y < w \) if and only if \( l(y) < l(w) \). Moreover, if \( D_m = \langle s, t \rangle \) with \( m < \infty \), we denote \( \text{Prod} (s, t; n) = ssts \cdots \), which has \( n \) factors with \( n \geq 0 \).

**Definition 1.1.2** (see [3]) A map \( L: W \to \mathbb{Z} \) is said to be a weight function for \( W \), if \( L(ww') = L(w) + L(w') \) for any \( w, w' \in W \) such that \( l(ww') = l(w) + l(w') \). Note that \( L \) is determined by its values \( L(s) \) on \( S \) which are subject only to the condition that \( L(s) = L(s') \) for any \( s \neq s' \) in \( S \) such that \( m_{s,s'} \) is finite and odd. It is clear that \( L(1) = 0 \) and \( L(w) = L(w^{-1}) \) for all \( w \in W \).

**Definition 1.1.3** (see [3]) Let \( A = \mathbb{Z}[v, v^{-1}] \) be the Laurent polynomial ring, where \( v \) is an indeterminate. For \( s \in S \), we set \( v_s = v^{L(s)} \in A \).

**Definition 1.1.4** (see [3]) Let \( \mathcal{H} \) be the \( \mathcal{A} \)-algebra with 1 defined by the generators \( T_s (s \in S) \) and relations:
(a) \( (T_s - v_s)(T_s + v_s^{-1}) = 0 \) for all \( s \in S \);
(b) \( T_s T_s' T_s \cdots = T_{s'} T_s T_{s'} \cdots \), both sides have \( m_{s,s'} \) factors, for any \( s \neq s' \) in \( S \) such that \( m_{s,s'} < \infty \).
\( \mathcal{H} \) is called the Hecke algebra or Iwahori-Hecke algebra with unequal parameters.

For all \( s \in S \), we can see that the element \( T_s \) is invertible in \( \mathcal{H} \) by (a), and have \( T_s^{-1} = T_s - (v_s - v_s^{-1}) \). By (b) and Matsumoto’s theorem (see [4]), we have \( T_w \) is independent of the choices of a reduced expression of \( w \). Moreover, if \( w = s_1 s_2 \cdots s_q \in W \) is a reduced expression, then \( T_w^{-1} = T_{s_q}^{-1} \cdots T_{s_2}^{-1} T_{s_1}^{-1} \).

**Definition 1.1.5** (see [3]) Let \( \tilde{\cdot}: \mathcal{A} \to \mathcal{A} \) be the ring involution which takes \( v^n \) to \( v^{-n} \) for any \( n \in \mathbb{Z} \).

**Lemma 1.1.6** (see [3]) (a) There is a unique ring homomorphism \( \tilde{\cdot}: \mathcal{H} \to \mathcal{H} \) which is \( \mathcal{A} \)-semilinear with respect to \( \cdot: \mathcal{A} \to \mathcal{A} \) and satisfies \( \tilde{T}_s = T_s^{-1} \) for all \( s \in S \).
(b) This homomorphism is involutive which takes \( T_w \) to \( T_w^{-1} \) for any \( w \in W \).

**Proposition 1.1.7** (see [3]) For any \( w \in W \), we can write uniquely \( T_w = \sum_{y \in W} r_{y,w} T_y \), where \( r_{y,w} \in \mathcal{A} \) are zero for all but finitely many \( y \). In particular, \( r_{w,w} = 1 \) for all \( w \in W \), while \( r_{y,w} = 0 \) unless \( y \leq w \) for \( y, w \in W \).

**Proposition 1.1.8** (see [3]) Let \( w \in W \) and \( s \in S \) be such that \( sw < w \). For any \( y \in W \), we have
The \( r \)-polynomials in Hecke algebras with unequal parameters

(1) \( r_{y,w} = r_{sy,sw} \), if \( sy < y \);
(2) \( r_{y,w} = r_{sy,sw} + (v_s - v_s^{-1})r_{y,sw} \), if \( sy > y \).

