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Beckmann's Formula for Multiclass Traffic Equilibrium Problems

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Abstract

In this paper, we introduce the multiclass traffic equilibrium problem which includes the traffic equilibrium problem as a special case, and extend Beckmann's formula to the case of multiclass traffic equilibrium problems. By an example, we illustrate the calculation process of the multiclass traffic equilibrium flow using multiclass Beckmann's formula and show that multiclass Beckmann's formula is a sufficient condition only for the multiclass traffic equilibrium flow.

Keywords: The multiclass traffic equilibrium problem, Multiclass traffic equilibrium flow, Beckmann's formula

1 Introduction

Wardrop (1952) introduced a traffic equilibrium problem with a scalar cost function. Beckmann et al.(1956) constructed a mathematical programming problem which is equivalent to Wardrop's traffic equilibrium problem. In real world, there often are multiclass vehicle on road, such as car, truck, bus, motorcycle and so on, so in recent years, the multiclass traffic equilibrium problem has attracted much attention. In this paper, we introduce the multiclass traffic equilibrium problem (briefly, MTEP) which includes the traffic equilibrium problem (briefly, TEP) as a special case, and extend Beckmann's formula to

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the case of the multiclass traffic equilibrium problem. By an example, we illustrate the calculation process of the multiclass traffic equilibrium flow using multiclass Beckmann's formula and show that multiclass Beckmann's formula is a sufficient condition only for the multiclass traffic equilibrium flow. For other results with respect to multiclass traffic equilibrium problems, we refer to Dafermos (1972,1973), Nagurney (2000), Nagurney and Dong (2002), Yang and Huang (2004), Li and Chen (2006), Zhang et. al. (2008), Zhu et. al. (2012), Xu et al. (2014) and the references therein.

2 Preliminaries

For a traffic network, assume that there are q classes of vehicles, for example, truck, bus, car and so on and denote by $Q = \{1, 2, \cdots, q\}$. Let V denote the set of nodes and E the set of directed arcs, and W the set of origin-destination (O-D) pairs. For each $\omega \in W$, let P_ω denote the set of available paths joining O-D pair ω and $m = \sum_{\omega\in W} \mid P_\omega \mid$. Let $D = (d_{(\omega,s)})_{s\in Q,\omega\in W}$ denote the demand vector, where $d_{(\omega,s)}(> 0)$ denotes the traffic demand of class s of vehicles on O-D pair ω. For each $\alpha \in E$, the arc flow $f_\alpha = (f_{(\alpha,1)}, f_{(\alpha,2)}, \cdots, f_{(\alpha,q)})^T \in R^q_+$, where $f_{(\alpha,s)}(s \in Q)$ denotes the flow of class s of vehicles on arc α. For each $s \in Q, \omega \in W, k \in P_\omega$, let $f_{(k,s)}(\geq 0)$ denote the traffic flow of class s of vehicles on path k. $f = (f_{(k,s)})^T_{s\in Q,w\in W,k\in P_\omega} = (f_{(1,1)}, \cdots, f_{(m,1)}, f_{(1,2)}, \cdots, f_{(m,2)}, \cdots, f_{(1,q)}, \cdots, f_{(m,q)})^T \in R^{mq}_+$ is said to be a path flow (briefly, flow). Clearly, for $\alpha \in E$, $s \in Q$, $f_{(\alpha,s)} = \sum_{\omega\in W} \sum_{k\in P_\omega} \delta_{\alpha k} f_{(k,s)}$, where $\delta_{\alpha k} = 1$ if arc α belongs to path k, otherwise $\delta_{\alpha k} = 0$, thus $f_\alpha = f_\alpha(f)$. A traffic network is usually denoted by $\aleph = \{V, E, W, D\}$. For each $s \in Q, \omega \in W$, the flow f needs to satisfy the demand constraint: $\sum_{k\in P_\omega} f_{(k,s)} = d_{(\omega,s)}$. A flow f satisfying the demand constraints is called a feasible path flow (briefly, feasible flow). Let $A = \{f \in R^{mq}_+ : \forall s \in Q, \omega \in W, \sum_{\omega\in P_\omega} f_{(k,s)} = d_{(\omega,s)}\}$. Clearly, A is convex, compact and $A \neq \emptyset$. For each $\alpha \in E, s \in Q$, let $t_{(\alpha,s)} = t_{(\alpha,s)}(f_\alpha)$ be a cost of class s of vehicles on arc a and for each $\omega \in W,\ k \in P_w$, the cost $t_{(k,s)}$ of class s of vehicles along path k is assumed to be the sum of all the arc cost along k, i.e., $t_{(k,s)}(f) = \sum_{\alpha\in E} \delta_{\alpha k} t_{(\alpha,s)}(f)$. Denote that for each $s \in Q$, $t_s(f) = (t_{(1,s)}(f), t_{(2,s)}(f), \cdots, t_{(m,s)}(f))$ and

