International Mathematical Forum, Vol. 10, 2015, no. 10, 477 - 486
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2015.5757

Existence and Uniqueness of Solution for Caginalp Hyperbolic Phase Field System with Polynomial Growth Potential

Mayeul Evrard Isseret Goyaud, Fidèle Moukamba[1], Daniel Moukoko and Franck Davhys Reval Langa

Faculté des Sciences et Techniques
Université Marien NGouabi, B.P. 69, Brazzaville, Congo

Abstract

Our aim in this article is to study the existence and the uniqueness of solution for Caginalp hyperbolic phase-field system, with initial conditions, homogenous Dirichlet boundary conditions and polynomial growth potential in bounded and smooth domain.

Mathematics Subject Classification: Primary: 35D05, 35D10, 35D20, 35D25, 35D30, Secondary: 46F30

Keywords: Caginalp hyperbolic phase-field system, polynomial growth potential, Dirichlet boundary conditions, phase space

[1]Corresponding author

1 Introduction

We consider the folllowing Caginalp hyperbolic phase-field system

$$\epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}, \tag{1.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -u - \frac{\partial u}{\partial t}, \tag{1.2}$$

with homogenous Dirichlet boundary conditions

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0, \tag{1.3}$$

and initial conditions

$$u|_{t=0} = u_0,\ \frac{\partial u}{\partial t}|_{t=0} = u_1,\ \alpha|_{t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1, \tag{1.4}$$

where $\epsilon > 0$ is a relaxation parameter. $u = u(x,t)$ express a phase-field or order parameter. $\alpha = \alpha(x,t)$ is the thermal displacement or primitive of relative temperature θ,
with $\alpha(x,t) = \int_0^t \theta(x,\tau)d\tau + \alpha_0$. Ω is a bounded and smooth domain, of class C^2 in $\mathbb{R}^n$
$(1 \leq n \leq 3)$, and $\partial\Omega$ the smooth boundary, of classe C^2 of Ω.

Hypothesis of potential f.

$$f \ is \ of \ class \ C^2, \tag{1.5}$$

$$f(0) = 0, \tag{1.6}$$

$$-c_0 \leq F(s) \leq f(s)s + c_1,\ c_0 \geq 0,\ c_1 \geq 0,\ s \in \mathbb{R}, \tag{1.7}$$

$$\text{with } F(s) = \int_0^s f(\tau)d\tau,$$

$$|f'(s)| \leq c_2(|s|^{2p} + 1),\ c_2 > 0,\ p > 0,\ s \in \mathbb{R}, \tag{1.8}$$

$$f'(s) \geq -c_3,\ c_3 \geq 0,\ s \in \mathbb{R}. \tag{1.9}$$

We will precise restrictions on p when $n = 3$ about certain stapes, when being in an obligation.

Such studies have already been carried out by authors; in the case of parabolic-hyperbolic systems (see [1], [2], [4], [5], [6], [9]). We can also mention the recent work of Daniel Moukoko, for example [7] et [8] in which this system was the subject of a study with regular potential $f(s) = s^3 - s$, and also those of Doumbé Bongola brice Landry, [3] in which the parabolic-hyperbolic system governed by a potential polynomial growth is studied.

Notations.
*(.,.) denotes the scalar product on $L^2(\Omega)$, and $\|.\|$ associated norm.
*$(.,.)_X$ denotes the scalar product on X, et $\|.\|_X$ associated norm.
*$|\Omega|$ is a measure of Ω.

