Some New Bounds on the Minimum Eigenvalue of Nonsingular $M$-Matrices

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Abstract

If $A$ is a nonsingular $M$-matrix, some new upper and lower bounds on the minimum eigenvalue $\tau(A)$ of $A$ are given. These bounds improve some results of Tian and Huang and are more accurate than Li et al. in some cases. In addition, we obtain some new bounds on the minimum eigenvalue of a nonsingular $M$-matrix $A$, which can give affirmative conclusions about whether bounds for the minimum eigenvalue of a general nonsingular $M$-matrix $A$ are only depending on the entries of $A$ and they are easy to calculate.

Mathematics Subject Classification: 15A06, 15A42, 15A48

Keywords: nonsingular $M$-matrix, Hadamard product, minimum eigenvalue, eigenvector

1 Introduction

Nonsingular $M$-matrices are widely used in many fields of mathematics, such as Markov chain issues of probabilities and statistics, the linear complementarily problems of optimization and problems of input and output in economics et al., which are closely related to the eigenvalues of nonsingular $M$-matrices. Thus, estimating the bounds for $\tau(A)$ has earned people’s attention and various
related bounds can be found in [1, 2, 3, 4, 5]. In this paper, we establish some
new bounds of $\tau(A)$ for a nonsingular $M$-matrices. These new bounds improve
the latest results in [1, 2, 3]. Numerical examples show the advantages of the
results obtained.

For a positive integer $n$, $N$ denotes the set $\{1, 2, \cdots, n\}$. The set of all $n \times n$
complex matrices is denoted by $\mathbb{C}^{n \times n}$ and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$
real matrices throughout.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $n \times n$ matrices. Then $A \geq B (> B)$
if $a_{ij} \geq b_{ij} (> b_{ij})$ for all $1 \leq i \leq n$, $1 \leq j \leq n$. If $O$ is the null matrix
and $A \geq O (> O)$, we say that $A$ is a nonnegative matrix. The Perron-
Frobenius theorem (see [6]) tells us that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ is the set of
all eigenvalues of $A$.

For $n \geq 2$, a matrix $A \in \mathbb{C}^{n \times n}$ is reducible if there exists a permutation
matrix $P$ such that

$$
P^T A P = \begin{pmatrix} A_{11} & A_{12} \\
0 & A_{22} \end{pmatrix},
$$

where $A_{11}$ is an $r \times r$ submatrix and $A_{22}$ is an $(n-r) \times (n-r)$ submatrix, where
$1 \leq r < n$. If no such permutation matrix exists, then $A$ is irreducible. If $A$
is a $1 \times 1$ complex matrix, then $A$ is irreducible if its single entry is nonzero,
and reducible otherwise.

Let $A$ be an irreducible nonnegative matrix. It is well known that there
exists a positive vector $u$ such that $Au = \rho(A) u$, $u$ being called right Perron
eigenvector of $A$.

Let $Z_n$ denote the set of $n \times n$ real matrices all of whose off-diagonal entries
are nonpositive. A matrix $A$ is called a nonsingular $M$-matrix (see [6]) if $A \in Z_n$
and the inverse of $A$, denoted by $A^{-1}$, is nonnegative. If $A$ is a
nonsingular $M$-matrix, then there exists a positive eigenvalue of $A$ equal to
$\tau(A) = [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the spectral radius of the nonnegative
matrix $A^{-1}$. $\tau(A) = \min\{\lambda : \lambda \in \sigma(A)\}$ is called the minimum eigenvalue of
$A$ (see [2]). The Perron-Frobenius theorem tells us that $\tau(A)$ is an eigenvalue
of $A$ corresponding to a nonnegative eigenvector $x = (x_1, x_2, \cdots, x_n)^T$.
If, in
addition, $A$ is irreducible, then $\tau(A)$ is simple and $x > 0$. Let $D$ is the diagonal
matrix of $A$ and $C = D - A$, then the spectral radius of the Jacobi iterative
matrix $J_A = D^{-1}C$ of $A$, denoted by $\rho(J_A)$, is less than $1$ (see [7]).

For two matrices $A = (a_{ij}) \in \mathbb{C}^{m \times n}$ and $B = (b_{ij}) \in \mathbb{C}^{m \times n}$, the Hadamard
product of $A$ and $B$ is the matrix $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{m \times n}$. If $A \geq O$
and $B$ is a nonsingular $M$-matrices, then it is clear that $A \circ B^{-1} \geq O$ (see [6]).

Let $\zeta(A)$ represent the set of all simple circuits in the digraph $\Gamma(A)$ of
$A$. Recall that a circuit of length $k$ in $\Gamma(A)$ is an ordered sequence $\gamma =
(i_1, i_2, \cdots, i_{k+1})$, where $i_1, i_2, \cdots, i_{k+1} \in N$ are all distinct, $i_1 = i_{k+1}$. The set
$\{i_1, i_2, \cdots, i_k\}$ is called the support of $\gamma$ and is denoted by $\gamma$. The length
of the circuit is denoted by $|\gamma|$ (see [8]).
For convenience, we employ the following notations throughout. Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, we denote, for any $i, j \in N$,

$$R_i(A) = R_i = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad C_i(A) = C_i = \sum_{j=1, j \neq i}^{n} |a_{ji}|,$$

$$R'_i(A) = R'_i = \sum_{j=1, j \neq i}^{n} a_{ij}, \quad C'_i(A) = C'_i = \sum_{j=1, j \neq i}^{n} a_{ji},$$

$$\sigma_i(A) = \sigma_i = \frac{R_i}{|a_{ii}|}, \quad \delta_i(A) = \delta_i = \frac{C_i}{|a_{ii}|}.$$

Recall that $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is called diagonally dominant by rows(by columns) if $\sigma_i \leq 1, \delta_i \leq 1$, respectively) for all $i \in N$. If $\sigma_i < 1, \delta_i < 1$ for all $i \in N$, we say that $A$ is a strictly diagonally dominant by rows (by columns, respectively) (see [6]).

In this paper, we first present some lemmas which will be useful in the following proofs in section 2. An upper bound for the spectral radius of the Hadamard product of a nonnegative matrix and the inverse of a nonsingular $M$-matrix is obtained in section 3, which is an improvement on some related results in [1] and [3] and more accurate than some estimations in [2] in certain cases. In section 4, we will give some new upper and lower bounds of $\tau(A)$ which improve some related results in [1] and are more accurate than some estimations in [2] in certain cases. Meanwhile, we also establish two new bounds of $\tau(A)$ for a nonsingular $M$-matrix $A$, which are only depending on the entries of matrix $A$ and are easy to calculate.