$$t(f) = \begin{bmatrix} t_{(1,1)}(f) & \cdots & t_{(m,1)}(f) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_{(1,2)}(f) & \cdots & t_{(m,2)}(f) & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & t_{(1,q)}(f) & \cdots & t_{(m,q)}(f) \end{bmatrix}.$$

Definition 1. (Multiclass equilibrium principle). A flow $f \in A$ is said to be in equilibrium if:

$$\forall s \in Q, \omega \in W, \forall k, j \in P_\omega,\ t_{(k,s)}(f) - t_{(j,s)}(f) > 0$$

$$\Rightarrow f_{(k,s)} = 0.$$

f is said to be a multiclass equilibrium flow (briefly, equilibrium flow) or multiclass equilibrium.

A MTEP is usually denoted by $\Gamma = \{\aleph, A, t\}$. x is said to be a solution of Γ if x is a multiclass equilibrium flow of Γ.

3 The multiclass Beckmann's formula

For the MTEP $\Gamma = \{\aleph, A, t\}$, construct the following mathematical programming problem MP:

$$\begin{array}{ll} Min z(f) = \Sigma_{s=1}^{q} \Sigma_{\alpha \in E} \int_0^{f_{(\alpha,s)}} t_{(\alpha,s)}(x)dx & \\ s.t. \quad \left\{ \begin{array}{ll} \sum_{k \in P_\omega} f_{(k,s)} = d_{(\omega,s)}, & \forall s \in Q, \omega \in W \\ f_{(k,s)} \geq 0, & \forall \omega \in W, k \in P_\omega. \end{array} \right. \end{array}$$

Above formula is a generalization of Beckmann's formula, which is called the multiclass Beckmann's formula. Next theorem shows that each solution of the multiclass Beckmann's formula is an equilibrium flow for Γ.

Theorem 1. Consider a MTEP $\Gamma = \{\aleph, A, t\}$. Assume that for each $s \in Q, \alpha \in E$, $t_{(\alpha,s)}(f)$ is continuous on R_+^m, then the flow $f \in A$ is in equilibrium if f solves the mathematical programming problem MP.

proof. Denote that $h_{(\omega,s)} = \sum_{k \in P_\omega} f_{(k,s)} - d_{(\omega,s)}$. The problem MP's Kuhn-Tucker conditions are:

$$\left\{ \begin{array}{ll} \frac{\partial z[f]}{\partial f_{(k.s)}} - \Sigma_{s \in Q, \omega \in W} \lambda_{(\omega,s)} \frac{\partial h_{(\omega,s)}}{\partial f_{(k,s)}} - \beta_{(k,s)} = 0 & \forall s \in Q, \omega \in W, k \in P_\omega \\ \beta_{(k,s)} f_{(k,s)} \geq 0, & \forall s \in Q, \omega \in W, k \in P_\omega \\ \lambda_{((\omega,s)} \geq 0, \beta_{(k,s)} \geq 0, & \forall s \in Q, \omega \in W, \alpha \in E, k \in P_\omega \end{array} \right.$$

where $\lambda_{(\omega,s)}$ and $\beta_{(k,s)}$ are Lagrange multipliers.

