Structure Torsions of Trajectories on Real Hypersurfaces of Exceptional Type in Complex Projective Space

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Abstract

The structure torsions for trajectories play an important role on study of trajectories. In this paper we consider the condition for trajectories to be curves of order two on Real hypersurfaces of exceptional type in complex projective space through the structure torsions.

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1. Introduction

When we study Riemannian manifolds it is one of natural ways to investigate properties of curves on them. There are strong relations between properties of Riemannian manifolds and curves on them. It is needless to say that Riemannian geometry was developed by investigations of geodesics. In this context, Adachi and the author [2] studied the trajectories under Sasakian magnetic

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field on real hypersurfaces of type (A) in a nonflat complex space forms. They showed that the condition for trajectories to be circles, and called them circular trajectories. They characterize these real hypersurfaces by properties of circular trajectories. In their papers [3] and [4] they estimated the length of circular trajectories on real hypersurface of type (A$_1$) in a complex projective space and in a complex hyperbolic space. They studied the trajectories under Sasakian magnetic field on real hypersurfaces of type (B) in a complex hyperbolic space in [5] and [6]. Except the above real hypersurfaces of types (A) and (B) there are real hypersurface of types (C), (D) and (E) in a complex projective space, which are called real hypersurfaces of exceptional type. The study on the real hypersurface of type (A) goes so smoothly, because it have the nice properties that its shape operator and the characteristic tensors are simultaneously diagonalizable and its principal curvature number is at most three. But for real hypersurface of type (B), the structure torsions of trajectories for Sasakian magnetic fields are not constant, hence we studied their behaviors. For a real hypersurfaces of exceptional type in a complex projective space, structure torsions of trajectories under Sasakian magnetic fields are not constant just like for trajectories on real hypersurface of type (B). Moreover being different from real hypersurfaces of types (A) and (B), real hypersurfaces of exceptional type have five principal curvatures. Hence we are interested in trajectories for Sasakian magnetic fields on real hypersurfaces of exceptional type in a complex projective space.

2. **Sasakian magnetic fields**

A closed 2-form on a Riemannian manifolds is said to be a magnetic field. The magnetic field is a generalizations of uniform static magnetic fields on a Euclidean 3-space, the static magnetic field gives the Lorentz force $v \times B = \Omega_B(v)$ on a unit charged particle when its velocity vector is $v$, here the skew symmetric matrix $\Omega_B(v)$ is given by

$$
\begin{pmatrix}
0 & B_3 & -B_2 \\
-B_3 & 0 & B_1 \\
B_2 & -B_1 & 0
\end{pmatrix}.
$$

and satisfies $B(v, w) = \langle v, \Omega_B(w) \rangle$ for all $v, w \in T_p M$ at an arbitrary point $p \in M$. We say a smooth curve $\gamma$ parameterized by its arclength to be a trajectories for magnetic fields if it satisfies the equation that $\nabla_\dot{\gamma} \dot{\gamma} = \Omega_B(\dot{\gamma})$.

There are two magnetic fields with nice properties. One is a Kähler magnetic field, and the other is a Sasakian magnetic field. For a Kähler manifold with complex structure $J$ and Riemannian metric $\langle \cdot, \cdot \rangle$, we call a constant multiple of Kähler form $B(v, w) = \langle v, J(w) \rangle$ a Kähler magnetic field. When smooth curve $\gamma$ parameterized by its arclength satisfies the equation that $\nabla_\dot{\gamma} \dot{\gamma} = \kappa J(\dot{\gamma})$, we call this curve a trajectory for Kähler magnetic field. Since trajectories for Kähler magnetic fields are closely related with complex structure, there are many result from properties of trajectories for Kähler magnetic fields to
Structure torsions of trajectories showed some properties of Kähler manifold (see [1], for example). Since Kähler manifolds are of real even dimensional. In this way, the author interested in a study of magnetic fields on odd dimensional manifolds. As odd dimensional objects corresponding to complex space forms we have Sasakian space forms, a real hypersurface in complex space forms which are complex projective space, complex Euclidean spaces and complex hyperbolic spaces.