2 A priori estimates

Multiplying (1.1) and (1.2) respectively by $\frac{\partial u}{\partial t}$ and $\frac{\partial \alpha}{\partial t}$, and then integrating over Ω, we get the following respective equations

$$\begin{aligned} \frac{d}{dt}\left(\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+2\left(F(u),1\right)\right)+2\|\frac{\partial u}{\partial t}\|^2 &= 2(\frac{\partial \alpha}{\partial t},\ \frac{\partial u}{\partial t}), \\ \frac{d}{dt}\left(\|\alpha\|_{H^1}^2\ +\ \|\frac{\partial \alpha}{\partial t}\|^2\right)+2\|\frac{\partial \alpha}{\partial t}\|^2+\ 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2 &= -2(u,\frac{\partial \alpha}{\partial t})-2(\frac{\partial u}{\partial t},\ \frac{\partial \alpha}{\partial t}), \end{aligned} \quad (2.1)$$

whose sum is

$$\frac{d}{dt}\left(\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+2\left(F(u),1\right)+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2\right)+2\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2$$
$$=-2(u,\frac{\partial \alpha}{\partial t}),$$

from which we deduce

$$\frac{d}{dt}\left(\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+2\left(F(u)+c_0,1\right)+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2\right)+2\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2$$
$$\leq 2|(u,\frac{\partial \alpha}{\partial t})|.$$

Now, apply Holder and Young inequalities. Then we get,

$$\frac{d}{dt}E_1+2\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial \alpha}{\partial t}\|^2+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2\leq C\|u\|_{H^1}^2, \quad (2.2)$$

where

$$E_1=\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+2\left(F(u)+c_0,1\right)+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2.$$

Thanks to the estimate (1.7), we obtain

$$(F(u)+c_0,1)\geq 0. \quad (2.3)$$

The estimates (2.2) and (2.3) imply

$$\frac{d}{dt}E_1+\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial \alpha}{\partial t}\|^2+\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\leq k_1E_1.$$

Applying the Gronwall's lemma, we find

$$\|u\|_{H^1}^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2+\int_0^t\left(\|\frac{\partial u}{\partial t}(\tau)\|^2+\|\frac{\partial \alpha}{\partial t}(\tau)\|^2+\|\frac{\partial \alpha}{\partial t}(\tau)\|_{H^1}^2\right)d\tau\leq E_1(0)e^{k_1T}, \quad (2.4)$$

for all, $t\in[0,T]$.

Multiplying (1.1) by $-\Delta\frac{\partial u}{\partial t}$ and integrating over Ω. We get the equation

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2\right)+2\|\frac{\partial u}{\partial t}\|_{H^1}^2=-2(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t})+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\frac{\partial u}{\partial t}),$$

from which we deduce the inequality

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2\right)+2\|\frac{\partial u}{\partial t}\|_{H^1}^2\leq 2\int_\Omega|f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx+2(\nabla\frac{\partial \alpha}{\partial t},\nabla\frac{\partial u}{\partial t}). \quad (2.5)$$

Thanks to the estimate (1.8), we have

$$\int_\Omega|f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq c_1\int_\Omega\left(|u|^{2p}+1\right)|\nabla u||\nabla\frac{\partial u}{\partial t}|dx. \quad (2.6)$$

On the other hand, if $n=2$ and $p>0$, one finds, through the Hölder's inequality,

$$\int_\Omega|u|^{2p}|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq\|u\|_{L^{2(2p+1)}}^{2p}\|\nabla u\|_{L^{2(2p+1)}}\|\nabla\frac{\partial u}{\partial t}\|.$$

The continuous injection $H^1(\Omega)\subset L^{2(2p+1)}(\Omega)$ implies

$$\int_\Omega|u|^{2p}|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq c\|u\|_{H^1}^{2p}\|u\|_{H^2}\|\frac{\partial u}{\partial t}\|_{H^1}. \quad (2.7)$$

If $n=3$, in this case $p\leq 1$ we find, through Holder's inequality (for $p=1$)

$$\int_\Omega|u|^2|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla\frac{\partial u}{\partial t}\|,$$

which implies

$$\int_\Omega|u|^2|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq c\|u\|_{H^1}^2\|u\|_{H^2}\|\frac{\partial u}{\partial t}\|_{H^1}. \quad (2.8)$$

So starting from (2.6), (2.7) and (2.8) we deduce

$$\int_\Omega|f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx\leq c\|u\|_{H^1}^q\left(\|u\|_{H^2}+1\right)\|\frac{\partial u}{\partial t}\|_{H^1}. \quad (2.9)$$

