Some Higher Separation Axioms in Soft Čech Closure Spaces

R. Gowri
Department of Mathematics
Govt. College for Women (A)
Kumbakonam, India

G. Jegadeesan
Department of Mathematics
Anjalai Ammal Mahalingam College
Kovilvenni, India

Copyright © 2015 R. Gowri and G. Jegadeesan. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The aim of present paper is to introduce and characterise some higher separation axioms in soft Čech closure spaces.

Mathematics Subject Classification: 54A05, 54B05, 54D10

Keywords: Soft quasi regular, Soft semi regular, Soft pseudo regular, Soft regular, Soft semi normal, Soft normal, Completely soft normal

1 Introduction

The separation properties in closure spaces were introduced by E.Čech [2]. According to him any point $x$ can be separated by distinct neighbourhoods in a regular space. In a paper on biclosure spaces [3] the authors have proved results concerning pairwise $T_0, T_1, T_2$ quasi Hausdorff and Uryshon space. R.Gowri and G.Jegadeesan [8] discussed some properties of separation axioms on soft Čech closure spaces and also studied the relation between soft Čech
closure space \((F_A, k)\) and those in the associated soft topological space \((F_A, \tau)\). The present paper is devoted to study of some higher separation properties of soft Čech closure space.

2 Preliminaries

In this section, we recall the basic definitions of soft Čech closure spaces.

**Definition 2.1** Let \(X\) be an initial universe set, \(A\) be a set of parameters. Then the function \(k : P(X_{F_A}) \rightarrow P(X_{F_A})\) defined from a soft power set \(P(X_{F_A})\) to itself over \(X\) is called Čech Closure operator if it satisfies the following axioms:

1. \((C1)\) \(k(\emptyset_A) = \emptyset_A\).
2. \((C2)\) \(F_A \subseteq k(F_A)\).
3. \((C3)\) \(k(F_A \cup G_A) = k(F_A) \cup k(G_A)\).

Then \((X, k, A)\) or \((F_A, k)\) is called a soft Čech closure space.

**Definition 2.2** A soft subset \(U_A\) of a soft Čech closure space \((F_A, k)\) is said to be soft \(k\)-closed (soft closed) if \(k(U_A) = U_A\).

**Definition 2.3** A soft subset \(U_A\) of a soft Čech closure space \((F_A, k)\) is said to be soft \(k\)-open (soft open) if \(k(U^C_A) = U_A^C\).

**Definition 2.4** A soft set \(\text{Int}(U_A)\) with respect to the closure operator \(k\) is defined as \(\text{Int}(U_A) = F_A - k(F_A - U_A) = [k(U_A^C)]^C\). Here \(U_A^C = F_A - U_A\).

**Definition 2.5** A soft subset \(U_A\) in a soft Čech closure space \((F_A, k)\) is called Soft neighbourhood of \(e_F\) if \(e_F \in \text{Int}(U_A)\).

**Definition 2.6** If \((F_A, k)\) be a soft Čech closure space, then the associate soft topology on \(F_A\) is \(\tau = \{U_A^C : k(U_A) = U_A\}\).

**Definition 2.7** Let \((F_A, k)\) be a soft Čech closure space. A soft Čech closure space \((G_A, k^*)\) is called a soft subspace of \((F_A, k)\) if \(G_A \subseteq F_A\) and \(k^*(U_A) = k(U_A) \cap G_A\), for each soft subset \(U_A \subseteq G_A\).

3 Some Separation Axioms in Soft Čech Closure Space

In this section, we introduce and characterise some higher separation axioms in Soft Čech closure spaces.
Definition 3.1 A soft Čech closure space \((F_A, k)\) is said to be soft quasi regular if for every soft point \((x, u)\) and soft closed set \(V_A\) not containing \((x, u)\), there exists soft open set \(U_A\) such that \((x, u) \in U_A\) and \(k[U_A] \cap V_A = \emptyset_A\).

Definition 3.2 A soft Čech closure space \((F_A, k)\) is said to be soft semi regular if for every soft point \((x, u)\) and soft closed set \(V_A\) not containing \((x, u)\), there exists soft open set \(U_A\) such that \(V_A \subset U_A\) and \((x, u) \notin k[U_A]\).