A real hypersurface \( M \) in a complex space forms \( \widetilde{M} \) with complex structure \( J \) and Riemannian metric \( \langle \cdot , \cdot \rangle \) admits an almost contact metric structure. It is quartet \((\varphi, \xi, \eta, \langle \cdot , \cdot \rangle)\) of a vector field \( \xi \), a \((1,1)\)-tensor \( \varphi \), a function \( \eta \) and an induced metric on \( M \), they are defined as

\[
\xi = -JN, \quad \eta(v) = \langle v, \xi \rangle, \quad \varphi(v) = Jv - \eta(v)N, \quad v \in T_pM
\]

with a unit normal vector field \( N \) of \( M \) in \( \widetilde{M} \). We call \( \xi \) and \( \varphi \) the characteristic vector and the characteristic tensor, respectively. For a real hypersurface \( M \) in a Kähler manifold we denote by \( A \) its shape operator. Eigenvalues and eigenvectors for \( A \) are called principal curvatures and principal curvature vectors, respectively. The covariant differentials of the characteristic vector and the characteristic tensor are given as

\[
\nabla_X \xi = \varphi AX \quad \text{and} \quad (\nabla_X \varphi)Y = \langle Y, \xi \rangle AX - \langle AX, Y \rangle \xi,
\]

for vector fields \( X, Y \) on \( M \) (see [9]).

We define a 2-form \( F_\varphi \) on \( M \) by \( F_\varphi(v, w) = \langle v, \varphi(w) \rangle \). One can easily check that it is a closed 2-form, hence is a magnetic field (see [2]). We say a constant multiple \( F_\kappa = \kappa F_\varphi \) a Sasakian magnetic field. The equation of trajectories for \( F_\kappa \) is

\[
\nabla_{\dot{\gamma}} \dot{\gamma} = \kappa \varphi \dot{\gamma}.
\]

In this paper we restrict ourselves to real hypersurfaces \( M \) in complex projective space \( \mathbb{C}P^n(c) \) of constant holomorphic sectional curvature \( c(c > 0) \). In his paper (see [7]), R. Takagi classified homogenous real hypersurfaces in a complex projective space. Among such real hypersurface we here study the real hypersurface of type (C), (D) and (E). If \( M \) is a homogeneous real hypersurface of radius \( r \), the subbundles \( V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}, V_{\lambda_4} \) correspond to the principal curvatures

\[
\lambda_1 = (\sqrt{c}/2) \cot(\sqrt{c}r/2), \quad \lambda_2 = -(\sqrt{c}/2) \tan(\sqrt{c}r/2),
\lambda_3 = (\sqrt{c}/2) \cot(\sqrt{c}r/2 - \pi/4), \quad \lambda_4 = (\sqrt{c}/2) \cot(\sqrt{c}r/2 + \pi/4),
\]

respectively. We note that the principal curvature of \( \xi \) is \( \nu = \sqrt{c} \cot \sqrt{c}r \). The characteristic tensor \( \varphi \) acts on \( TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R} \xi \) as

\[
\varphi(V_{\lambda_1}) = V_{\lambda_1}, \quad \varphi(V_{\lambda_2}) = V_{\lambda_2}, \quad \varphi(V_{\lambda_3}) = V_{\lambda_4}, \quad \varphi(V_{\lambda_4}) = V_{\lambda_3}, \quad \varphi(\mathbb{R} \xi) = 0.
\]

In order to make clear the radius \( r \) of \( M \) as a tube, we shall denote it as \( M(r) \) in this paper.
3. Curves of order 2

A smooth curve \( \gamma \) on a complete Riemannian manifold which is parameterized by its arclength is called a curve of order 2 if it satisfies the equation

\[
\| \nabla \dot{\gamma} \|^2 \{ \nabla \dot{\gamma} \nabla \ddot{\gamma} + \| \nabla \dot{\gamma} \|^2 \dot{\gamma} \} = \langle \nabla \dot{\gamma}, \nabla \dot{\gamma} \nabla \ddot{\gamma} \rangle \nabla \dot{\gamma}.
\]  
(3.1)

Under the assumption that \( \nabla \dot{\gamma} \) does not vanish at each point, by putting \( k(t) = \| \nabla \dot{\gamma} \| \) and \( Y = \nabla \dot{\gamma} / \| \nabla \dot{\gamma} \| \), we find the equation (3.1) is equivalent to the system of equations

\[
\nabla \dot{\gamma} \ddot{\gamma}(t) = k(t)Y(t) \quad \text{and} \quad \nabla \dot{\gamma} Y(t) = -k(t)\dot{\gamma}(t)
\]

(see [8]). When \( k(t) \) is a positive constant function, we call \( \gamma \) a circle of positive geodesic curvature. One can easily see that geodesics also satisfy the equation (3.1). Hence, the notion of curves of order 2 is a generalization of the notion of circles.