Inserting (2.9) into (2.5). We get

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2\right)+2\|\frac{\partial u}{\partial t}\|_{H^1}^2\leq c\|u\|_{H^1}^q\left(\|u\|_{H^2}+1\right)\|\frac{\partial u}{\partial t}\|_{H^1}+2\|\frac{\partial \alpha}{\partial t}\|_{H^1}\|\frac{\partial u}{\partial t}\|_{H^1}.$$

which implies

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2\right)+\frac{1}{2}\|\frac{\partial u}{\partial t}\|_{H^1}^2\leq C\|u\|_{H^1}^{2q}\left(\|u\|_{H^2}^2+1\right)+\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2.$$

Since $u\in L^\infty(0,\,T\,;H_0^1(\Omega))$, we deduce the estimate

$$\frac{d}{dt}\left(\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2\right)+\frac{1}{2}\|\frac{\partial u}{\partial t}\|_{H^1}^2\leq C'\|u\|_{H^2}^2+\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2+C',\tag{2.10}$$

with $C'>0$.

Equation (2.1) gives the estimate

$$\frac{d}{dt}\left(\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2\right)+\|\frac{\partial \alpha}{\partial t}\|^2+2\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\leq C_1\|u\|_{H^1}^2+C_2\|\frac{\partial u}{\partial t}\|_{H^1}^2,\tag{2.11}$$

with $C_1>0$ and $C_2>0$.

Adding (2.10) and (2.11), with $u\in L^\infty(0,\,T\,;H_0^1(\Omega))$, we get the estimate

$$\frac{d}{dt}E_3+\frac{1}{2}\|\frac{\partial u}{\partial t}\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2+\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\leq k_3E_3+C'',$$

where

$$E_3=\|u\|_{H^2}^2+\epsilon\|\frac{\partial u}{\partial t}\|_{H^1}^2+\|\alpha\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}\|^2.$$

Appling the Gronwall's lemma with $t\in[0,T]$. We get

$$E_3+\int_0^t\left(\frac{1}{2}\|\frac{\partial u}{\partial t}(\tau)\|_{H^1}^2+\|\frac{\partial \alpha}{\partial t}(\tau)\|^2+\|\frac{\partial \alpha}{\partial t}(\tau)\|_{H^1}^2\right)d\tau\leq E_3(0)e^{k_3T}+C''',\tag{2.12}$$

Multiplying (1.1) by $\frac{\partial^2 u}{\partial t^2}$, We get

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2+2\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2=2(\Delta u,\frac{\partial^2 u}{\partial t^2})\ -2(f(u),\frac{\partial^2 u}{\partial t^2})\ +2(\frac{\partial \alpha}{\partial t},\frac{\partial^2 u}{\partial t^2}).$$

We deduce inequality

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2+2\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2\leq 2(\Delta u,\frac{\partial^2 u}{\partial t^2})\ +2\int_\Omega|f(u)||\frac{\partial^2 u}{\partial t^2}|dx+2(\frac{\partial \alpha}{\partial t},\frac{\partial^2 u}{\partial t^2}).\tag{2.13}$$

With the assumption (1.8), we find the following estimate

$$|f(u)|\leq\ c_2\left(\int_0^{|u|}|s|^{2p}ds+\int_0^{|u|}ds\right)=C\left(|u|^{2p+1}+|u|\right),$$

which implies

$$\int_\Omega|f(u)||\frac{\partial^2 u}{\partial t^2}|dx\leq C\int_\Omega\left(|u|^{2p}+1\right)|u||\frac{\partial^2 u}{\partial t^2}|dx.$$

We deduce the estimate

$$\int_{\Omega} |f(u)||\frac{\partial^2 u}{\partial t^2}|dx \leq C\|u\|_{H^1}^{q} \left(\|u\|_{H^1} + 1\right) \|\frac{\partial^2 u}{\partial t^2}\|.$$