## 2 Preliminaries

In this section, we start with some lemmas which will be useful in the proofs.

**Lemma 2.1** [9] Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an irreducible nonsingular $M$-matrix. Then there exists two positive vectors $q = (q_1, q_2, \cdots, q_n)^T$ and $d^T = (d_1, d_2, \cdots, d_n)$, where $q$ and $d^T$ are right eigenvector and left eigenvector of $\rho(J_A)$, respectively. That is, there exists two positive diagonal matrices $Q = \text{diag}(q_1, q_2, \cdots, q_n)$ and $D = \text{diag}(d_1, d_2, \cdots, d_n)$, for all $i \in N$, such that

$$\sum_{k=1, k \neq i}^{n} |a_{ik}|q_k = \rho(J_A)a_{ii}q_i, \quad \sum_{k=1, k \neq i}^{n} |a_{ki}|d_k = \rho(J_A)a_{ii}d_i,$$

where $\rho(J_A)$ is the spectral radius of the Jacobi iterative matrix $J_A$ of $A$.

**Lemma 2.2** [10] Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, then
(i) If \( A \) is a strictly diagonally dominant matrix by rows, then \( A^{-1} = (v_{ij})_{n \times n} \) exists, and
\[
|v_{ji}| \leq \sigma_j |v_{ii}|, \quad \text{for all } i \neq j;
\]

(ii) If \( A \) is a strictly diagonally dominant matrix by columns, then \( A^{-1} = (v_{ij})_{n \times n} \) exists, and
\[
|v_{ij}| \leq \delta_j |v_{ii}|, \quad \text{for all } i \neq j.
\]

Lemma 2.3 [6] Let \( A, B \in \mathbb{C}^{m \times n} \), suppose that \( D \) and \( Q \) are \( m \times m \) and \( n \times n \) diagonal matrices, respectively, then
\[
D(A \circ B)Q = (DAQ) \circ B = (DA \circ BQ) = (AQ) \circ (DB) = A \circ (DBQ).
\]

Lemma 2.4 [8] Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, and let \( \zeta(A) \neq \phi \). Then for any diagonal matrix \( B \in \mathbb{R}^{n \times n} \) with positive diagonal entries, we have
\[
\min_{\gamma \in \zeta(A)} \left[ \prod_{i \in \gamma} R_i(B^{-1}AB) \right]^{\frac{1}{n}} \leq \rho(A) \leq \max_{\gamma \in \zeta(A)} \left[ \prod_{i \in \gamma} R_i(B^{-1}AB) \right]^{\frac{1}{n}}.
\]

Lemma 2.5 [6] Let \( A \in \mathbb{Z}_n \), \( A \) is a nonsingular M-matrix if and only if all its leading principal minors are positive.

Lemma 2.6 [6] Let \( A = (a_{ij})_{n \times n} \) be a nonsingular M-matrix, \( B = (b_{ij})_{n \times n} \in \mathbb{Z}_n \) and \( A \leq B \). Then \( B \) is a nonsingular M-matrix and \( A^{-1} \geq B^{-1} \geq O \).

Lemma 2.7 [6] Let \( A, B \in \mathbb{R}^{n \times n} \) and \( O \leq A \leq B \), then \( \rho(A) \leq \rho(B) \).

Lemma 2.8 [7] Let \( A \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, and \( B \in \mathbb{R}^{n \times n} \) be a nonsingular M-matrix, then
\[
|\det(A \circ B^{-1})| \leq \rho^n(A \circ B^{-1}).
\]

Lemma 2.9 [5] Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular M-matrix and \( A^{-1} = (v_{ij})_{n \times n} \). Then
\[
\min_{i \in N} \frac{1}{n} \sum_{j=1}^{n} a_{ij} \leq \frac{1}{n} \sum_{i \in N} v_{ij} \leq \tau(A) \leq \frac{1}{n} \sum_{i \in N} \min_{j \in N} v_{ij} \leq \max_{i \in N} \sum_{j=1}^{n} a_{ij}.
\]

Lemma 2.10 [1] Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular M-matrix and \( A^{-1} = (v_{ij})_{n \times n} \). Then, (i) If \( A \) is a strictly diagonally dominant matrix by rows, then
\[
\frac{1}{a_{ii}} \leq v_{ii} \leq \frac{1}{a_{ii}} + \sum_{j=1, j \neq i}^{n} a_{ij} \sigma_j \leq \frac{1}{R_i(A)}, \quad \text{for all } i \in N;
\]
(ii) If \( A \) is a strictly diagonally dominant matrix by columns, then
\[
\frac{1}{a_{ii}} \leq v_{ii} \leq \frac{1}{a_{ii} + \sum_{j=1, j \neq i}^{n} a_{ij} \delta_{j}} \leq \frac{1}{C_{i}(A)}, \quad \text{for all } i \in N.
\]

Lemma 2.11 [9] Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular M-matrix, and \( B \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, then
(i) If \( A \) is a strictly diagonally dominant matrix by rows, then
\[
\min_{1 \leq i \leq n} R_{i}^{r}(B) / R_{i}^{r}(A) \leq \rho(A^{-1}B) \leq \max_{1 \leq i \leq n} R_{i}^{r}(B) / R_{i}^{r}(A);
\]
(ii) If \( A \) is a strictly diagonally dominant matrix by columns, then
\[
\min_{1 \leq i \leq n} C_{i}^{r}(B) / C_{i}^{r}(A) \leq \rho(A^{-1}B) \leq \max_{1 \leq i \leq n} C_{i}^{r}(B) / C_{i}^{r}(A).
\]

Lemma 2.12 [2] Let \( B = (a_{ij}) \in \mathbb{C}^{n \times n} \), \( n \geq 2 \), for any \( \lambda \in \sigma(A) \), there is a pair of distinct integers \( i \) and \( j \) in \( N \) such that
\[
\lambda \in K_{i,j}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq R_{i}R_{j} \}.
\]

Lemma 2.13 [11] Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a nonnegative matrix, then
\[
\rho(A) \leq \frac{1}{2} \max_{i \neq j} \{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4R_{i}R_{j}]^{1/2}\}.
\]

Lemma 2.14 [12] Let \( A \in \mathbb{R}^{n \times n} \) be an irreducible nonsingular M-matrix, then
\[
\rho(A) = \min_{x \geq 0, x \neq 0} \max_{x_i \neq 0} \frac{(Ax)_i}{x_i}.
\]

Lemma 2.15 [13] Let \( A \in \mathbb{R}^{n \times n} \) be an irreducible nonsingular M-matrix, and \( Az \geq kz \) for a nonnegative nonzero vector \( z \), then \( \tau(A) \geq k \).

