Symmetric Identities for Carlitz’s Generalized Twisted $q$-Bernoulli Numbers and Polynomials

Associated with $p$-Adic $q$-Integral on $\mathbb{Z}_p$

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Abstract

In this paper, we study the symmetry for the generalized twisted $q$-Bernoulli numbers $\beta_{n,X,q,\zeta}$ and polynomials $\beta_{n,X,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the generalized twisted $q$-Bernoulli polynomials $\beta_{n,X,q,\zeta}(x)$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$.

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1 Introduction

Many mathematicians have studied different kinds of the Euler, Bernoulli, Genocchi, Tangent numbers and polynomials (see [1-11]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. Recently, Y. Hu studied several identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials in complex field (see [1]). D. Kim et al. [4] derived some identities of symmetry for generalized Carlitz’s $q$-Bernoulli numbers and polynomials by
using the $p$-adic $q$-integrals on $\mathbb{Z}_p$ in $p$-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and generalized twisted $q$-Bernoulli polynomials $\beta_{n, x, q, \zeta}(x)$ using the symmetric properties for the $p$-adic $q$-invariant integral on $\mathbb{Z}_p$.

Let $p$ be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (\text{cf. [1, 2, 3, 4]}).$$

Hence, $\lim_{q \to 1} [x] = x$ for any $x$ with $|x|_p \leq 1$ in the present $p$-adic case. For $g \in UD(\mathbb{Z}_p) = \{ g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \}$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{|p^N|_q} \sum_{x=0}^{p^N-1} g(x) q^x \quad (\text{cf. [3]}). \quad (1.1)$$

Let a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp \quad (a, p) = 1} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$\int_X g(x) d\mu_q(x) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x), \quad (\text{see [3]}). \quad (1.3)$$

We assume that $h \in \mathbb{Z}$. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{ \zeta | \zeta^{p^N} = 1 \}$ is the cyclic group of order $p^N$. For $\zeta \in T_p$, we denote by $\phi_{\zeta} : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \zeta^x$ (see [11]).

## 2 Symmetric identities for Carlitz’s twisted $q$-Bernoulli numbers and polynomials

D. Kim et al. [4] investigated interesting properties of symmetry $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ for generalized $q$-Bernoulli polynomials. By using same method of [4], expect for obvious modifications, we obtain some symmetric properties for generalized twisted $q$-Bernoulli polynomials. If we take $\zeta = 1$ in
Carlitz’s generalized twisted $q$-Bernoulli polynomials

all equations of this article, then [4] are the special case of our results. Let $\chi$ be Dirichlet’s character with conductor $d \in \mathbb{N}$ with $(d, p) = 1$. For $q \in \mathbb{C}_p$ with $|q - 1|_p < p^{-\frac{1}{r+1}}$, the twisted $q$-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ are defined by

$$
\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_{\zeta}(y) [x + y]_q^n d\mu_q(y).
$$

When $x = 0$, $\beta_{n,q,\zeta} = \beta_{n,q,\zeta}(0)$ is called the $n$-th twisted $q$-Bernoulli numbers $\beta_{n,q,\zeta}$. We introduce the generalized twisted $q$-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to $\chi$. The generalized twisted $q$-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to $\chi$ are defined by

$$
\beta_{n,\chi,q,\zeta}(x) = \int_{X} \chi(y) \phi_{\zeta}(y) [x + y]_q^n d\mu_q(y).
$$

When $x = 0$, $\beta_{n,\chi,q,\zeta} = \beta_{n,\chi,q,\zeta}(0)$ is called the $n$-th generalized twisted $q$-Bernoulli numbers $\beta_{n,\chi,q,\zeta}$.

By using $p$-adic $q$-integral, we obtain

$$
\beta_{n,\chi,q,\zeta}(x) = [d]_q^{n-1} \sum_{i=0}^{d-1} \chi(i) q^i \zeta^i \beta_{n,\chi,q,\zeta} \left( \frac{x}{d} \right).
$$

We note that

$$
\sum_{n=0}^{\infty} \beta_{n,\chi,q,\zeta} \frac{t^n}{n!} = \int_{X} \chi(y) \zeta^y e^{[x+y]_q t} d\mu_q(x).
$$

Let $w_1$ and $w_2$ be natural numbers. Then, by (1.3), we obtain

$$
\frac{1}{[w_1]_q} \int_{X} \chi(y) \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} d\mu_{q^{w_1}}(y)
$$

$$
= \lim_{N \to \infty} \frac{1}{[w_1]_q} \frac{1}{[dw_1 w_2 p^N]_q^{w_1}} \sum_{y=0}^{dw_2 p^N-1} \chi(y) \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} q^{w_1 y}
$$

$$
= \lim_{N \to \infty} \frac{1}{[dw_1 w_2 p^N]_q} \sum_{i=0}^{dw_2-1} \chi(i) q^{w_1 i} \zeta^{w_1 i} \sum_{y=0}^{p^N-1} \zeta^{dw_1 w_2 y} d\mu_{q^{w_1}}(y)
$$