It is sometimes useful to have alternate versions of (1) and (2), with \( s \) occurring on the right hand, rather than the left. For the right-handed version, we have the similar relations as follows:

(3) if \( y < w, ys < y, ws < w \) (forcing \( ys < ws \)), then \( r_{y,w} = r_{ys,ws} \);
(4) if \( y < w, y < ys \) (forcing \( ys \leq w \) and \( y \leq wy \)), then \( r_{y,w} = r_{ys,ws} + (v_s - v_s^{-1})r_{y,sw} \).

Now we consider a special case: \( l(w) - l(y) = 1 \). Let \( w = s_1s_2 \cdots s_r \) be a reduced expression, we can take \( y \) by omitting a single \( s_i \) with \( 1 \leq i \leq r \). Consider the \( i \)-th position of \( s_i \) with 3 cases. If \( i = 1 \), taking \( y = s_2s_3 \cdots s_r \), we can get \( r_{y,w} = r_{1,s_1} = v_{s_1} - v_{s_1}^{-1} \) by (3); if \( i = r \), taking \( y = s_1s_2 \cdots s_{r-1} \), we can get \( r_{y,w} = r_{1,s_r} = v_{s_r} - v_{s_r}^{-1} \) by (1); at last, if \( 1 < i < r \), taking \( y = s_1 \cdots s_i \cdots s_r \), we can have \( r_{y,w} = r_{1,s_i} = v_{s_i} - v_{s_i}^{-1} \) by (1) and (3).

So, for any \( 1 \leq i \leq r \), we can get the conclusion \( r_{y,w} = v_{s_i} - v_{s_i}^{-1} \).

Similarly for another special case: \( l(w) - l(y) = 2 \). we can get that \( r_{y,w} = (v_{s_i} - v_{s_i}^{-1})(v_{s_j} - v_{s_j}^{-1}) \) for \( 1 \leq i < j \leq r \).

However, if \( l(w) - l(y) \geq 3 \), the \( r \)-polynomials are rapidly become less manageable, because of the more complicated possibilities for subexpressions when more than two factors are omitted. For example, when \( W \) is of the type \( B_3 \), let \( y = 1 \) and \( w = s_1s_2s_1 \) in \( W \), we can get that 

\[
r_{y,w} = (v_{s_1} - v_{s_1}^{-1})(v_{s_2} - v_{s_2}^{-1}) + (v_{s_1} - v_{s_1}^{-1}).
\]

2. SOME PROPERTIES OF \( R \)-POLYNOMIALS

**Proposition 2.1** Let \( S = \{ s_1, s_2, \ldots, s_n \} \), and \( J \) be the set consisting of all the Coxeter elements of \( W_I \) for all \( I \subseteq S \). For \( w = s_1s_2 \cdots s_r \) we have \( r_{1,w} = \prod_{i=1}^{l(w)}(v_{s_i} - v_{s_i}^{-1}) \) if and only if \( w \in J \).

Proof. We first prove by induction \( l(w) \) that if \( w \in J \), then \( r_{1,w} = \prod_{i=1}^{l(w)}(v_{s_i} - v_{s_i}^{-1}) \). It is obvious when \( l(w) = 0, 1 \) or 2. Let \( l(w) = k \in [3, n] \), the equivalence \( r_{1,w} = \prod_{j=1}^{l(w)}(v_{s_i} - v_{s_i}^{-1}) \) hold. If \( l(w) = k + 1 \), then we can get

\[
r_{1,w} = r_{1,s_1s_2 \cdots s_{k+1}} = r_{s_1s_2 \cdots s_{k+1}} + (v_{s_1} - v_{s_1}^{-1})r_{s_1s_2 \cdots s_{k+1}}
\]