4. Structure torsion of trajectories

For a trajectory \( \gamma \) for a Sasakian magnetic field \( F_\kappa \) on a real hypersurface \( M(r) \), we set \( \rho_\gamma = \langle \dot{\gamma}, \xi \rangle \) and call it the structure torsion of \( \gamma \). As we have \( \| \nabla \dot{\gamma} \| = |\kappa|\sqrt{1-\rho_\gamma^2} \), we can guess that this function plays an important role in our study of trajectories.

For the differential of this function of structure torsion, we have

\[
\rho_\gamma' = \langle \kappa \phi \dot{\gamma}, \xi \rangle + \langle \dot{\gamma}, \nabla \dot{\gamma} \xi \rangle = \langle \dot{\gamma}, \phi A \dot{\gamma} \rangle \left( = \frac{1}{2} \langle \dot{\gamma}, (\phi A - A \phi) \dot{\gamma} \rangle \right).
\]

When \( M \) is a hypersurface of type (A), as it is characterized by the property that its shape operator and its characteristic vector field are simultaneously diagonalizable, we find that the structure torsion of each trajectory is constant on this hypersurface. Thus we also find that if \( M \) is not a real hypersurface of type (A), not all of trajectories have constant structure torsion. If a trajectory \( \gamma \) on a real hypersurface has a point that \( \rho_\gamma(t_0) = \pm 1 \), as we have \( \kappa \phi \dot{\gamma}(t_0) = \pm \kappa \phi \xi = 0 \), it is a geodesic. We therefore need to consider the case \( |\rho_\gamma| < 1 \).

5. MAIN RESULTS

Our main result in this paper is the following.

**Theorem 1.** Let \( F_\kappa \) be a non-trivial Sasakian magnetic field on a real hypersurface \( M(r) \) of exceptional type in \( \mathbb{C}P^n(c) \). When \( 0 < |\kappa| \leq \lambda(r, c) \), there are no circular trajectories for \( F_\kappa \) having non-null structure torsions.

In order to show this theorem we study structure torsions of trajectories on real hypersurfaces of exceptional types.

**Proposition 1.** Let \( \gamma \) be a trajectory for a non-trivial Sasakian magnetic field \( F_\kappa \) on a real hypersurface \( M \) of type one of \( (C), (D) \) and \( (E) \) in \( \mathbb{C}P^n(c) \). Suppose \( \gamma \) is also a curve of order 2 and \( |\rho_\gamma| < 1 \). If we decompose its velocity
vector as $\gamma = X_\gamma + Y_\gamma + Z_\gamma + W_\gamma + \rho_\gamma \xi \in V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R} \xi$, we find one of the following holds:

1) $\rho_\gamma \equiv 0$,

2) $\kappa \rho_\gamma \equiv \lambda_1$, $Y_\gamma = Z_\gamma = W_\gamma \equiv 0$ and $\|X_\gamma\| = \sqrt{1 - \rho_\gamma^2}$,

3) $\kappa \rho_\gamma \equiv \lambda_2$, $X_\gamma = Z_\gamma = W_\gamma \equiv 0$ and $\|Y_\gamma\| = \sqrt{1 - \rho_\gamma^2}$,

4) $X_\gamma = Y_\gamma \equiv 0$ and the following equalities hold. In particular, $Z_\gamma$ and $W_\gamma$ are parallel and have

$$\|Z_\gamma\|^2 = \frac{(1 - \rho_\gamma^2)(\lambda_3 - \kappa \rho_\gamma)}{\lambda_4 - \lambda_3}, \quad \|W_\gamma\|^2 = \frac{(1 - \rho_\gamma^2)(\kappa \rho_\gamma - \lambda_3)}{\lambda_4 - \lambda_3},$$

and $\rho_\gamma$ satisfies $\lambda_3 \leq \kappa \rho_\gamma \leq \lambda_3$.

In the case 1) the situations $\|Y_\gamma(t)\| = \|Z_\gamma(t)\| = 0$ and $\|X_\gamma(t)\| = \|W_\gamma(t)\| = 0$ do not occur at any point $t$.