Since for each $s \in Q, \alpha \in E$, $t_{(\alpha,s)}(f)$ is continuous on R_+, we have

$$\frac{\partial z[f]}{\partial f_{(k,s)}} = \frac{\partial}{\partial f_{(k,s)}} (\Sigma_{\alpha \in E} \int_0^{f_{(\alpha,s)}} t_{(\alpha,s)}(x)dx) = \Sigma_{\alpha \in E} \frac{\partial}{\partial f_{(\alpha,s)}} \int_0^{f_{(\alpha,s)}} t_{(\alpha,s)}(x)dx \cdot \frac{\partial f_{(\alpha,s)}}{\partial f_{(k,s)}}$$

$$= \Sigma_{\alpha \in E} t_{(\alpha,s)}(f) \delta_{\alpha k} = t_{(k,s)}, \qquad \Sigma_{\omega \in W} \lambda_{(\omega,s)} \frac{\partial h_{(\omega,s)}}{\partial f_{(k,s)}} = \lambda_{(\omega,s)}.$$

Thus, we have $f_{(k,s)}(t_{(k,s)} - \lambda_{(\omega,s)}) = 0, \forall s \in Q, \omega \in W, k \in P_\omega$, i.e.,

$$\begin{array}{lll} \text{if } f_{(k,s)} > 0, & t_{(k,s)} = \lambda_{(\omega,s)} & \forall s \in Q, \omega \in W, k \in P_\omega \\ \text{if } f_{(k,s)} = 0, & t_{(k,s)} \geq \lambda_{(\omega,s)} & \forall s \in Q, \omega \in W, k \in P_\omega \end{array}.$$

In other words, for each path k, we have $t_{(k,s)} \geq \lambda_{(\omega,s)}$. Hence, for $\forall s \in Q, \omega \in W, \forall k, j \in P_\omega,$, if $t_{(k,s)}(f) - t_{(j,s)}(f) > 0$, then $f_{(k,s)} = 0$, otherwise $f_{(k,s)} > 0$, which implies that $t_{(k,s)} = \lambda_{(\omega,s)} \leq t_{(j,s)}$, a contradiction. By Definition 1, the proof is finished.

By Theorem 1, it is easy to construct algorithms to calculate the equilibrium flow for MTEP.

Example 1. Consider the MTEP (see Figure 1), where $V = \{1, 2, 3, 4\}$, $E = \{e_1, e_2, e_3, e_4\}$, $,Q = \{1, 2\}$, $W = \{\omega\} = \{(1, 4)\}$, $D = (d_{(\omega,1)}, d_{(\omega,2)}) = (3, 4)$, and

$$t_{(e_1,1)}(f_{e_1}) = 3f_{(e_1,1)} + 2f_{(e_1,2)} + 40, t_{(e_2,1)}(f_{e_2}) = 4f_{(e_2,1)} + 5f_{(e_2,2)} + 10,$$

$$t_{(e_3,1)}(f_{e_3}) = 5f_{(e_3,1)} + 2f_{(e_3,2)} + 30, t_{(e_4,1)}(f_{e_4}) = 7f_{(e_4,1)} + 6f_{(e_4,2)} + 40,$$

$$t_{(e_1,2)}(f_{e_1}) = 10f_{(e_1,1)} + 5f_{(e_1,2)} + 35, t_{(e_2,2)}(f_{e_2}) = 6f_{(e_2,1)} + 3f_{(e_2,2)} + 47,$$

$$t_{(e_3,2)}(f_{e_3}) = 5f_{(e_3,1)} + 4f_{(e_3,2)} + 42, t_{(e_4,2)}(f_{e_4}) = 4f_{(e_4,1)} + 6f_{(e_4,2)} + 38.$$