Since \( s_i \neq s_j \) for any \( 1 \leq i < j \leq k + 1 \), it implies that \( s_1 \) is not the subexpression of \( s_2 \cdots s_{k+1} \), we can get

\[
r_{s_1s_2 \cdots s_{k+1}} = 0.
\]
On the other hand, by induction, we get
\[ r_{1,s_2\ldots s_{k+1}} = \prod_{j=2}^{k+1} (v_{s_i} - v_{s_i}^{-1}). \]

Finally, we obtain
\[ r_{1,w} = (v_{s_1} - v_{s_1}^{-1}) \prod_{j=2}^{k+1} (v_{s_i} - v_{s_i}^{-1}) = \prod_{j=1}^{l(w)} (v_{s_i} - s_i) \]
as required.

Conversely, assume that \( r_{1,w} = \prod_{j=1}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) \), we need to show \( w \in J \).
Consider \( l(w) \) on two steps:
First, if \( l(w) = 1 \) or 2, it is clear.
Second, if \( l(w) \geq 3 \), then we can get that
\[ r_{1,w} = r_{1,s_1s_2\ldots s_r} = r_{s_1,s_2\ldots s_r} + (v_{s_1} - v_{s_1}^{-1})r_{1,s_2\ldots s_r}. \]
Since \( l(s_2\ldots s_r) = r - 1 \leq l(w) \), it implies \( r_{1,s_2\ldots s_r} = \prod_{i=2}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) \) by
\[ r_{1,w} = \prod_{i=1}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) \]. Hence,
\[ r_{1,w} = r_{s_1,s_2\ldots s_r} + (v_{s_1} - v_{s_1}^{-1}) \prod_{i=2}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) = r_{s_1,s_2\ldots s_r} + \prod_{i=1}^{l(w)} (v_{s_i} - v_{s_i}^{-1}). \]

So we can get \( r_{s_1,s_2\ldots s_r} = 0 \) by using \( r_{1,w} = \prod_{i=1}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) \) again. Since \( s_1 \) is not a subexpression of \( s_2\ldots s_r \), we have \( s_1 \neq s_i \) for any \( 2 < i \leq r \). By the same argument, we can obtain \( s_m \neq s_n \) for any \( 2 < m < n \leq r \). Hence, we can have \( s_i \neq s_j \) and so on for any \( 1 \leq i < j \leq r \), so \( w \) is a Coxeter element of \( W_I \) for some \( I \subseteq S \). ie. \( w \in J \).

**Corollary 2.2** Let \( S = \{s_1, s_2, \ldots, s_n\} \), and \( J \) be the set consisting of all the Coxeter elements of \( W_I \) for all \( I \subseteq S \). Finding \( w_1 \in W \) and \( w = s_{i_2}\ldots s_{i_1} \ldots s_i \) in \( J \) such that \( l(w_1w) = l(w_1) + l(w) \) (resp. \( l(ww_1) = l(w) + l(w_1) \)), we have \( r_{w_1,w_1w} = \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \) (resp. \( r_{w_1,ww_1} = \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \)).

**Proposition 2.3** Let \( S = \{s_1, s_2, \ldots, s_n\} \), and let \( J \) be the set consisting of all the Coxeter elements of \( W_I \) for all \( I \subseteq S \) with \( |I| \geq 3 \). Assume that \( w = s_{i_2}\ldots s_{i_1} \ldots s_i \) for \( w \in J \), then we have \( r_{s_{i_1},w} = \prod_{j=2}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \).