3 Upper bound for the spectral radius of the Hadamard product of a nonnegative matrix and the inverse of a nonsingular M-matrix

In this section, we give a new upper bound for \( \rho(A \circ B^{-1}) \), where \( A \) is a nonnegative matrix and \( B \) is a nonsingular M-matrix.
Theorem 3.1  Suppose that $A = (a_{ij})_{n \times n}$ is a nonnegative matrix and $B = (b_{ij})_{n \times n}$ is a nonsingular M-matrix. Let $B^{-1} = (w_{ij})_{n \times n}$. Then

$$\rho(A \circ B^{-1}) \leq \max_{\gamma \in \zeta(A \circ B^{-1})} [\prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}))]^{\frac{1}{|\gamma|}}. \quad (1)$$

Proof. Case 1: Both $A$ and $B$ are irreducible. From the Perron-Frobenius theorem, there exists a positive vector $u = (u_1, u_2, \cdots, u_n)^T$ such that $Au = \rho(A)u$. By Lemma 2.1, there exists a diagonal matrix $D = \text{diag}(d_1, d_2, \cdots, d_n)$ with positive diagonal entries, such that

$$\sum_{k=1, k \neq i}^n |b_{ki}|d_k = \rho(J_B)b_{ii}d_i \quad \text{for all } i \in N, \quad (2)$$

and $\rho(J_B) < 1$, then $DB$ is a strictly diagonally dominant matrix by columns. Notice that $(DB)^{-1} = (w_{ij}/d_j)_{n \times n}$, by Lemma 2.2 (ii) and Equation (2), for $j \neq i$, we have

$$\frac{w_{ij}}{d_j} \leq \delta_j(DB)\frac{w_{ii}}{d_i} = \sum_{k=1, k \neq j}^n \frac{|b_{kj}|d_k}{b_{jj}d_j} \frac{w_{ii}}{d_i} = \rho(J_B)\frac{w_{ii}}{d_i}, \ i.e., \ \frac{w_{ij}d_i}{d_j} \leq w_{ii}\rho(J_B). \quad (3)$$

Let $U = \text{diag}(u_1, u_2, \cdots, u_n)$, $W = D^{-1}U$, then $W$ is a diagonal matrix with positive diagonal entries. By Lemma 2.3, we have

$$W^{-1}(A \circ B^{-1})W = (U^{-1}D)(A \circ B^{-1})(D^{-1}U) = U^{-1}(A \circ (DB^{-1}D^{-1}))U = (U^{-1}AU) \circ (DB^{-1}D^{-1}) = \hat{A} \circ \hat{B}^{-1},$$

where $\hat{A} = (\hat{a}_{ij})_{n \times n} = U^{-1}AU$, $\hat{B} = (\hat{b}_{ij})_{n \times n} = DBD^{-1}$. By Inequality (3), we have

$$R'_i(\hat{A} \circ \hat{B}^{-1}) = R'_i(W^{-1}(A \circ B^{-1})W)$$

$$= a_{ii}w_{ii} + \sum_{j=1, j \neq i}^n \hat{a}_{ij}\hat{w}_{ij} = a_{ii}w_{ii} + \sum_{j=1, j \neq i}^n a_{ij}w_{ij}u_jd_i$$

$$\leq a_{ii}w_{ii} + w_{ii}\rho(J_B)\sum_{j=1, j \neq i}^n \frac{a_{ij}u_j}{u_i} = a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}).$$

By Lemma 2.4, we have

$$\rho(A \circ B^{-1}) \leq \max_{\gamma \in \zeta(A \circ B^{-1})} [\prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}))]^{\frac{1}{|\gamma|}}.$$
Case 2: One of $A$ and $B$ is reducible. By replacing the zeros of $A$ and $B$ with $\varepsilon$ and $-\varepsilon$, respectively, we obtain the nonnegative matrix $A(\varepsilon)$ and a Z-matrix $B(-\varepsilon)$, both irreducible. By Lemma 2.5, we know that all leading principal minors of $B$ are positive, then all leading principal minors of $B(-\varepsilon)$ are positive if $\varepsilon$ is a sufficiently small positive number. Thus, $B(-\varepsilon)$ is a nonsingular $M$-matrix. Now we substitute $A(\varepsilon)$ and $B(-\varepsilon)$ for $A$ and $B$, respectively, in the previous case. Let $\varepsilon \to 0$, the result follows by continuity.

**Corollary 3.1** Suppose that $A = (a_{ij})_{n \times n}$ is a nonnegative matrix and $B = (b_{ij})_{n \times n}$ is a nonsingular $M$-matrix. Let $B^{-1} = (w_{ij})_{n \times n}$. Then

$$|\det(A \circ B^{-1})| \leq \rho^n(A \circ B^{-1})$$

$$\leq \left\{ \max_{\gamma \in (A \circ B^{-1})} \prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})) \right\}^{\frac{1}{n}}.$$ 

**Remark 3.1** The upper bound in Theorem 3.1 is sharper than the upper bounds derived in Lemma 2.6 in [1] and Theorem 5.74 in [3]. Since

$$\rho(A \circ B^{-1}) \leq \max_{\gamma \in (A \circ B^{-1})} \prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}))$$

$$\leq \max_{\gamma \in (A \circ B^{-1})} \left\{ \prod_{i \in \gamma} \max_{1 \leq i \leq n} [(a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}))] \right\}^{\frac{1}{n}}$$

$$= \max_{1 \leq i \leq n} \{a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})\}.$$ 

Furthermore, since $\rho(J_B) < 1$, it follows that

$$\max_{1 \leq i \leq n} \{a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})\} \leq \rho(A) \max_{1 \leq i \leq n} w_{ii}.$$ 

Since $\text{diag}(w_{11}, w_{22}, \cdots, w_{nn}) \leq B^{-1}$, by Lemma 2.7, we can obtain $\max_{1 \leq i \leq n} w_{ii} \leq \rho(B^{-1})$. Thus

$$\max_{1 \leq i \leq n} \{a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})\} \leq \rho(A) \max_{1 \leq i \leq n} w_{ii} \leq \rho(A) \rho(B^{-1}).$$