Definition 3.3 A soft Čech closure space \((F_A, k)\) is said to be soft pseudo regular if it is both soft quasi regular and soft semi regular.

Definition 3.4 A soft Čech closure space \((F_A, k)\) is said to be soft regular if for every soft point \((x, u)\) and soft closed set \(V_A\) not containing \((x, u)\), there exists soft open sets \(U_A\) and \(W_A\) such that \(V_A \subset U_A, (x, u) \in W_A\) and \(U_A \cap W_A = \emptyset_A\).

Example 3.5 Let the initial universe set \(X = \{u_1, u_2\}\) and \(E = \{x_1, x_2, x_3\}\) be the parameters. Let \(A = \{x_1, x_2\} \subseteq E\) and \(F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_1, u_2\})\}\). Then \(P(X_{F_A})\) are \(F_{1A} = \{(x_1, \{u_1\})\}, F_{2A} = \{(x_1, \{u_2\})\}, F_{3A} = \{(x_1, \{u_1, u_2\})\}\), \(F_{4A} = \{(x_2, \{u_1\})\}, F_{5A} = \{(x_2, \{u_2\})\}, F_{6A} = \{(x_2, \{u_1, u_2\})\}\), \(F_{7A} = \{(x_1, \{u_1\}), (x_2, \{u_1\})\}, F_{8A} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\}\), \(F_{9A} = \{(x_1, \{u_2\}), (x_2, \{u_1\})\}, F_{10A} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}\), \(F_{11A} = \{(x_1, \{u_1\}), (x_2, \{u_1, u_2\})\}, F_{12A} = \{(x_1, \{u_2\}), (x_2, \{u_1, u_2\})\}\), \(F_{13A} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_1\})\}, F_{14A} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}\), \(F_{15A} = F_A, F_{16A} = \emptyset_A\).

An operator \(k : P(X_{F_A}) \rightarrow P(X_{F_A})\) is defined from soft power set \(P(X_{F_A})\) to itself over \(X\) as follows.
\[
\begin{align*}
  k(F_{1A}) &= k(F_{2A}) = k(F_{3A}) = F_{3A},
  k(F_{4A}) &= k(F_{5A}) = k(F_{6A}) = F_{6A},
  k(F_{7A}) &= k(F_{8A}) = k(F_{9A}) = k(F_{10A}) = k(F_{11A}) = k(F_{12A}) = k(F_{13A})
  k(F_{14A}) &= k(F_{15A}) = F_A, k(\emptyset_A) = \emptyset_A.
\end{align*}
\]

Here, the Soft Čech closure space \((F_A, k)\) is soft regular.

Theorem 3.6 Every soft subspace of a soft regular space is also soft regular.

Proof. Let \((F_A, k)\) be a soft regular Čech closure space. Let \((G_A, k^*)\) be soft subspace of \((F_A, k)\). Let \((x, u) \in F_A\) and \(V_A\) be soft \(k^*\) – closed set in \(G_A\). Therefore \(V_A = L_A \cap G_A\), where \(L_A\) is soft \(k\) – closed in \(F_A\). Since, \((F_A, k)\) is soft regular, there exist soft \(k\) – open sets \(U_A\) and \(W_A\) such that \(L_A \subset U_A, (x, u) \in W_A\) and \(U_A \cap W_A = \emptyset_A\). Therefore, there exist \(k^*\) neighbourhood \(U_A \cap G_A\) of \(V_A(= L_A \cap G_A)\) and \(k^*\) neighbourhood \(W_A \cap G_A\) of \((x, u)\) such that \((U_A \cap G_A) \cap (W_A \cap G_A) = \emptyset_A\). Thus \((G_A, k^*)\) is also soft regular.
Theorem 3.7 If \((F_A, \tau)\) is soft regular, then \((F_A, k)\) is also soft regular.

Proof. Let \((F_A, \tau)\) be soft regular. Then for each soft point \((x, u)\) in \(F_A\) and each soft \(\tau\) – closed set \(V_A\) not containing \((x, u)\), there exists soft \(\tau\) – open sets \(U_A\) and \(W_A\) such that \((x, u) \in W_A, V_A \subseteq U_A\) and \(U_A \cap W_A = \emptyset\). Since, soft neighbourhood in \((F_A, \tau)\) is also a soft neighbourhood in \((F_A, k)\). \(U_A\) and \(W_A\) are soft neighbourhoods of \(V_A\) and \((x, u)\) in \((F_A, k)\) respectively such that \(U_A \cap W_A = \emptyset\). Therefore, \((F_A, k)\) is also soft regular.