**Proof.** By use of the decomposition $\dot{\gamma} = X_\gamma + Y_\gamma + Z_\gamma + W_\gamma + \rho_\gamma \xi \in V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R} \xi$, we find that the trajectory $\gamma$ satisfies the following:

$$\nabla_\gamma \dot{\gamma} = \kappa \phi \dot{\gamma} = \kappa (\phi \dot{X}_\gamma + \phi \dot{Y}_\gamma + \phi \dot{W}_\gamma + \phi \dot{Z}_\gamma),$$

$$\nabla_\gamma \nabla_\gamma \dot{\gamma} = \kappa \{ (\nabla_\gamma \phi) \dot{\gamma} + \phi \nabla_\gamma \dot{\gamma} \} = \kappa \{ \rho_\gamma A_\gamma \dot{\gamma} - \langle A_\gamma, \dot{\gamma} \rangle \xi + \kappa \phi^2 \dot{\gamma} \}
= \kappa \{ (\rho_\gamma \lambda_1 - \kappa) X_\gamma + (\rho_\gamma \lambda_2 - \kappa) Y_\gamma + (\rho_\gamma \lambda_3 - \kappa) Z_\gamma + (\rho_\gamma \lambda_4 - \kappa) W_\gamma
- (\lambda_1 \|X_\gamma\|^2 + \lambda_2 \|Y_\gamma\|^2 + \lambda_3 \|Z_\gamma\|^2 + \lambda_4 \|W_\gamma\|^2) \xi \}.$$

Since $\gamma$ is also a curve of order 2, we substitute these into the equality

$$\|\nabla_\gamma \dot{\gamma}\|^2 (\nabla_\gamma \nabla_\gamma \dot{\gamma} + \|\nabla_\gamma \dot{\gamma}\|^2 \dot{\gamma}) = \langle \nabla_\gamma \dot{\gamma}, \nabla_\gamma \nabla_\gamma \dot{\gamma} \rangle \nabla_\gamma \dot{\gamma}.$$

As we have $\|\nabla_\gamma \dot{\gamma}\| = |\kappa| \sqrt{1 - \rho_\gamma^2}$, considering the action of $\phi$, we get

$$\kappa^3 (1 - \rho_\gamma^2) \{ \rho_\gamma (\lambda_1 - \kappa \rho_\gamma) X_\gamma + \rho_\gamma (\lambda_2 - \kappa \rho_\gamma) Y_\gamma + \rho_\gamma (\lambda_3 - \kappa \rho_\gamma) Z_\gamma + \rho_\gamma (\lambda_4 - \kappa \rho_\gamma) W_\gamma + (\kappa \rho_\gamma (1 - \rho_\gamma^2) - \lambda_1 \|X_\gamma\|^2 - \lambda_2 \|Y_\gamma\|^2 - \lambda_3 \|Z_\gamma\|^2 - \lambda_4 \|W_\gamma\|^2) \xi \}
= \kappa^3 \rho_\gamma (\lambda_3 - \lambda_4) \langle Z_\gamma, \phi W_\gamma \rangle (\phi X_\gamma + \phi Y_\gamma + \phi W_\gamma + \phi Z_\gamma).$$

Comparing each components of subbundles of principal curvature vectors, we find the following hold:

$$\left\{ \begin{array}{l}
\rho_\gamma (1 - \rho_\gamma^2) (\lambda_1 - \kappa \rho_\gamma) X_\gamma = \rho_\gamma (\lambda_3 - \lambda_4) \langle Z_\gamma, \phi W_\gamma \rangle \phi X_\gamma, \\
\rho_\gamma (1 - \rho_\gamma^2) (\lambda_2 - \kappa \rho_\gamma) Y_\gamma = \rho_\gamma (\lambda_3 - \lambda_4) \langle Z_\gamma, \phi W_\gamma \rangle \phi Y_\gamma, \\
\rho_\gamma (1 - \rho_\gamma^2) (\lambda_3 - \kappa \rho_\gamma) Z_\gamma = \rho_\gamma (\lambda_3 - \lambda_4) \langle Z_\gamma, \phi W_\gamma \rangle \phi Z_\gamma, \\
\rho_\gamma (1 - \rho_\gamma^2) (\lambda_4 - \kappa \rho_\gamma) W_\gamma = \rho_\gamma (\lambda_3 - \lambda_4) \langle Z_\gamma, \phi W_\gamma \rangle \phi W_\gamma, \\
\kappa \rho_\gamma (1 - \rho_\gamma^2) = \lambda_1 \|X_\gamma\|^2 + \lambda_2 \|Y_\gamma\|^2 + \lambda_3 \|Z_\gamma\|^2 + \lambda_4 \\
\end{array} \right. \quad (5.2)$$
Since $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$, we find that the function $\rho_\gamma$ may vanish on some interval. By homogeneity of $M$ we see $\rho_\gamma \equiv 0$ in this case. In the rest of cases it might occur $\rho_\gamma = 0$ on some discrete subset of $\mathbb{R}$. We consider on the open dense subset $T = \{ t \in \mathbb{R} \mid \rho_\gamma(t) \neq 0 \}$ in $\mathbb{R}$. We choose a point $t_0 \in T$. By smoothness of $\|X_\gamma\|$ we find that $\kappa \rho_\gamma \equiv \lambda_1$ and $Y_\gamma = Z_\gamma = W_\gamma \equiv 0$ on $\mathbb{R}$. Similarly, if $Y_\gamma(t_0) \neq 0$, then we have $\kappa \rho_\gamma \equiv \lambda_2$ and $X_\gamma = Z_\gamma = W_\gamma \equiv 0$ on $\mathbb{R}$. We next consider the case $X_\gamma = Y_\gamma \equiv 0$. We finally study the case $\rho_\gamma \equiv 0$. The fifth equation in (5.2) and the definition of $\rho_\gamma$ show that principal curvatures satisfy $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$. We here note that these equalities show that $Z_\gamma$ is parallel to $\phi W_\gamma$. We hence see $\langle Z_\gamma, \phi W_\gamma \rangle^2 = \|Z_\gamma\|^2 \|W_\gamma\|^2$. Since $\rho_\gamma \neq \pm 1$, we see either $Z_\gamma$ or $W_\gamma$ does not vanish. If $Z_\gamma(t_0) = 0$ the second equality shows that $\kappa \rho_\gamma(t_0) = \lambda_4$. Similarly, if $W_\gamma(t_0) = 0$ the first equality shows that $\kappa \rho_\gamma(t_0) = \lambda_3$. If $Z_\gamma(t_0) \neq 0, W_\gamma(t_0) \neq 0$, by taking inner products of both sides of the first equality with $Z_\gamma$ and with $\phi W_\gamma$, we obtain

$$
\|Z_\gamma(t_0)\|^2 = \frac{(1 - \rho_\gamma^2)(\kappa \rho_\gamma - \lambda_3)}{\lambda_4 - \lambda_3}, \quad \|W_\gamma(t_0)\|^2 = \frac{(1 - \rho_\gamma^2)(\kappa \rho_\gamma - \lambda_3)}{\lambda_4 - \lambda_3}.
$$

We can see that this expression include the cases when $\|Z_\gamma\|$ vanishes and when $\|W_\gamma\|$ vanishes. As $\lambda_4 > 0 > \lambda_3$, these expressions guarantee that $\lambda_3 \leq \kappa \rho_\gamma \leq \lambda_4$.

We finally study the case $\rho_\gamma \equiv 0$. The fifth equation in (5.2) and the definition of $\rho_\gamma$ show that

$$
\begin{align*}
\lambda_1 \|X_\gamma\|^2 + \lambda_2 \|Y_\gamma\|^2 + \lambda_3 \|Z_\gamma\|^2 + \lambda_4 \|W_\gamma\|^2 &= 0, \\
\|X_\gamma\|^2 + \|Y_\gamma\|^2 + \|Z_\gamma\|^2 + \|W_\gamma\|^2 &= 1.
\end{align*}
$$

As $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$, we can conclude that situations $\|Y_\gamma\| = \|Z_\gamma\| = 0$ and $\|X_\gamma\| = \|W_\gamma\| = 0$ do not occur. This complete the proof.