According to the above inequalities, we have

$$\rho(A \circ B^{-1}) \leq \max_{\gamma \in (A \circ B^{-1})} \left\{ \prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})) \right\}^{\frac{1}{n}}$$

$$\leq \max_{1 \leq i \leq n} \{a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})\}$$

$$\leq \rho(A) \max_{1 \leq i \leq n} w_{ii} \leq \rho(A) \rho(B^{-1}).$$ 

Hence the bound in Theorem 3.1 is sharper than that in Theorem 5.74 in [3].
Recently, Li et al. [2] improved the result in Lemma 2.6 in [1] and gave an upper bound for $\rho(A \circ B^{-1})$, that is
\[
\rho(A \circ B^{-1}) \leq \frac{1}{2} \max_{i \neq j} \{a_{ii}w_{ii} + a_{jj}w_{jj} + [(a_{ii}w_{ii} - a_{jj}w_{jj})^2 \\
+ 4(\rho(A) - a_{ii})(\rho(A) - a_{jj})\rho^2(J_B)w_{ii}w_{jj}]^{\frac{1}{2}}\}.
\]

We could not verify that the result in Theorem 3.1 is better than the above inequality in theoretical analysis, but the following numerical example shows that the result derived in Theorem 3.1 is better than that derived in Theorem 3.1 in [2] in certain cases.

**Example 3.1** Let
\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1.2 & -0.1 & -0.1 \\ -0.3 & 1 & -0.1 \\ -0.2 & -0.4 & 0.8 \end{pmatrix}.
\]

It is easy to calculate that $\rho(A) = 3$, $\rho(J_B) = 0.368$. By Lemma 2.6 in [1], we have
\[
\rho(A \circ B^{-1}) \leq \max_{1 \leq i \leq n} \{a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})\} = 2.3783.
\]

By Theorem 3.1 in [2], we have
\[
\rho(A \circ B^{-1}) \leq \frac{1}{2} \max_{i \neq j} \{a_{ii}w_{ii} + a_{jj}w_{jj} + [(a_{ii}w_{ii} - a_{jj}w_{jj})^2 \\
+ 4(\rho(A) - a_{ii})(\rho(A) - a_{jj})\rho^2(J_B)w_{ii}w_{jj}]^{\frac{1}{2}}\} = 2.1492.
\]

By Theorem 3.1 in this paper, we obtain
\[
\rho(A \circ B^{-1}) \leq \max_{\gamma \in \zeta(A \circ B^{-1})} \left[\prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii}))\right]^\frac{1}{|\gamma|} = 2.1318.
\]

In fact, $\rho(A \circ B^{-1}) = 1.263$.

### 4 Bounds for the minimum eigenvalue of a Nonsingular $M$-matrix

In this section, some bounds for $\tau(A)$ are presented, where $A$ is a nonsingular $M$-matrix. We first give a bound for $\tau(A)$ which is sharper than that in Theorem 3.1 in [1].
**Theorem 4.1** Let $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular $M$-matrix and $B^{-1} = (w_{ij})_{n \times n}$. Then

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + \rho(J_B)(n-1)} \left( \max_{\gamma \in \zeta(B^{-1})} \left( \prod_{i \in \gamma} w_{ii} \right)^{\frac{1}{|\gamma|}} \right)^{-1},$$

where $\rho(J_B)$ is the spectral radius of the Jacobi iterative matrix $J_B$ of $B$.

**Proof.** Let $A$ be the $n \times n$ matrix of all elements one. By Theorem 3.1, notice that $\rho(A) = n$, we have

$$\rho(B^{-1}) = \rho(A \circ B^{-1}) \leq \max_{\gamma \in \zeta((A \circ B^{-1}))} \left( \prod_{i \in \gamma} (a_{ii}w_{ii} + w_{ii}\rho(J_B)(\rho(A) - a_{ii})) \right)^{\frac{1}{|\gamma|}} \leq [1 + \rho(J_B)(n-1)] \max_{\gamma \in \zeta(B^{-1})} \left( \prod_{i \in \gamma} w_{ii} \right)^{\frac{1}{|\gamma|}},$$

Then, from Inequality (5), we have

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + \rho(J_B)(n-1)} \left( \max_{\gamma \in \zeta(B^{-1})} \left( \prod_{i \in \gamma} w_{ii} \right)^{\frac{1}{|\gamma|}} \right)^{-1}.$$  

The proof is completed.

**Remark 4.1** From Inequality (5), we have

$$\rho(B^{-1}) \leq [1 + \rho(J_B)(n-1)] \max_{\gamma \in \zeta(B^{-1})} \left( \prod_{i \in \gamma} w_{ii} \right)^{\frac{1}{|\gamma|}} \leq (1 + \rho(J_B)(n-1)) \max_{1 \leq i \leq n} w_{ii}.$$  

Then,

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + \rho(J_B)(n-1)} \frac{1}{\max_{1 \leq i \leq n} w_{ii}}.$$  

Thus the bound in Theorem 4.1 is better than that in Theorem 3.1 in [1].

Li et al. obtained lower bound for $\rho(A \circ B^{-1})$ derived in Theorem 4.1 in [2], that is

$$\tau(B) \geq \frac{1}{\max_{i \neq j} \left\{ w_{ii} + w_{jj} + [(w_{ii} - w_{jj})^2 + 4(n-1)^2\rho(J_B)^2w_{ii}w_{jj}]^{\frac{1}{2}} \right\}}.$$  

We could not verify that result in Theorem 4.1 is better than the above inequality in theoretical analysis, but the following numerical example shows that the result derived in Theorem 4.1 is better than that derived in Theorem 4.1 in [2] in certain cases.
Example 4.1 [2] Let

\[
B = \begin{pmatrix}
1 & -0.2 & -0.1 & -0.2 & -0.1 \\
-0.4 & 1 & -0.2 & -0.1 & -0.1 \\
-0.9 & -0.2 & 1 & -0.1 & -0.1 \\
-0.3 & -0.7 & -0.3 & 1 & -0.1 \\
-1 & -0.3 & -0.2 & -0.4 & 1
\end{pmatrix}.
\]

It is easy to verify that \(B\) is a nonsingular M-matrix and \(\tau(B) = 0.00812\). By Theorem 3.1 in [1], we have

\[
\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + \rho(J_B)(n-1)} \max_{1 \leq i \leq n} w_{ii} = 0.00598.
\]

By Theorem 4.1 in [2], we have

\[
\tau(B) \geq \frac{1}{\max_{i \neq j} \{w_{ii} + w_{jj} + [(w_{ii} - w_{jj})^2 + 4(n-1)^2 |\rho(J_B)|^2 w_{ii} w_{jj} |^2\}} = 0.00688.
\]

By Theorem 4.1 in this paper, we obtain

\[
\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + \rho(J_B)(n-1)} \left( \max_{\gamma \in \zeta(B^{-1})} \left( \prod_{i \in \bar{\gamma}} w_{ii} \right)^{\frac{1}{|\gamma|}} \right)^{-1} = 0.00690.
\]

Combining Theorem 4.1 and Lemma 2.9, we obtain the following theorem.