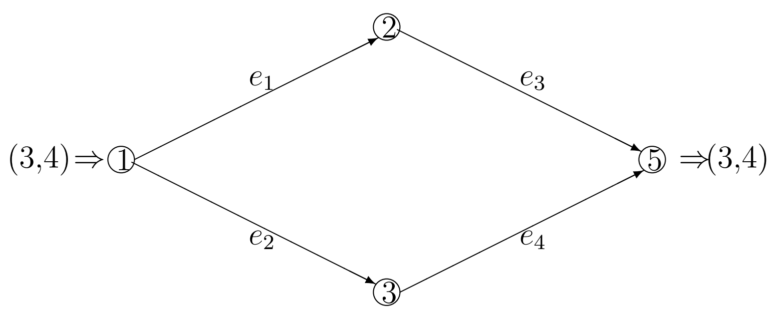

Figure 1: A traffic network

For O-D pairs $\omega = (1, 4)$: P_ω contains paths $l_1 = (e_1e_3)$, $l_2 = (e_2e_4)$. Denote by $f_{(l_i,s)} = f_{is}(i = 1, 2; s = 1, 2)$, where $f_{(l_i,s)}$ denotes the flow of class s of vehicles on path l_i.

Let $f = (f_{11}, f_{21}, f_{12}, f_{22})^T \in R^4_+$. Then we have

$f_{(e_1,1)} = f_{(e_3,1)} = f_{11}, f_{(e_1,2)} = f_{(e_3,2)} = f_{12}, f_{(e_2,1)} = f_{(e_4,1)} = f_{21}, f_{(e_2,2)} = f_{(e_4,2)} = f_{22}$.

Note that

$$\Sigma_{s=1}^{q} \Sigma_{\alpha \in E} \int_0^{f_{(\alpha,s)}} t_{(\alpha,s)}(x)dx = \int_0^{f_{11}} (3x+2f_{12})+40)dx + \int_0^{f_{11}} (5x+2f_{(12)}+30)dx$$

$$+\int_0^{f_{21}}(4x+5f_{(22)}+10)dx+\int_0^{f_{21}}(7x+6f_{(22)}+40)dx+\int_0^{f_{12}}(10f_{(11)}+5x+35)dx$$
$$+\int_0^{f_{12}}(5f_{11)}+4x+42)dx+\int_0^{f_{22}}(6f_{(21)}+3x+47)dx+\int_0^{f_{22}}(4f_{(21)}+6x+38)dx$$
$$=4f_{11}^2+19f_{11}f_{12}+70f_{11}+\frac{11}{2}f_{21}^2+21f_{21}f_{22}+50f_{21}+\frac{9}{2}f_{12}^2+77f_{12}+\frac{9}{2}f_{22}^2+85f_{22}.$$

By Theorem 1, we obtain the following mathematical programming problem MP_1:

$$Minz(f)=4f_{11}^2+19f_{11}f_{12}+70f_{11}+\tfrac{11}{2}f_{21}^2+21f_{21}f_{22}+50f_{21}+\tfrac{9}{2}f_{12}^2+77f_{12}+\tfrac{9}{2}f_{22}^2+85f_{22}$$
$$s.t.\quad \begin{cases} f_{11}+f_{21}=3\\ f_{12}+f_{22}=4\\ f_{is}\geq 0(i=1,2;s=1,2.)\end{cases}$$

It is easy to verify that $f=(f_{11},f_{21},f_{12},f_{22})^T=(0,3,4,0)^T$ is the solution of the multiclass Beckmann's formula MP_1 ($Minz(f)=579.5$). Clearly, f is a multiclass equilibrium flow.

Note that $g=(3,0,0,4)^T$ ($z(g)=658$) is also a multiclass equilibrium flow of MTEP , but it is not a solution of above mathematical programming problem MP_1, i.e., Theorem 1 is a sufficient condition only, not a necessary condition.

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