We here note that principal curvatures satisfy $\lambda_1 > \lambda_4 > 0 > \lambda_2 > \lambda_3$ and

$$
\begin{align*}
\lambda_1 > |\lambda_3| > \lambda_4 > |\lambda_2|, & \quad \text{if } 0 < r < \pi/(4 \sqrt{c}), \\
\lambda_1 = |\lambda_3| > \lambda_4 = |\lambda_2|, & \quad \text{if } r = \pi/(4 \sqrt{c}), \\
|\lambda_3| > \lambda_1 > |\lambda_2| > \lambda_4, & \quad \text{if } \pi/(4 \sqrt{c}) r < \pi/(2 \sqrt{c}).
\end{align*}
$$

we can conclude the following propositions.

**Proposition 2.** Let $M(r)$ be a real hypersurface of exceptional type and of radius $r < \pi/(4 \sqrt{c})$ in $\mathbb{C}P^n(c)$. Suppose there exist a trajectory $\gamma$ for a non-trivial Sasakian magnetic field $F_\kappa$ which is also a curve of order 2 and is not
a geodesic on $M(r)$. Then its structure torsion $\rho_\gamma$ satisfies the following with

\[
\begin{align*}
\lambda_1 &= (\sqrt{c}/2) \cot(\sqrt{c}/2), \\
\lambda_2 &= -\left(\sqrt{c}/2\right) \tan(\sqrt{c}/2), \\
\lambda_3 &= (\sqrt{c}/2) \cot(\sqrt{c}/2 - \pi/4), \\
\lambda_4 &= (\sqrt{c}/2) \cot(\sqrt{c}/2 + \pi/4).
\end{align*}
\]

(I) When $0 < |\kappa| \leq |\lambda_2|$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma$ is strictly monotone increasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = -1$ and
   $\lim_{t \to \infty} \rho_\gamma(t) = 1$;
3) $\rho_\gamma$ is strictly monotone decreasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = 1$ and
   $\lim_{t \to \infty} \rho_\gamma(t) = -1$.

(II) When $|\lambda_2| < |\kappa| \leq |\lambda_4|$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;
3) $\rho_\gamma$ is strictly monotone increasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = -1$ and
   $\lim_{t \to \infty} \rho_\gamma(t) = 1$;
4) $\rho_\gamma$ is strictly monotone decreasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = 1$ and
   $\lim_{t \to \infty} \rho_\gamma(t) = -1$.

(III) When $\lambda_4 < \kappa \leq |\lambda_3|$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;
3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;
4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = -1$ and there is $t_0$ satisfying that $\rho_\gamma$
   is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly
   monotone decreasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(IV) When $-\lambda_4 > \kappa \geq \lambda_3$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;
3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;
4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = 1$ and there is $t_0$ satisfying that $\rho_\gamma$
   is strictly monotone decreasing on the interval $(-\infty, t_0)$ and strictly
   monotone increasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(V) When $|\lambda_3| < |\kappa| \leq \lambda_1$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2)$;
3) $\rho_\gamma \equiv \lambda_3/\kappa$, $(X_\gamma = Y_\gamma = W_\gamma \equiv 0, \|Z_\gamma\|^2 = 1 - \rho_\gamma^2)$;
4) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2)$;
5) $\rho_\gamma$ is a periodic function satisfying $\lambda_3 \leq \kappa \rho_\gamma \leq \lambda_4$.

(VI) When $|\kappa| > \lambda_1$, 
1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_1 / \kappa$, \quad (Y_\gamma = Z_\gamma = W_\gamma \equiv 0, \|X_\gamma\|^2 = 1 - \rho_\gamma^2);
3) $\rho_\gamma \equiv \lambda_2 / \kappa$, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2);
4) $\rho_\gamma \equiv \lambda_3 / \kappa$, \quad (X_\gamma = Y_\gamma = W_\gamma \equiv 0, \|Z_\gamma\|^2 = 1 - \rho_\gamma^2);
5) $\rho_\gamma \equiv \lambda_4 / \kappa$, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2);
6) $\rho_\gamma$ is a periodic function satisfying $-\lambda_1 \leq \kappa \rho_\gamma \leq \lambda_1$.

**Proposition 3.** Let $M(\pi/(4\sqrt{c}))$ be a real hypersurface of exceptional type of radius $r = \pi/(4\sqrt{c})$ in $\mathbb{C}P^n(c)$. Suppose there exist a trajectory $\gamma$ for a non-trivial Sasakian magnetic field $F_\kappa$ which is also a curve of order 2 and is not a geodesic on $M(\pi/(4\sqrt{c}))$. Then its structure torsion $\rho_\gamma$ satisfies the following with $\lambda_1 = -\lambda_3 = \sqrt{c}((\sqrt{2} + 1)/2$, $-\lambda_2 = \lambda_4 = \sqrt{c}((\sqrt{2} - 1)/2$.