Theorem 4.2 Let \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) be a nonsingular M-matrix and \(A^{-1} = (v_{ij})_{n \times n}\). Then

\[
\tau(A) \geq \max \left\{ \frac{1}{1 + \rho(J_A)(n-1)} \left( \max_{\gamma \in \zeta(A^{-1})} \left( \prod_{i \in \bar{\gamma}} v_{ii} \right)^{\frac{1}{|\gamma|}} \right)^{-1}, \left( \max_{j=1}^{n} \sum_{j=1}^{n} v_{ij} \right)^{-1} \right\}.
\]

Next, we give some lower bounds for \(\tau(A)\) when \(A\) is a strictly diagonally dominant nonsingular M-matrix, which are only depending on the entries of \(A\) and are sharper than the ones derived in corollary 3.4 in [1].

Theorem 4.3 Let \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) be a nonsingular M-matrix. Then

(i) If \(A\) is a strictly diagonally dominant matrix by rows, then

\[
\tau(A) \geq \frac{1}{1 + (n-1) \max_{1 \leq i \leq n} \sigma_i} \min_{\gamma \in \zeta(A^{-1})} \left( \prod_{i \in \bar{\gamma}} (a_{ii} + \sum_{k=1,k \neq i}^{n} a_{ik} \sigma_k) \right)^{\frac{1}{|\gamma|}};
\]

(ii) If \(A\) is a strictly diagonally dominant matrix by columns, then

\[
\tau(A) \geq \frac{1}{1 + (n-1) \max_{1 \leq i \leq n} \delta_i} \min_{\gamma \in \zeta(A^{-1})} \left( \prod_{i \in \bar{\gamma}} (a_{ii} + \sum_{k=1,k \neq i}^{n} a_{ik} \delta_k) \right)^{\frac{1}{|\gamma|}}.
\]
Proof. Since $A$ is a nonsingular $M$-matrix, $D$ is a nonsingular $M$-matrix and $D - A$ is a nonnegative matrix. Notice that $D$ is a strictly diagonally dominant matrix both by rows and columns, by Lemma 2.11, we have

$$\rho(J_A) = \rho(D^{-1}(D - A)) \leq \max_{1 \leq i \leq n} \frac{R_i'(D - A)}{a_{ii}} = \max_{1 \leq i \leq n} \frac{R_i(A)}{a_{ii}} = \max_{1 \leq i \leq n} \sigma_i,$$

and

$$\rho(J_A) = \rho(D^{-1}(D - A)) \leq \max_{1 \leq i \leq n} \frac{C_i'(D - A)}{a_{ii}} = \max_{1 \leq i \leq n} \frac{C_i(A)}{a_{ii}} = \max_{1 \leq i \leq n} \delta_i.$$

By Theorem 4.1 and Lemma 2.10, the conclusion follows.

Remark 4.2 Similar to Remark 4.1, we know that the results derived in Theorem 4.3 are better than the ones derived in corollary 3.4 in [1], but we could not verify that the results in Theorem 4.3 are better than the those derived in corollary 4.4 in [2] in theoretical analysis, but the following numerical example shows that the results derived in Theorem 4.3 are better than the results derived in corollary 3.4 in [1] and corollary 4.4 in [2] in certain cases.

Example 4.2 [2] Let

$$A = \begin{pmatrix} 1.2 & -0.6 & -0.1 \\ -0.3 & 1 & -0.1 \\ -0.2 & -0.4 & 0.8 \end{pmatrix}.$$ 

It is easy to verify that $A$ is a strictly diagonally dominant nonsingular $M$-matrix by rows and $\tau(A) = 0.4781$. By corollary 3.4 in [1], we have

$$\tau(A) \geq 0.2093.$$ 

By corollary 4.4 in [2], we have

$$\tau(A) \geq 0.2463.$$ 

By Theorem 4.3 in this paper, we obtain

$$\tau(A) \geq 0.2506.$$ 

Tian and Huang [1] gave bounds for $\tau(A)$ in terms of the spectral radius of the Jacobi iterative matrix and its corresponding eigenvector.

Lemma 4.4 Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an irreducible nonsingular $M$-matrix, then

$$\left(1 - \rho(J_A)\right) \min_{1 \leq i \leq n} \frac{a_{ii}q_i}{\max_{1 \leq i \leq n} q_i} \leq \tau(A) \leq \left(1 - \rho(J_A)\right) \max_{1 \leq i \leq n} \frac{a_{ii}q_i}{\min_{1 \leq i \leq n} q_i},$$

where $\rho(J_A)$ is the spectral radius of the Jacobi iterative matrix $J_A$ of $A$ and $q = (q_1, q_2, \cdots, q_n)^T$ is its eigenvector corresponding to $\rho(J_A)$. 

Subsequently, Li et al. [2] improved the lower bound in (7) and obtained the following result.

**Lemma 4.5** Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be an irreducible nonsingular M-matrix, then
\[
\frac{\min_{1 \leq i \leq n} \{a_{ii}q_i + a_{jj}q_j - \Omega_{ij}\}}{2 \max_{1 \leq i \leq n} q_i} \leq \tau(A) \leq \frac{\max_{1 \leq i \leq n} \{a_{ii}q_i + a_{jj}q_j + \Omega_{ij}\}}{2 \min_{1 \leq i \leq n} q_i},
\]
where \( \Omega_{ij} = [(a_{ii}q_i - a_{jj}q_j)^2 + 4[\rho(J_A)]^2a_{ii}a_{jj}q_iq_j]^{\frac{1}{2}} \).

However, we could not verify that the upper bound in (8) is better than that in (7). Next, we exhibit some new bounds for \( \tau(A) \), and prove that the bounds in Theorem 4.6 are sharper than those in (7) and are better than (8) in some cases. Moreover, the lower bound for \( \tau(A) \) in Theorem 4.6 has no relation with \( q = (q_1, q_2, \cdots, q_n)^T \), then that is more convenient to calculate than the bounds in (7) and (8).