Theorem 3.8 In a soft Čech closure space \((F_A, k)\), every soft closed subspace of a soft pseudo regular space is also soft pseudo regular.

Proof. Let \((F_A, k)\) be a soft pseudo regular Čech closure space. Let \((G_A, k^*)\) be soft closed subspace of \((F_A, k)\). Let \(V_A\) be \(k^*\) soft closed in \(G_A\). Then \(V_A\) is \(k\) soft closed in \(F_A\). Let \((y, u) \in G_A\) be a soft point not in \(V_A\). Since, \((F_A, k)\) is a soft pseudo regular, there exist soft open sets \(U_A\) of \(V_A\) and \(W_A\) of \((y, u)\) such that \((y, u) \in W_A, k^*[W_A] \cap V_A = \emptyset\) and \(V_A \subseteq U_A\), \((y, u) \notin k[U_A]\). Then, \(U_A \cap G_A\) and \(W_A \cap G_A\) are \(k^*\) soft open sets of \(V_A\) and \((y, u)\) in \(G_A\) respectively such that \((y, u) \in W_A \cap G_A, k^*[W_A \cap G_A] \cap V_A = \emptyset\) and \(V_A \subseteq U_A \cap G_A\), \((y, u) \notin k^*[U_A \cap G_A]\). Thus \((G_A, k^*)\) is soft pseudo regular.

Theorem 3.9 A soft Čech closure space \((F_A, k)\) is soft quasi regular iff for each soft point \((x, u) \in F_A\) and soft \(k\) – open neighbourhood \(U_A\) of \((x, u)\), there exists a soft \(k\) – open neighbourhood \(V_A\) of \((x, u)\) such that \((x, u) \in V_A \subseteq k[V_A] \subseteq U_A\).

Proof. Let \((F_A, k)\) be soft quasi regular. Let \((x, u) \in F_A\) and \(U_A\) be soft \(k\) – open neighbourhood of \((x, u)\). Then \(F_A \setminus U_A\) is soft \(k\) – closed. Since, \((F_A, k)\) is soft quasi regular, for each soft point \((x, u)\) in \(F_A\) and soft \(k\) – closed set \(F_A \setminus U_A\), there exists a soft \(k\) – open set \(V_A\) such that \((x, u) \in V_A\) and \((F_A \setminus U_A) \cap k[V_A] = \emptyset\). Therefore, \(k[V_A] \subseteq U_A\). Thus \((x, u) \in V_A \subseteq k[V_A] \subseteq U_A\). Conversely, suppose the condition holds. Let \((x, u)\) in \(F_A\) and \(U_A\) be soft \(k\) – closed set not containing \((x, u)\) in \(F_A\). Then \((F_A \setminus U_A)\) is soft \(k\) – open set containing \((x, u)\). That is \((F_A \setminus U_A)\) is soft \(k\) – open neighbourhood of \((x, u)\). Therefore, there exists a soft \(k\) – open set \(V_A\) such that \((x, u) \in V_A \subseteq k[V_A] \subseteq F_A \setminus U_A\). That is \((x, u) \in V_A \text{ and } k[V_A] \cap U_A = \emptyset\). Therefore, \((F_A, k)\) is soft quasi regular.

Definition 3.10 A soft Čech closure space \((F_A, k)\) is said to be soft semi normal, if for each pair of disjoint soft closed sets \(U_A\) and \(V_A\), there exists a soft open set \(W_A\) such that \(U_A \subseteq W_A\) and \(k[W_A] \cap V_A = \emptyset\) or there exists a soft open set \(L_A\) such that \(V_A \subseteq L_A\) and \(k[L_A] \cap U_A = \emptyset\).

If both conditions hold, then \((F_A, k)\) is said to be soft pseudo normal.
Definition 3.11 A soft Čech closure space \((F_A, k)\) is said to be soft normal, if for each pair of disjoint soft closed sets \(U_A\) and \(V_A\), there exists disjoint soft open sets \(L_A\) and \(W_A\) such that \(U_A \subseteq L_A\) and \(V_A \subseteq W_A\).