(I) When $0 < |\kappa| \leq \lambda_4$,

1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma$ is strictly monotone increasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = -1$ and $\lim_{t \to \infty} \rho_\gamma(t) = 1$;
3) $\rho_\gamma$ is strictly monotone decreasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = 1$ and $\lim_{t \to \infty} \rho_\gamma(t) = -1$.

(II) When $\lambda_4 < \kappa \leq \lambda_1$,

1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv -\lambda_4 / \kappa$, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2);
3) $\rho_\gamma \equiv \lambda_4 / \kappa$, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2);
4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = -1$ and there is $t_0$ satisfying that $\rho_\gamma$ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4 / \kappa$.

(III) When $-\lambda_4 > \kappa \geq -\lambda_1$,

1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv -\lambda_4 / \kappa$, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2);
3) $\rho_\gamma \equiv \lambda_4 / \kappa$, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2);
4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = 1$ and there is $t_0$ satisfying that $\rho_\gamma$ is strictly monotone decreasing on the interval $(-\infty, t_0)$ and strictly monotone increasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4 / \kappa$.

(IV) When $|\kappa| > \lambda_1$,

1) $\rho_\gamma \equiv 0$;
2) $\rho_\gamma \equiv \lambda_1 / \kappa$, \quad (Y_\gamma = Z_\gamma = W_\gamma \equiv 0, \|X_\gamma\|^2 = 1 - \rho_\gamma^2);
3) $\rho_\gamma \equiv -\lambda_1 / \kappa$, \quad (X_\gamma = Z_\gamma = W_\gamma \equiv 0, \|Y_\gamma\|^2 = 1 - \rho_\gamma^2);
4) $\rho_\gamma \equiv -\lambda_1 / \kappa$, \quad (X_\gamma = Y_\gamma = W_\gamma \equiv 0, \|Z_\gamma\|^2 = 1 - \rho_\gamma^2);
5) $\rho_\gamma \equiv \lambda_4 / \kappa$, \quad (X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\|^2 = 1 - \rho_\gamma^2);
6) $\rho_\gamma$ is a periodic function satisfying $-\lambda_1 \leq \kappa \rho_\gamma \leq \lambda_4$. 
Proposition 4. Let $M(r)$ be a real hypersurface of exceptional type and of radius $r > \pi/(4\sqrt{c})$ in $\mathbb{C}P^n(c)$ Suppose there exist a trajectory $\gamma$ for a non-trivial Sasakian magnetic field $F_\kappa$ which is also a curve of order 2 and is not a geodesic on $M(r)$. Then its structure torsion $\rho_\gamma$ satisfies the following with

$$\lambda_1 = (\sqrt{c}/2) \cot(\sqrt{c} r/2), \quad \lambda_2 = -((\sqrt{c}/2) \tan(\sqrt{c} r/2),$$

$$\lambda_3 = (\sqrt{c}/2) \cot(\sqrt{c} r/2 - \pi/4), \quad \lambda_4 = (\sqrt{c}/2) \cot(\sqrt{c} r/2 + \pi/4).$$

(I) When $0 < |\kappa| \leq \lambda_4$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma$ is strictly monotone increasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = -1$ and $\lim_{t \to \infty} \rho_\gamma(t) = 1$;

3) $\rho_\gamma$ is strictly monotone decreasing and satisfies $\lim_{t \to -\infty} \rho_\gamma(t) = 1$ and $\lim_{t \to \infty} \rho_\gamma(t) = -1$.