$$
\times e^{[w_1 w_2 x + w_2 j + w_1 i + dw_1 w_2 y]_q t}.
$$

From (2.1), we can derive the following equation (2.2):

$$
\frac{1}{[w_1]_q} \sum_{j=0}^{dw_2-1} \chi(j) \zeta^{w_2 j} q^{w_2 j} \int_{X} \chi(y) \zeta^{w_1 y} e^{[w_1 w_2 x + w_2 j + w_1 y]_q t} d\mu_{q^{w_1}}(y)
$$

$$
= \lim_{N \to \infty} \frac{1}{[dw_1 w_2 p^N]_q} \sum_{j=0}^{dw_2-1} \sum_{i=0}^{p^N-1} \chi(i) \chi(j) \zeta^{w_2 j} \zeta^{w_1 i} q^{w_2 j} q^{w_1 i}
$$

$$
\times e^{[w_1 w_2 x + w_2 j + w_1 i + dw_1 w_2 y]_q t} \zeta^{dw_1 w_2 y} q^{dw_1 w_2 y}.
$$
By the same method as (2.2), we obtain

\[
\frac{1}{[w_2]^q} \sum_{j=0}^{dw_2-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y}e^{[w_1w_2x+w_1j+w_2y]q}t^j \, d\mu_{q}^{w_2}(y)
\]

\[
= \lim_{N \to \infty} \frac{1}{[dw_1w_2pN]^q} \sum_{j=0}^{dw_2-1} \sum_{i=0}^{dw_1-1} \sum_{y=0}^{pN-1} \chi(i)\chi(j)\zeta^{w_1i}q^{w_1j}q^{w_2i} \times e^{[w_1w_2x+w_1j+w_2i+dw_1w_2y]q}t^j \, d\mu_{q}^{w_2}(y)
\]

Therefore, by (2.2) and (2.3), we have the following theorem.

**Theorem 2.1** For \( w_1, w_2 \in \mathbb{N}, \) we have

\[
\frac{1}{[w_1]^q} \sum_{j=0}^{dw_1-1} \chi(j)\zeta^{w_2j}q^{w_2j} \int_X \chi(y)\zeta^{w_1y}e^{[w_1w_2x+w_2j+w_1y]q}t^j \, d\mu_{q}^{w_1}(y)
\]

\[
= \frac{1}{[w_2]^q} \sum_{j=0}^{dw_2-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y}e^{[w_1w_2x+w_1j+w_2y]q}t^j \, d\mu_{q}^{w_2}(y).
\]

By substituting Taylor series of \( e^{xt} \) into (2.4) and after elementary calculations, we obtain the following corollary.

**Corollary 2.2** For \( w_1, w_2 \in \mathbb{N}, n \geq 0, \) we have

\[
[w_1]^{n-1} \sum_{j=0}^{dw_1-1} \chi(j)\zeta^{w_2j}q^{w_2j} \int_X \chi(y)\zeta^{w_1y} \left[ w_2x + \frac{w_2}{w_1}j + y \right]_q^n \, d\mu_{q}^{w_1}(y)
\]

\[
= [w_2]^{n-1} \sum_{j=0}^{dw_2-1} \chi(j)\zeta^{w_1j}q^{w_1j} \int_X \chi(y)\zeta^{w_2y} \left[ w_1x + \frac{w_1}{w_2}j + y \right]_q^n \, d\mu_{q}^{w_2}(y).
\]

By Corollary 2.2, we have the following theorem.

**Theorem 2.3** For \( w_1, w_2 \in \mathbb{N}, n \geq 0, \) we have

\[
[w_1]^{n-1} \sum_{j=0}^{dw_1-1} \chi(j)\zeta^{w_2j}q^{w_2j} \beta_{n, x^{w_1} \zeta^{w_2}} \left( w_2x + \frac{w_2}{w_1}j \right)
\]

\[
= [w_2]^{n-1} \sum_{j=0}^{dw_2-1} \chi(j)\zeta^{w_1j}q^{w_1j} \beta_{n, x^{w_2} \zeta^{w_2}} \left( w_1x + \frac{w_1}{w_2}j \right).
\]
By (2.4), we can derive the following equation (2.5):
\[
\int_X \chi(y) \zeta^{w_{1}y} \left[ w_{2}x + \frac{w_{2}}{w_{1}} j + y \right]^{n} \mu_{q^{w_{1}}}(y) \\
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_{2}]_{q}}{[w_{1}]_{q}} \right)^{i} \left[ j \right]_{q^{w_{2}}}^{w_{2}(n-i)j} \int_X \chi(y) \zeta^{w_{1}y} \left[ w_{2}x + y \right]^{n-i} \mu_{q^{w_{1}}}(y) \\
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[w_{2}]_{q}}{[w_{1}]_{q}} \right)^{i} \left[ j \right]_{q^{w_{2}}}^{w_{2}(n-i)j} \beta_{n-i, \chi, q^{w_{1}}, \zeta^{w_{1}}} \left( w_{2}x \right).
\]