**Theorem 4.6** Let \( A = (a_{ij})_{n \times n} \) be a nonsingular M-matrix, \( n \geq 2 \). Then
\[
\frac{1}{2} \min_{i \neq j} \{a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4[\rho(J_A)]^2a_{ii}a_{jj}]^{\frac{1}{2}}\} \leq \tau(A) \leq \frac{2(1 - \rho(J_A))}{\min_{1 \leq i \leq n} q_i \min_{i \neq j} (\frac{1}{a_{ii}q_i} + \frac{1}{a_{jj}q_j})},
\]
where \( \rho(J_A) \) is the spectral radius of the Jacobi iterative matrix \( J_A \) of \( A \) and \( q = (q_1, q_2, \cdots, q_n)^T \) is an eigenvector of \( J_A \) corresponding to \( \rho(J_A) \).

**Proof.** Since \( A \) is an irreducible nonsingular M-matrix, \( J_A \) is obviously an irreducible and nonnegative matrix. Then there exists a positive vector \( p = (p_1, p_2, \cdots, p_n)^T \), such that
\[
\sum_{k=1, k \neq i}^{n} \frac{|a_{ik}|p_k}{p_i} = \rho(J_A)a_{ii} \quad \text{for all } i \in N.
\]
Let \( \bar{A} = (\bar{a}_{ij})_{n \times n} = P^{-1}AP \), where \( P = \text{diag}(p_1, p_2, \cdots, p_n) \). Notice that \( \bar{A} \) is also an irreducible nonsingular M-matrix and \( \tau(A) = \tau(\bar{A}) \), by Lemma 2.12, we have
\[
|\tau(A) - a_{ii}| |\tau(A) - a_{jj}| \leq \sum_{k=1, k \neq i}^{n} \frac{|a_{ik}|p_k}{p_i} \sum_{k=1, k \neq j}^{n} \frac{|a_{jk}|p_k}{p_j} = [\rho(J_A)]^2a_{ii}a_{jj}.
\]
Since \( A \leq \text{diag}(a_{11}, a_{22}, \cdots, a_{nn}) \) and \( \text{diag}(a_{11}, a_{22}, \cdots, a_{nn}) \in \mathbb{Z}_n \), by Lemma 2.6, we have
\[
A^{-1} \geq [\text{diag}(a_{11}, a_{22}, \cdots, a_{nn})]^{-1} \geq O.
\]
By Lemma 2.7, we obtain

$$\rho(A^{-1}) \geq \frac{1}{a_{ii}}, \text{ i.e., } \tau(A) \leq a_{ii} \text{ for all } i \in N.$$  

Hence

$$\tau(A) - a_{ii} \geq (\rho(A))a_{ii}a_{jj}.$$  

Thus, from Inequality (10), we have

$$\tau(A) \geq \frac{1}{2} \min_{i \neq j} \{a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4\rho(J_A)]a_{ii}a_{jj}\}.$$  

From Inequality (11), we can obtain the left part of the inequality in this theorem. Since $A$ is an irreducible nonsingular $M$-matrix, by Lemma 2.1, there exists a positive diagonal matrix $Q = \text{diag}(q_1, q_2, \ldots, q_n)$, such that

$$\sum_{k=1, k \neq i}^n |a_{ik}|q_k = \rho(J_A)a_{ii}q_i \text{ for all } i \in N.$$  

Notice that $AQ$ is also a nonsingular $M$-matrix, then $(AQ)^{-1} \geq O$, and there exists a positive vector $u = (u_1, u_2, \ldots, u_n)^T$, such that $(AQ)^{-1}u = \rho((AQ)^{-1})u$, i.e., $AQu = \tau(AQ)u$, then

$$\sum_{j=1, j \neq i}^n \frac{|a_{ij}|q_ju_j}{a_{ii}q_iu_i} = 1 - \frac{\tau(AQ)}{a_{ii}q_i}.$$  

Let $W = \text{diag}(q_1u_1, q_2u_2, \ldots, q_nu_n)$ and $\tilde{J}_A = W^{-1}J_AW$, then

$$t_{ij} = \frac{a_{ij}q_ju_j}{a_{ii}q_iu_i} \quad (j \neq i), \quad t_{ii} = 0.$$  

Since $J_A$ is a nonnegative and irreducible matrix, $\tilde{J}_A$ is also a nonnegative and irreducible matrix. Then $\rho(J_A) = \rho(\tilde{J}_A)$. By Lemma 2.13, we have

$$\rho(\tilde{J}_A) \leq \frac{1}{2} \max_{i \neq j} \{t_{ii} + t_{jj} + [(t_{ii} - t_{jj})^2 + 4 \sum_{k=1, k \neq i}^n t_{ik} \sum_{k=1, k \neq j}^n t_{jk}]^{\frac{1}{2}}\}$$

$$= \max_{i \neq j} \{\sum_{k=1, k \neq i}^n t_{ik} \sum_{k=1, k \neq j}^n t_{jk}]^{\frac{1}{2}}.$$  

From Equation (13) and Inequality (14), we have

$$\rho(\tilde{J}_A) \leq \max_{i \neq j} \sqrt{(1 - \frac{\tau(AQ)}{a_{ii}q_i})(1 - \frac{\tau(AQ)}{a_{jj}q_j})} \leq \frac{1}{2} \max_{i \neq j} [(1 - \frac{\tau(AQ)}{a_{ii}q_i}) + (1 - \frac{\tau(AQ)}{a_{jj}q_j})]$$

$$\leq 1 - \tau(AQ) \frac{1}{2} \min_{i \neq j} \left(\frac{1}{a_{ii}q_i} + \frac{1}{a_{jj}q_j}\right).$$
Then

\[ \tau(AQ) \leq \frac{2(1 - \rho(J_A))}{\min_{i \neq j} \left( \frac{1}{a_{ii}q_{ii}} + \frac{1}{a_{jj}q_{jj}} \right)} = \frac{2(1 - \rho(J_A))}{\min_{i \neq j} \left( \frac{1}{a_{ii}q_{ii}} + \frac{1}{a_{jj}q_{jj}} \right)}. \]  

(15)

Since \((AQ)^{-1} = Q^{-1}A^{-1} \leq \max_{1 \leq i \leq n} \tilde{q}_i A^{-1}\), we have

\[ \tau(AQ) = \frac{1}{\rho((AQ)^{-1})} \geq \max_{1 \leq i \leq n} \frac{1}{\tilde{q}_i} \rho(A^{-1}) = \tau(A) \min_{1 \leq i \leq n} q_i. \]  

(16)

Thus, from Inequalities (15) and (16), the conclusion follows.