(II) When $-\lambda_4 > \kappa \geq \lambda_2$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\| = 1 - \rho^2_\gamma)$;

3) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = -1$ and there is $t_0$ satisfying that $\rho_\gamma$ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(III) When $|\lambda_2| \leq \kappa \leq \lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\| = 1 - \rho^2_\gamma)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\| = 1 - \rho^2_\gamma)$;

4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = -1$ and there is $t_0$ satisfying that $\rho_\gamma$ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(IV) When $\lambda_2 > \kappa \geq -\lambda_1$,

1) $\rho_\gamma \equiv 0$;

2) $\rho_\gamma \equiv \lambda_2/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\| = 1 - \rho^2_\gamma)$;

3) $\rho_\gamma \equiv \lambda_4/\kappa$, $(X_\gamma = Y_\gamma = Z_\gamma \equiv 0, \|W_\gamma\| = 1 - \rho^2_\gamma)$;

4) $\lim_{t \to -\infty} \rho_\gamma(t) = \lim_{t \to \infty} \rho_\gamma(t) = 1$ and there is $t_0$ satisfying that $\rho_\gamma$ is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly monotone decreasing on the interval $(t_0, \infty)$, and $\rho_\gamma(t_0) = \lambda_4/\kappa$.

(V) When $\lambda_1 < \kappa \leq |\lambda_3|$, 


1) $\rho_{\gamma} \equiv 0$;
2) $\rho_{\gamma} \equiv \lambda_{1}/\kappa$, \quad (Y_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|X_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
3) $\rho_{\gamma} \equiv \lambda_{2}/\kappa$, \quad (X_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|Y_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
4) $\rho_{\gamma} \equiv \lambda_{3}/\kappa$, \quad (X_{\gamma} = Y_{\gamma} = Z_{\gamma} \equiv 0, \quad \|W_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
5) $\lim_{t \to -\infty} \rho_{\gamma}(t) = \lim_{t \to -\infty} \rho_{\gamma}(t) = -1$ and there is $t_0$ satisfying that $\rho_{\gamma}$
is strictly monotone increasing on the interval $(-\infty, t_0)$ and strictly
monotone decreasing on the interval $(t_0, \infty)$, and $\rho_{\gamma}(t_0) = \lambda_4/\kappa$.

(VI) When $-\lambda_1 > \kappa \geq \lambda_3$,

1) $\rho_{\gamma} \equiv 0$;
2) $\rho_{\gamma} \equiv \lambda_{1}/\kappa$, \quad (Y_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|X_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
3) $\rho_{\gamma} \equiv \lambda_{2}/\kappa$, \quad (X_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|Y_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
4) $\rho_{\gamma} \equiv \lambda_{3}/\kappa$, \quad (X_{\gamma} = Y_{\gamma} = Z_{\gamma} \equiv 0, \quad \|W_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
5) $\lim_{t \to -\infty} \rho_{\gamma}(t) = \lim_{t \to -\infty} \rho_{\gamma}(t) = 1$ and there is $t_0$ satisfying that $\rho_{\gamma}$
is strictly monotone decreasing on the interval $(-\infty, t_0)$ and strictly
monotone increasing on the interval $(t_0, \infty)$, and $\rho_{\gamma}(t_0) = \lambda_4/\kappa$.

(VII) When $|\kappa| > |\lambda_3|$,

1) $\rho_{\gamma} \equiv 0$;
2) $\rho_{\gamma} \equiv \lambda_{1}/\kappa$, \quad (Y_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|X_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
3) $\rho_{\gamma} \equiv \lambda_{2}/\kappa$, \quad (X_{\gamma} = Z_{\gamma} = W_{\gamma} \equiv 0, \quad \|Y_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
4) $\rho_{\gamma} \equiv \lambda_{3}/\kappa$, \quad (X_{\gamma} = Y_{\gamma} = W_{\gamma} \equiv 0, \quad \|Z_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
5) $\rho_{\gamma} \equiv \lambda_{4}/\kappa$, \quad (X_{\gamma} = Y_{\gamma} = Z_{\gamma} \equiv 0, \quad \|W_{\gamma}\|^2 = 1 - \rho_{\gamma}^2);
6) $\rho_{\gamma}$ is a periodic function satisfying $\lambda_3 \leq \kappa \rho_{\gamma} \leq \lambda_4$.

We set

$$\lambda(r,c) = \begin{cases} 
\left(\sqrt{c}/2\right)\tan\left(\sqrt{c}r/2\right), & \text{if } 0 < r \leq \pi/(4\sqrt{c}), \\
\left(\sqrt{c}/2\right)\cot\left(\sqrt{c}r/2 + \pi/4\right), & \text{if } \pi/(4\sqrt{c}) < r < \pi/(2\sqrt{c}).
\end{cases}$$

As a consequence of these propositions we complete our proof of Theorem 1.

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