Proof. We argue it by induction on \( l(w) \), starting with the fact \( l(w) = 3 \). So,
\[ r_{s_{i_1},s_{i_2}s_{i_1}s_{i_3}} = r_{s_{i_2}s_{i_1}s_{i_1}s_{i_3}} + (v_{s_{i_2}} - v_{s_{i_2}}^{-1})r_{s_{i_1},s_{i_1}s_{i_3}} = (v_{s_{i_2}} - v_{s_{i_2}}^{-1})(v_{s_{i_3}} - v_{s_{i_3}}^{-1}) = \]
Consider the case \( l(w) > 3 \), we can find \( s = s_i \) such that \( sw < w \) while \( sy > y \). \( w \) has two possibilities:
(1) if \( w = s_i s_i s_i \cdots s_i \), we have
\[
 r_{s_i w} = r_{s_i s_i s_i \cdots s_i} + (v_{s_i} - v_{s_i}^{-1}) r_{s_i s_i s_i \cdots s_i} = r_{s_i s_i s_i s_i s_i \cdots s_i} + (v_{s_i} - v_{s_i}^{-1}) r_{s_i s_i s_i \cdots s_i}
\]
Since it is clear that \( s_i s_i \) is not a subexpression of \( s_i s_i s_i \cdots s_i \), we can get the first term
\[
 r_{s_i s_i s_i s_i s_i \cdots s_i} = 0.
\]
By Proposition 2.1, we can have the second term
\[
 r_{s_i s_i s_i \cdots s_i} = \prod_{j=3}^{l(w)} (v_{s_i} - v_{s_i}^{-1}).
\]
Combing these, we get that
\[
 r_{s_i w} = (v_{s_i} - v_{s_i}^{-1}) \prod_{j=3}^{l(w)} (v_{s_i} - v_{s_i}^{-1}) = \prod_{j=2}^{l(w)} (v_{s_i} - v_{s_i}^{-1}).
\]
as required.
(2) if \( w = s_i s_i \cdots s_i \cdots s_i \), we get
\[
 r_{s_i s_i w} = r_{s_i s_i s_i \cdots s_i s_i} + (v_{s_i} - v_{s_i}^{-1}) r_{s_i s_i s_i \cdots s_i s_i} + (v_{s_i} - v_{s_i}^{-1}) r_{s_i s_i s_i \cdots s_i}
\]
Since \( s_i s_i \) is not the subexpression of \( s_i s_i \cdots s_i \cdots s_i \), we have \( r_{s_i s_i s_i s_i s_i \cdots s_i s_i} = 0 \).
By the induction hypothesis, we have
\[
 r_{s_i s_i s_i \cdots s_i s_i} = \prod_{j=3}^{l(w)} (v_{s_i} - v_{s_i}^{-1})
\]
Combining these, we have
\[
 r_{s_i w} = \prod_{j=2}^{l(w)} (v_{s_i} - v_{s_i}^{-1}),
\]
as required.

**Corollary 2.4** Let \( W \) be the Coxeter group \( B_3 \) as in Definition 1.1.1, if \( y = s_2 \) and \( w = s_1 s_3 s_2 s_3 \) which are both in \( W \), then \( r_{y,w} \neq (v_{s_1} - v_{s_1}^{-1})(v_{s_3} - v_{s_3}^{-1})^2 \).

**Proof.** First we have
\[
 r_{y,w} = r_{s_2 s_1 s_3 s_2 s_3} = r_{s_2 s_1 s_3 s_2 s_3} + (v_{s_1} - v_{s_1}^{-1}) r_{s_2 s_1 s_3 s_2 s_3}
\]
by taking \( s = s_2 \). Since \( s_2 s_1 \) is not a subexpression of \( s_3 s_2 s_3 \), it implies \( r_{s_2 s_1, s_3 s_2 s_3} = 0 \). So,

\[
y_{v, w} = (v_{s_1} - v_{s_1}^{-1}) r_{s_2, s_3 s_2 s_3} = (v_{s_1} - v_{s_1}^{-1}) [r_{s_2 s_3, s_2 s_3} + (v_{s_3} - v_{s_3}^{-1}) r_{s_2 s_2 s_3}]
\]

On one hand, \( r_{s_2 s_3, s_2 s_3} = 1 \). On the other hand, \( r_{s_2, s_3 s_2} = v_{s_3} - v_{s_3}^{-1} \).