Inserting above estimate into (2.13), we find the following estimate

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + 2\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2 \leq 2\|u\|_{H^2}\|\frac{\partial^2 u}{\partial t^2}\| + C\|u\|_{H^1}^{q} \left(\|u\|_{H^1} + 1\right) \|\frac{\partial^2 u}{\partial t^2}\| + 2\|\frac{\partial \alpha}{\partial t}\|\|\frac{\partial^2 u}{\partial t^2}\|.$$

Appling Holder and Young inequalities, we get

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2 \leq \frac{6}{\epsilon}\|u\|_{H^2}^2 + \frac{C}{\epsilon}\left(\|u\|_{H^1}^2 + 1\right)\| + \frac{6}{\epsilon}\|\frac{\partial \alpha}{\partial t}\|^2. \qquad (2.14)$$

Multiplying (1.2) by $-\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we get the following estimate

$$\frac{d}{dt}\left(\|\alpha\|_{H^2}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right) + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|_{H^2}^2 \leq 2\|u\|_{H^1}^2 + 2\|\frac{\partial u}{\partial t}\|_{H^1}^2. \qquad (2.15)$$

Multiplying (1.2) by $\frac{\partial^2 \alpha}{\partial t^2}$, and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right) + 2\|\frac{\partial^2 \alpha}{\partial t^2}\|^2 = 2(\Delta\alpha, \frac{\partial^2 \alpha}{\partial t^2}) + 2|(u, \frac{\partial^2 \alpha}{\partial t^2})| + 2|(\frac{\partial u}{\partial t}, \frac{\partial^2 \alpha}{\partial t^2})|.$$

Appling Holder and Young inequalities, we find the estimate

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2\right) + \|\frac{\partial^2 \alpha}{\partial t^2}\|^2 \leq C_1\|\alpha\|_{H^2}^2 + C_2\|u\|_{H^1}^2 + C_3\|\frac{\partial u}{\partial t}\|^2. \qquad (2.16)$$

In this study, we have three main results; existence theorem, uniqueness theorem, existence
theorem with more regularity.

3 Existence and Uniqueness of solution.

Theorem 3.1 (Existence). *We assume* $(u_0, u_1, \alpha_0, \alpha_1) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ *and* $F(u_0) < +\infty$*, then the problem (1.1)-(1.4) possesses at least one solution* (u, α) *such that*
$u, \alpha \in L^\infty(0, T; H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$,
$\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, *for all* $T > 0$.

The prove of this theorem is based on (2.4) and the standard Galerkin scheme (see [7]).

Theorem 3.2 (Uniqueness)**.** *Assume the hypothesis of Theorem 3.1 verified, then the problem (1.2)-(1.4) possesses a unique solution* $(u,\,\alpha)$, *such that* $u,\,\alpha \in L^\infty(0,\,T\,;H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0,\,T\,;L^2(\Omega)) \cap L^2(0,\,T\,;L^2(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0,\,T\,;L^2(\Omega)) \cap L^2(0,\,T\,;H_0^1(\Omega))$, *for all* $T > 0$, *with* $p \leq 1$ *and* $n = 3$.

Proof. *Let* $(u^{(1)},\,\alpha^{(1)})$ *and* $(u^{(2)},\,\alpha^{(2)})$ *two solutions of the problem (1.1)-(1.4), with* $(u_0^{(1)},\,u_1^{(1)},\,\alpha_0^{(1)},\,\alpha_1^{(1)}),\,(u_0^{(2)},\,u_1^{(2)},\,\alpha_0^{(2)},\,\alpha_1^{(2)}) \in H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ their respective initial data.
Let $u = u^{(1)} - u^{(2)}$ and $\alpha = \alpha^{(1)} - \alpha^{(2)}$. Then $(u,\,\alpha)$ satisfies

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u^{(1)}) - f(u^{(2)}) = \frac{\partial \alpha}{\partial t}, \tag{3.1}$$

$$\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t}, \tag{3.2}$$

$$\begin{aligned} u|_{\partial\Omega} &= \alpha|_{\partial\Omega} = 0, \\ u|_{t=0} &= u_0 = u_0^{(1)} - u_0^{(2)}, \quad \frac{\partial u}{\partial t}|_{t=0} = u_1 = u_1^{(1)} - u_1^{(2)}, \\ \alpha|_{t=0} &= \alpha_0 = \alpha_0^{(1)} - \alpha_0^{(2)}, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1 = \alpha_1^{(1)} - \alpha_1^{(2)}. \end{aligned}$$