**Remark 4.3** We next give a simple comparison between the upper and lower bounds in Theorem 4.6 and those derived in Lemma 4.4. Without loss of generality, for \( i \neq j \), we assume that

\[ a_{ii} - a_{ii} \rho(J_A) \leq a_{jj} - a_{jj} \rho(J_A). \]  

(17)

Thus, Inequality (17) can be written as

\[ a_{jj} \rho(J_A) \leq a_{jj} - a_{ii} + a_{ii} \rho(J_A). \]

Then, we have

\[
\frac{1}{2} \min_{i \neq j} \left\{ a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4\rho(J_A)^2 a_{ii} a_{jj}]^{\frac{1}{2}} \right\} \\
\geq \frac{1}{2} \min_{i \leq j} \left\{ a_{ii} + a_{jj} - [(a_{ii} - b_{jj})^2 + 4\rho(J_A)a_{ii}(a_{jj} - a_{ii} + a_{ii} \rho(J_A))]^{\frac{1}{2}} \right\} \\
= \frac{1}{2} \min_{i \neq j} \left\{ a_{ii} + a_{jj} - [(a_{jj} - a_{ii}) + 2\rho(J_A)a_{ii}] \right\} \\
= \min_{1 \leq i \leq n} \{ a_{ii} - \rho(J_A)a_{ii} \} = (1 - \rho(J_A)) \min_{1 \leq i \leq n} a_{ii}.
\]

Thus, we have

\[
(1 - \rho(J_A)) \frac{\min_{1 \leq i \leq n} a_{ii} q_i}{\max_{1 \leq i \leq n} q_i} \leq (1 - \rho(J_A)) \frac{\min_{1 \leq i \leq n} a_{ii} (\max_{1 \leq i \leq n} q_i)}{\max_{1 \leq i \leq n} q_i} = (1 - \rho(J_A)) \min_{1 \leq i \leq n} a_{ii} \\
\leq \frac{1}{2} \min_{i \neq j} \left\{ a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4\rho(J_A)^2]^\frac{1}{2} \right\}.
\]

Furthermore, consider the following inequality

\[
\frac{1}{2} \min_{i \neq j} \left( \frac{1}{a_{ii}q_i} + \frac{1}{a_{jj}q_j} \right) \geq \frac{1}{2} \min_{1 \leq i \leq n} \frac{1}{a_{ii}q_i} + \frac{1}{2} \min_{1 \leq j \leq n} \frac{1}{a_{jj}q_j} = \min_{1 \leq i \leq n} \frac{1}{a_{ii}q_i}.
\]

So the bounds in Theorem 4.6 are sharper than those in Theorem 3.6 in [1].
It is difficult for us to calculate the corresponding eigenvector of $\rho(J_A)$ in practice, we next give some new bounds for $\tau(A)$ and those bounds have no relation with $q = (q_1, q_2, \cdots, q_n)^T$.

**Theorem 4.7** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an irreducible nonsingular $M$-matrix, then

$$(1 - \rho(J_A)) \min_{1 \leq i \leq n} a_{ii} \leq \tau(A) \leq (1 - \rho(J_A)) \max_{1 \leq i \leq n} a_{ii},$$

where $\rho(J_A)$ is the spectral radius of the Jacobi iterative matrix $\rho(J_A)$ of $A$.

**Proof.** Since $A$ is an irreducible nonsingular $M$-matrix, $A^{-1} \geq O$ and $J_A$ is an irreducible nonnegative matrix. Then there exists positive vectors $t = (t_1, t_2, \cdots, t_n)^T$ and $g = (g_1, g_2, \cdots, g_n)^T$, such that

$$A^{-1}t = \rho(A^{-1})t, \quad J_A g = \rho(J_A)g,$$

i.e.,

$$At = \tau(A)t, \quad J_A g = \rho(J_A)g.$$

By Lemma 2.14, we have

$$\rho(J_A) = \min_{x \geq 0, x \neq 0} \max_{x_i \neq 0} \frac{(J_A x)_i}{x_i} \leq \max_{t_i \neq 0} \frac{(J_A t)_i}{t_i} = \max_{t_i \neq 0} \frac{\sum_{j=1}^{n} |a_{ij}| t_j}{a_{ii} t_i} = \max_{t_i \neq 0} (1 - \frac{\tau(A)}{a_{ii}}) \leq 1 - \frac{\tau(A)}{\max_{1 \leq i \leq n} a_{ii}}.$$

Thus, we have

$$\tau(A) \leq (1 - \rho(J_A)) \max_{1 \leq i \leq n} a_{ii}. \quad (18)$$

Furthermore, we have

$$(Ag)_i = a_{ii} g_i - \sum_{j=1, j \neq i}^{n} |a_{ij}| g_j = a_{ii} g_i - \rho(J_A) a_{ii} g_i = a_{ii} (1 - \rho(J_A)) g_i \geq \min_{1 \leq i \leq n} a_{ii} (1 - \rho(J_A)) g_i.$$

By Lemma 2.15, we obtain

$$(1 - \rho(J_A)) \min_{1 \leq i \leq n} a_{ii} \leq \tau(A). \quad (19)$$

From Inequalities (18) and (19), the conclusion follows.
Remark 4.4 Since
\[
\min_{1 \leq i \leq n} a_{ii}q_i \leq \min_{1 \leq i \leq n} a_{ii}(\max_{1 \leq i \leq n} q_i), \quad \max_{1 \leq i \leq n} a_{ii}q_i \geq \max_{1 \leq i \leq n} a_{ii}(\min_{1 \leq i \leq n} q_i),
\]
we have
\[
(1 - \rho(J_A)) \frac{\min_{1 \leq i \leq n} a_{ii}q_i}{\max_{1 \leq i \leq n} q_i} \leq (1 - \rho(J_A)) \frac{\min_{1 \leq i \leq n} a_{ii}(\max_{1 \leq i \leq n} q_i)}{\max_{1 \leq i \leq n} q_i} = (1 - \rho(J_A)) \min_{1 \leq i \leq n} a_{ii},
\]
and
\[
(1 - \rho(J_A)) \frac{\max_{1 \leq i \leq n} a_{ii}q_i}{\min_{1 \leq i \leq n} q_i} \geq (1 - \rho(J_A)) \frac{\max_{1 \leq i \leq n} a_{ii}(\min_{1 \leq i \leq n} q_i)}{\min_{1 \leq i \leq n} q_i} = (1 - \rho(J_A)) \max_{1 \leq i \leq n} a_{ii}.
\]
Hence the bounds in Theorem 4.7 are better than those in Theorem 3.6 in [1].