Theorem 3.12 If \((F_A, \tau)\) is soft normal, then \((F_A, k)\) is also soft normal.

Proof. The proof is similar to the proof of theorem 3.7.

Theorem 3.13 In a soft Čech closure space \((F_A, k)\), every soft closed subspace of a soft normal space is also soft normal.

Proof. The proof is similar to proof of theorem 3.8.

Theorem 3.14 A soft Čech closure space \((F_A, k)\) is soft normal implies that given soft closed set \(U_A\) and soft open set \(V_A\) such that \(U_A \subseteq V_A\), there exist soft open set \(L_A\) and soft closed set \(G_A\) such that \(U_A \subseteq L_A \subseteq G_A \subseteq V_A\).

Proof. Let \((F_A, k)\) be soft normal check Čech closure space. Let \(U_A \subseteq V_A\), where \(U_A\) is soft closed set and \(V_A\) is soft open set. Therefore, \(U_A \cap (F_A - V_A) = \emptyset_A\) and \((F_A - U_A)\) is soft closed. Therefore, there exist soft open sets \(L_A\) and \(W_A\) such that \(U_A \subseteq L_A\), \(F_A - V_A \subseteq W_A\) and \(L_A \cap W_A = \emptyset_A\). \(F_A - V_A \subseteq W_A \Rightarrow F_A - W_A \subseteq V_A\). \(L_A \cap W_A = \emptyset_A \Rightarrow L_A \subseteq F_A - W_A \subseteq V_A\). Also, \(U_A \subseteq L_A \subseteq F_A - W_A \subseteq V_A\). If we take \(F_A - W_A = G_A\), then \(G_A\) is soft closed set. Thus we have, \(U_A \subseteq L_A \subseteq G_A \subseteq V_A\).

Definition 3.15 A soft Čech closure space \((F_A, k)\) is said to be completely soft normal if for each pair of disjoint soft closed sets \(U_A\) and \(V_A\) in \(F_A\), there exist soft open sets \(L_A, G_A\) such that \(U_A \subseteq L_A\), \(V_A \subseteq G_A\) and \(k[L_A] \cap k[G_A] = \emptyset_A\).

Theorem 3.16 If \((F_A, k)\) is completely soft normal then it is soft normal.

Proof. If \((F_A, k)\) is completely soft normal Čech closure space. Let \(U_A\) and \(V_A\) are any two disjoint soft closed sets. Since, \((F_A, k)\) is completely soft normal, then there exist soft open sets \(L_A, G_A\) such that \(U_A \subseteq L_A\), \(V_A \subseteq G_A\) and \(k[L_A] \cap k[G_A] = \emptyset_A\). By the axiom of soft Čech closure space, we have, \(L_A \cap G_A = \emptyset_A\). Thus, \((F_A, k)\) is soft normal.

Theorem 3.17 If \((F_A, k)\) is completely soft normal then every soft subspace is soft normal.

Proof. Let \((F_A, k)\) is completely soft normal and \((G_A, k^*)\) be subspace. Let \(U_A\) and \(V_A\) are any two disjoint soft closed sets in \((G_A, k^*)\). Then \(U_A = M_A \cap G_A\) and \(V_A = N_A \cap G_A\), where \(M_A\) and \(N_A\) are soft closed sets in \((F_A, k)\). Then, there exist \(L_A\) and \(W_A\) soft open sets in \(F_A\) such that \(M_A \subseteq L_A\), \(N_A \subseteq W_A\) and \(L_A \cap W_A = \emptyset_A\). This implies \(M_A \cap G_A \subseteq L_A \cap G_A\), \(N_A \cap G_A \subseteq W_A \cap G_A\) and \((L_A \cap G_A) \cap (W_A \cap G_A) = \emptyset_A\). Thus, \((G_A, k^*)\) is soft normal.
References

http://dx.doi.org/10.1016/j.camwa.2011.05.016


http://dx.doi.org/10.12785/amis/070527

http://dx.doi.org/10.4995/agt.2014.2268

http://dx.doi.org/10.14445/22315373/ijmtt-v9p513

http://dx.doi.org/10.1016/s0898-1221(99)00056-5


http://dx.doi.org/10.1016/j.camwa.2011.02.006

Received: June 29, 2015; Published: October 10, 2015