Combing these, we can get that

\[
y_{v, w} = (v_{s_1} - v_{s_1}^{-1}) + (v_{s_1} - v_{s_1}^{-1})(v_{s_3} - v_{s_3}^{-1})^2 \neq (v_{s_1} - v_{s_1}^{-1})(v_{s_3} - v_{s_3}^{-1})^2.
\]

as required.

**Corollary 2.5** Let \( S = \{s_1, s_2, \ldots, s_n\} \), and \( J \) be the set consisting of all the Coxeter elements of \( W_I \) for all \( I \subseteq S \), for any \( w = s_{i_1} \cdots s_{i_r} \),

(a) Assume that there exists \( s = s_{i_j} \) such that \( sw > w \) (resp. \( ws > w \)), where \( 1 \leq j \leq r \), then \( r_{1, s w} \neq (v_s - v_s^{-1}) \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \) (resp. \( r_{1, w s} \neq (v_s - v_s^{-1}) \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \)) for \( w \in J \).

(b) Assume that there exists \( s = s_{i_j} \) such that \( s_{i_1} \cdots s_{i_j} s_{i_r} > s_{i_1} \cdots s_{i_r} \), where \( 1 \leq j \leq r \), then \( r_{1, s_{i_1} \cdots s_{i_j} s_{i_r}} \neq (v_s - v_s^{-1}) \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \).

Proof. (a) It is clear that \( s \neq s_{i_1} \) by \( sw > w \), and we can have

\[
r_{1, s w} = r_{s, w} + (v_s - v_s^{-1}) r_{1, w}.
\]

By proposition 2.1, we can obtain \( r_{1, w} = \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}) \). By proposition 2.3, we can get \( r_{s, w} = \prod_{l=1, l \neq j}^{l(w)} (v_{s_{i_l}} - v_{s_{i_l}}^{-1}) \).

Combining these, we compute

\[
r_{1, s w} = \prod_{l=1, l \neq j}^{l(w)} (v_{s_{i_l}} - v_{s_{i_l}}^{-1}) + (v_s - v_s^{-1}) \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}),
\]

ie.

\[
r_{1, s w} \neq (v_s - v_s^{-1}) \prod_{j=1}^{l(w)} (v_{s_{i_j}} - v_{s_{i_j}}^{-1}),
\]

as required.

(b) Suppose that \( s \) occurs in the \( m \)th position of \( r_{1, s_{i_1} \cdots s_{i_r}} \), where \( m \) satisfies \( 1 < m \leq \left[ \frac{r+1}{2} \right] + 1 \) when \( r \) is even, and \( 1 < m \leq \left[ \frac{r+1}{2} \right] \) when \( r \) is odd.

Proceed by induction on \( m \). If \( m = 2 \), then we get

\[
r_{1, s_{i_1} s_{i_2} \cdots s_{i_r}} = r_{s_{i_1} s_{i_2} \cdots s_{i_r}} + (v_{s_{i_1}} - v_{s_{i_1}}^{-1}) r_{1, s_{i_2} \cdots s_{i_r}} \]

Since \( s_{i_1} \cdots s_{i_r} > s_{i_1} s_{i_2} \cdots s_{i_r} \), it is clear that \( s \neq s_{i_1} \), which implies that \( s_{i_1} \) is not the subexpression of \( s_{i_2} \cdots s_{i_r} \), hence \( r_{s_{i_1}, s_{i_2} \cdots s_{i_r}} = 0 \). However, by
the case (a), we get $r_{1,s_1s_2\cdots s_{ir}} \neq (v_s - v_s^{-1}) \prod_{j=2}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1})$. Thus,

$$r_{1,s_1s_2\cdots s_{ir}} \neq (v_s - v_s^{-1}) \prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}).$$

Assume that $m = k$, where $1 < k \leq \left[\frac{r+1}{2}\right] + 1$ when $r$ is even, and $1 < k \leq \left[\frac{r+1}{2}\right]$ when $r$ is odd. The case (b) holds. Now $m = k + 1$, so

$$r_{1,s_1\cdots s_kss_{k+1}\cdots s_{ir}} = r_{1,s_1s_2\cdots s_kss_{k+1}\cdots s_{ir}} + (v_{s_{i1}} - v_{s_{i1}}^{-1})r_{1,s_1s_2\cdots s_kss_{k+1}\cdots s_{ir}}$$

two cases are possible:

Case (1). Suppose that $s = s_{i1}$, by Proposition 2.1 and Proposition 2.3, we have

$$r_{1,s_1s_2\cdots s_kss_{k+1}\cdots s_{ir}} = \prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}),$$

while

$$r_{1,s_1\cdots s_kss_{k+1}\cdots s_{ir}} = \prod_{j=2}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}).$$

Hence

$$r_{1,s_1\cdots s_{ir}} = \prod_{j=2}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}) + (v_{s_{i1}} - v_{s_{i1}}^{-1})\prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}),$$

ie.

$$r_{1,s_1\cdots s_{ir}} \neq (v_s - v_s^{-1})\prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1})$$

this proves (b).

Case (2). Suppose that $s \neq s_{i1}$, note that $s$ is not the subsequence of $s_{i2} \cdots s_kss_{k+1} \cdots s_{ir}$, so we have

$$r_{s_{i1}s_{i2}\cdots s_kss_{k+1}\cdots s_{ir}} = 0.$$  

On the other hand, by the induction hypothesis, we have

$$r_{1,s_1s_2\cdots s_{k-1}s_{k}s_{k+1}\cdots s_{ir}} \neq (v_s - v_s^{-1})\prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}),$$

it can deduce

$$r_{1,s_1s_2\cdots s_{k-1}s_{k}s_{k+1}\cdots s_{ir}} \neq (v_s - v_s^{-1})\prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}).$$

Combining these, we can get that

$$r_{1,s_1\cdots s_{ir}} \neq (v_s - v_s^{-1})\prod_{j=1}^{l(w)}(v_{s_{ij}} - v_{s_{ij}}^{-1}),$$

as required.

**Proposition 2.6** Let $W$ be the dihedral group $D_m = \langle s, t \rangle$ with $m < \infty$, for any $y, w \in W$, if $l(w) - l(y) \geq 3$, then $r_{y,w} \neq \text{Prod} (v_s - v_s^{-1}, v_t - v_t^{-1}; l(w) - l(y)).$

Proof. It is clear that we just need to consider these two cases as follows:

Case (1). $y = 1$ and $w = sts \cdots$ (resp. $w = tsts \cdots$) with $l(w) \geq 3$. Then we have

$$r_{1,w} = r_{s,ts\cdots} + (v_s - v_s^{-1})r_{1,ts\cdots}.$$ 

It is obvious $r_{s,ts\cdots} \neq 0$, forcing $r_{s,ts\cdots}$ is a factor of some term in $r_{1,w}$. Hence $r_{y,w} \neq \text{Prod} (v_s - v_s^{-1}, v_t - v_t^{-1}; l(w)).$
Case (2). \( y = sts\cdots s \) and \( w = tst\cdots s \) with \( l(w) - l(y) \geq 3 \), we have

\[
r_{y,w} = r_{ty,tw} + (v_t - v_t^{-1}) r_{y,tw}.
\]

Since \( l(w) - l(y) \geq 3 \), we can get \( l(tw) - l(ty) \geq 1 \) and \( ty \leq tw \), so \( r_{ty,tw} \neq 0 \), forcing \( r_{ty,tw} \) is a factor of some term of \( r_{y,w} \). Hence

\[
r_{y,w} \neq \text{Prod} \left( v_s - v_s^{-1}, v_t - v_t^{-1}; l(w) - l(y) \right).
\]

In each case, \( r_{y,w} \neq \text{Prod} \left( v_s - v_s^{-1}, v_t - v_t^{-1}; l(w) - l(y) \right) \), as required.

References


Received: August 14, 2015; Published: October 12, 2015