Example 4.3 [2] Let
\[
A = \begin{pmatrix}
2 & -1 & 0 \\
0 & 1 & -0.5 \\
-0.5 & -1 & 2
\end{pmatrix},
\]
where $A$ is an irreducible nonsingular $M$-matrix and $\tau(A) = 0.5402$.

By Theorem 3.6 in [1], we have
\[
0.3393 \leq \tau(A) \leq 1.1480.
\]

By Theorem 4.7 in [2], we have
\[
0.4164 \leq \tau(A) \leq 3.9001.
\]

By Theorem 4.6, we have
\[
0.5203 \leq \tau(A) \leq 0.9489.
\]

By Theorem 4.7, we obtain
\[
0.4043 \leq \tau(A) \leq 0.8086.
\]

Remark 4.5 Example 4.3 shows that the bounds for $\tau(A)$ in Theorem 4.6 are sharper than those in Theorem 3.6 of [1] and Theorem 4.7 of [2]. Although Theorem 4.6 and Theorem 4.7 can not contain each other, the bounds in Theorem 4.7 are more convenient to calculate than the ones in Theorem 4.6.
It is difficult for us to estimate \( \rho(J_A) \) in practice. In [1], authors put forward that obtaining upper and lower bounds for \( \tau(A) \) which are only depending on the entries of a nonsingular \( M \)-matrix \( A \) would be an interesting problem to be studied in further research. Next, we present some new bounds for \( \tau(A) \) of a nonsingular \( M \)-matrix \( A \), which are only depending on the entries of \( A \).

**Theorem 4.8** Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a nonsingular \( M \)-matrix, then

\[
\frac{1}{\max_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \min_j |a_{ij}|}} \leq \tau(A) \leq \frac{1}{\min_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \max_j |a_{ij}|}}
\]

where \( A_{ii} \) means algebraic cofactor of \( a_{ii} \).

**Proof.** Let \( A^{-1} = (v_{ij})_{n \times n} \). Since \( A^{-1}A = I \), we have \( 1 = \sum_{j=1}^n v_{ij}a_{ji} \) for all \( i \in N \). Then

\[
2v_{ii}a_{ii} - 1 = 2v_{ii}a_{ii} - \sum_{j=1}^n v_{ij}a_{ji} = \sum_{j=1}^n v_{ij}|a_{ji}| \leq \max_{j \in N} |a_{ji}| \sum_{j=1}^n v_{ij}.
\]

Thus, we have

\[
\sum_{j=1}^n v_{ij} \leq \frac{2v_{ii}a_{ii} - 1}{\max_{j \in N} |a_{ji}|} = \frac{2a_{ii}A_{ii} - 1}{\max_{j \in N} |a_{ji}|} = \frac{2a_{ii}A_{ii} - \det A}{\det A \max_j |a_{ij}|}.
\]

From \( 2v_{ii}a_{ii} - 1 = \sum_{j=1}^n v_{ij}|a_{ji}| \geq \min_{j \in N} |a_{ji}| \sum_{j=1}^n v_{ij} \), we have

\[
\sum_{j=1}^n v_{ij} \leq \frac{2a_{ii}v_{ii} - 1}{\min_{j \in N} |a_{ji}|} = \frac{2a_{ii}A_{ii} - 1}{\min_{j \in N} |a_{ji}|} = \frac{2a_{ii}A_{ii} - \det A}{\det A \min_j |a_{ij}|}.
\]

By Lemma 2.4, we have

\[
\min_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \max_j |a_{ij}|}^{\frac{1}{|\gamma|}} \leq \rho(A^{-1}) \leq \max_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \min_j |a_{ij}|}^{\frac{1}{|\gamma|}}.
\]

Hence

\[
\frac{1}{\max_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \min_j |a_{ij}|}^{\frac{1}{|\gamma|}}} \leq \frac{1}{\rho(A^{-1})} = \tau(A) \leq \frac{1}{\min_{\gamma \in \xi(A^{-1})} \prod_{i \in \gamma} \frac{2a_{ii}A_{ii} - \det A}{\det A \max_j |a_{ij}|}^{\frac{1}{|\gamma|}}}.
\]
Theorem 4.9 Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a nonsingular M-matrix, then

\[
\frac{\det A \min_{i,j \in N} |a_{ij}|}{2 \max_{i \in N} (a_{ii} A_{ii}) - \det A} \leq \tau(A) \leq \max_{i,j \in N} |a_{ij}|.
\]

where \( A_{ii} \) means algebraic cofactor of \( a_{ii} \).

Proof. Let \( A^{-1} = (v_{ij})_{n \times n} \). Since \( A^{-1} A = I \), we have \( 1 = \sum_{j=1}^{n} v_{ij} a_{ji} \) for all \( i \in N \). Then

\[
1 = v_{ii} a_{ii} + \sum_{j=1, j \neq i}^{n} v_{ij} a_{ji} \leq v_{ii} a_{ii}.
\]

Similar to the proof of Theorem 4.8, we have

\[
\sum_{j=1}^{n} v_{ij} \geq \frac{2v_{ii} a_{ii} - 1}{\max_{j \in N} |a_{ji}|} \geq \frac{1}{\max_{j \in N} |a_{ji}|} \geq \frac{1}{\max_{i,j \in N} |a_{ij}|}.
\]

\[
\sum_{j=1}^{n} v_{ij} \leq \frac{2a_{ii} A_{ii} - \det A}{\det A \min_{j \in N} |a_{ij}|} \leq \frac{2 \max_{i \in N} (a_{ii} A_{ii}) - \det A}{\det A \min_{i,j \in N} |a_{ij}|}.
\]

By Lemma 2.9, the conclusion follows.

Remark 4.6 We cannot affirm that which bounds are better in Theorem 4.8 and Theorem 4.9, but the bounds in Theorem 4.9 are more convenient to calculate than those in Theorem 4.8 and they are only depending on the entries of matrix \( A \).

5 Conclusion

In this paper, some upper and lower bounds for the minimum eigenvalue of nonsingular M-matrices are given. Furthermore, we prove that the results of this paper are sharper than the ones of [1, 2].

Acknowledgements. This work has been supported by the National Natural Science Foundations of China (No.10802068).

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New bounds on the minimum eigenvalue of matrices


Received: June 16, 2015; Published: August 12, 2015