Integrate over Ω the product of (3.1) by $\frac{\partial u}{\partial t}$. We find

$$\frac{d}{dt}\left(\|u\|_{H^1}^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2\right) + 2\|\frac{\partial u}{\partial t}\|^2 \leq 2\int_\Omega |f(u^{(1)}) - f(u^{(2)})||\frac{\partial u}{\partial t}|dx + 2(\frac{\partial \alpha}{\partial t},\,\frac{\partial u}{\partial t}). \tag{3.3}$$

Lagrange theorem gives an estimate

$$|f(u^{(1)}) - f(u^{(2)})| = |f'(\delta_0 u^{(1)} + (1-\delta_0)u^{(2)})||u|$$

$$\text{avec } 0 \leq \delta_0 \leq 1.$$

The hypothesis (1.8) allows to write, when $n = 3$ et $p \leq 1$, inequalities

$$|f'(\delta_0 u^{(1)} + (1-\delta_0)u^{(2)})| \leq c_2(|\delta_0 u^{(1)} + (1-\delta_0)u^{(2)}|^{2p} + 1) \leq c_2\left(2\left(\delta_0^2|u^{(1)}|^{2p} + (1-\delta_0)^2|u^{(2)}|^{2p}\right) + 1\right).$$

We deduce the estimate

$$|f'(\delta_0 u^{(1)} + (1-\delta_0)u^{(2)})| \leq C\left(|u^{(1)}|^{2p} + |u^{(2)}|^{2p} + 1\right).$$

Insert the above estimate into (3.3). We get

$$\frac{d}{dt}\left(\|u\|_{H^1}^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2\right) + 2\|\frac{\partial u}{\partial t}\|^2 \leq C\int_\Omega \left(\|u^{(1)}\|_{H^1}^{2p} + \|u^{(2)}\|_{H^1}^{2p} + 1\right)\|u\|_{H^1}^2\|\frac{\partial u}{\partial t}\| + 2\|\frac{\partial \alpha}{\partial t}\|\|\frac{\partial u}{\partial t}\|,$$

then we have

$$\frac{d}{dt}\left(\|u\|_{H^1}^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2\right) + \|\frac{\partial u}{\partial t}\|^2 \leq C\|u\|_{H^1}^2 + 2\|\frac{\partial \alpha}{\partial t}\|^2. \tag{3.4}$$

(u, α) verifies equation (2.1) which implies

$$\frac{d}{dt}\left(\|\alpha\|_{H^1}^2 + \|\frac{\partial \alpha}{\partial t}\|^2\right) + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \leq C\|u\|_{H^1}^2 + 2\|\frac{\partial u}{\partial t}\|^2. \tag{3.5}$$

Add together (3.4) and γ_1(3.5) with $\gamma_1 > 0$, such that 1 - $2\gamma_1 > 0$, we obtain the estimate

$$\frac{d}{dt}E_2 \,+\, C4\|\frac{\partial u}{\partial t}\|^2 \,+\, \gamma_1\|\frac{\partial \alpha}{\partial t}\|^2 \,+\, \gamma_1\|\frac{\partial \alpha}{\partial t}\|_{H^1}^2 \,\leq\, k_2 E_2$$

with $k_2 > 0$, independent of ϵ, where

$$E_2 = \|u\|_{H^1}^2 \,+\, \epsilon\|\frac{\partial u}{\partial t}\|^2 \,+\, \gamma_1\|\alpha\|_{H^1}^2 \,+\, \gamma_1\|\frac{\partial \alpha}{\partial t}\|^2.$$

Apply the Gronwall's lemma. We get

$$E_2 \,+\, \int_0^t \left(C_4\|\frac{\partial u}{\partial t}(\tau)\|^2 \,+\, \gamma_1\|\frac{\partial \alpha}{\partial t}(\tau)\|^2 \,+\, \gamma_1\|\frac{\partial \alpha}{\partial t}(\tau)\|_{H^1}^2\right) d\tau \,\leq\, E_2(0)e^{k_2 T},$$

for all $t \in [0, T]$.