By (2.5) and Theorem 2.3, we have
\[
[w_{1}]_{q}^{-n} \sum_{j=0}^{d_{w_{1}}-1} \chi(j) \zeta^{w_{2}j} q^{w_{2}j} \int_X \chi(y) \zeta^{w_{1}y} \left[ w_{2}x + \frac{w_{2}}{w_{1}} j + y \right]^{n} \mu_{q^{w_{1}}}(y) \\
= \sum_{j=0}^{d_{w_{1}}-1} \chi(j) \zeta^{w_{2}j} q^{w_{2}j} \sum_{i=0}^{n} \binom{n}{i} \left[ w_{2} \right]_{q}^{i} \left[ w_{1} \right]_{q}^{n-i-1} \left[ j \right]_{q^{w_{2}}}^{w_{2}(n-i)j} \times \beta_{n-i, \chi, q^{w_{1}}, \zeta^{w_{1}}} \left( w_{2}x \right) \\
= \sum_{j=0}^{d_{w_{1}}-1} \chi(j) \zeta^{w_{2}j} q^{w_{2}j} \sum_{i=0}^{n} \binom{n}{i} \left[ w_{2} \right]_{q}^{i} \left[ w_{1} \right]_{q}^{n-i-1} \beta_{n-i, \chi, q^{w_{1}}, \zeta^{w_{1}}} \left( w_{2}x \right) \sum_{j=0}^{d_{w_{1}}-1} \zeta^{w_{2}j} q^{w_{2}(n-i+1)j} \left[ j \right]_{q^{w_{2}}}^{i} \\
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_{2} \right]_{q}^{i} \left[ w_{1} \right]_{q}^{n-i-1} \beta_{n-i, \chi, q^{w_{1}}, \zeta^{w_{1}}} \left( w_{2}x \right) S_{n,i}(d_{w_{1}}, \zeta^{w_{2}}, q^{w_{2}}),
\]
where
\[
S_{n,i}(w_{1}, \zeta, q | \chi) = \sum_{j=0}^{w_{1}-1} \chi(j) \zeta^{j} q^{(n-i+1)j} \left[ j \right]_{q}^{i}.
\]

By the same method as (2.6), we obtain
\[
[w_{1}]_{q}^{-2} \sum_{j=0}^{d_{w_{2}}-1} \chi(j) \zeta^{w_{2}j} q^{w_{2}j} \int_X \chi(y) \zeta^{w_{1}y} \left[ w_{1}x + \frac{w_{1}}{w_{2}} j + y \right]^{n} \mu_{q^{w_{2}}}(y) \\
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_{1} \right]_{q}^{i} \left[ w_{2} \right]_{q}^{n-i-1} \beta_{n-i, \chi, q^{w_{2}}, \zeta^{w_{2}}} \left( w_{1}x \right) S_{n,i}(d_{w_{2}}, \zeta^{w_{1}}, q^{w_{2}} | \chi).
\]

By (2.6) and (2.7), we have the following theorem.

**Theorem 2.4** For \( w_{1}, w_{2} \in \mathbb{N}, n \geq 0 \), we have
\[
\sum_{i=0}^{n} \binom{n}{i} \left[ w_{1} \right]_{q}^{i} \left[ w_{2} \right]_{q}^{n-i-1} \beta_{n-i, \chi, q^{w_{1}}, \zeta^{w_{1}}} \left( w_{2}x \right) S_{n,i}(d_{w_{1}}, \zeta^{w_{2}}, q^{w_{2}} | \chi) \\
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_{1} \right]_{q}^{i} \left[ w_{2} \right]_{q}^{n-i-1} \beta_{n-i, \chi, q^{w_{2}}, \zeta^{w_{2}}} \left( w_{1}x \right) S_{n,i}(d_{w_{2}}, \zeta^{w_{1}}, q^{w_{1}} | \chi).
\]
Let $\chi$ be the trivial character in Theorem 2.4. Then we also obtain the interesting identity for twisted $q$-Euler numbers as follows (see [7]):

**Theorem 2.5** Let $w_1, w_2 \in \mathbb{N}, n \geq 0$, and $\chi$ be the trivial character. Then we have

$$
\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-1} \beta_{n-i,q_w_1,q_w_2} (w_2x) S_{n,i} (w_1 \xi,w_2,q_w) = \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i-1} \beta_{n-i,q_w_1,q_w_2} (w_1x) S_{n,i} (w_2 \xi,w_1,q_w).
$$

If we take $x = 0$ in Theorem 2.4, we also obtain the interesting identity for generalized $q$-Euler numbers as follows:

**Corollary 2.6** For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$
\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-1} \beta_{n-i,q_w_1,q_w_2} (w_2x) S_{n,i} (w_1 \xi,q_w) = \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i-1} \beta_{n-i,q_w_1,q_w_2} (w_1x) S_{n,i} (w_2 \xi,q_w).
$$

**References**


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