We deduce the continuous dependence of the solution relative to the initial conditions, hence the uniqueness of the solution. □

The existence and uniqueness of the solution of problem (1.2) - (1.4) being proven in a larger space, we will seek a solution with more regularity.

Theorem 3.3. *We assume that* (u, α) *is solution of problem (1.2)-(1.4), and in that* $p \leq 1$ *and* $n = 3$*, with* $(u_0, u_1, \alpha_0, \alpha_1) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ *then, the problem (1)-(4) has a unique solution* (u, α) *such that* $u, \alpha \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$,
$\frac{\partial u}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega))$ *and* $\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; L^2(\Omega))$*, for all* $T > 0$.

Proof. *By Theorems 3.1 and 3.2, the problem (1.2)-(1.4) has a unique solution* (u, α) *such that* $u, \alpha \in L^\infty(0, T; H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, for all $T > 0$. *Thanks to those hypothesis,*
one can affirm :

* *(2.12) implies* $u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.

* *Since* $\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ *and* $u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega))$, *then (2.14) implies* $\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega))$.

* *Since* $u \in L^\infty(0, T; H_0^1(\Omega))$ and $\frac{\partial u}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$,*(2.15) implies*
$\alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)))$, $\frac{\partial \alpha}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega))$.

* *Since* $u \in L^\infty(0, T; H_0^1(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega))$ *and* $\alpha \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$, *(2.16) implies* $\frac{\partial^2 \alpha}{\partial t^2} \in L^2(0, T; L^2(\Omega))$. □

References

[1] Haim Brezis, *Analyse Fonctionnelle, Theorie et Application,* Masson, Paris, 1983.

[2] Laurence Cherfils, Stefania Gatti and Alain Miranville, Corrigendum to "Exitence and globalsolutions to the Caginalp phase-fields system with dynamic boundary conditions and singular potentials", *J. Math. Anal. Appl.*, **348** (2008), no. 2, 1029-1030. http://dx.doi.org/10.1016/j.jmaa.2008.07.058

[3] Doumbé Bongola brice Landry, *Etude de modèles de champ de phases de type Caginalp.* Thèse soutenue à la Faculté des Sciences Fondamentales et Appliquées de Poitiers, le 30 mai 2013.

[4] Alain Miranville and Ramon Quintanilla, Some generalizations of the Caginalp phase-field system, *Applicable Analysis*, **88** (2009), no. 6, 877-894. http://dx.doi.org/10.1080/00036810903042182

[5] Alain Miranville and Ramon Quintanilla, A generalizations of the Caginalp phase-field system based on the Cattaneo law, *Nonlinear Analysis: Theory, Methods and Applications*, **71** (2009), 2278-2290.
http://dx.doi.org/10.1016/j.na.2009.01.061

[6] A. Miranville and R. Quintanilla, A Caginalp phase-field system with nonlinear coupling, *Nonlinear Analysis: Real World Appl.*, **11** (2010), 2849-2861. http://dx.doi.org/10.1016/j.nonrwa.2009.10.008

[7] Moukoko Daniel, Well-posedness and long time behavior of a hyperbolic Caginalp system, *Journal of Applied Analysis and Computation*, **4** (2014), no. 2, 151-196.

[8] Daniel Moukoko, *Etude de modèles hyperboliques de champ de phases de Caginalp,* These unique, Faculté des Sciences et Techniques, Université Marien NGOUABI, 23 janvier 2015.

[9] Daniel Moukoko, Fidele Moukamba, Langa Franck Davhys Reval, Global Attractor for Caginalp Hyperbolic Field-phase System with Sin-

gular Potential, *Journal of Mathematics Research*, **7** (2015), 165-177. http://dx.doi.org/10.5539/jmr.v7n3p165

Received: August 11, 2015; Published: September 11, 2015