Graphical Realisation of Filters

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Abstract
A filter \( \mathcal{F} \) on a set \( X \) is said to be graphically realisable if there exists a connected graph \( G = (V,E) \) and a set indexer \( f : V \to 2^X \) such that \( f(V) = \mathcal{F} \). The graph \( G \) is said to be the graphical realisation of the filter \( \mathcal{F} \). Here we discuss the non isomorphic graphical realisations of finite filters.

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1 Introduction
For all terminology and notation in graph theory, we refer the reader to F.Harary [8]. The graphs considered in this paper are finite.

The graph labeling problems gained a new direction after the introduction of the notion of set-valuation or set-labeling of graphs in [2]. In such labelings, the vertices of a given graph \( G \) are labeled by certain subsets of a given ground set \( X \) and every edge of \( G \) by the symmetric difference of the set-labels of its
end vertices. That is, a set valuation of a graph $G$ is an injective set-valued function $f : V(G) \to 2^X$ such that the induced edge-function $f^\oplus : E(G) \to 2^X - \emptyset$ is defined by $f^\oplus(uv) = f(u) \oplus f(v) \forall uv \in E(G)$, where $2^X$ is the power set of $X$ and $\oplus$ denotes the symmetric difference of two sets.

For a $(p, q)$-graph $G(V, E)$ and a non-empty ground set $X$ of cardinality $n$, a set-indexer of $G$ as an injective set-valued function $f : V(G) \to 2^X$ such that the induced edge-function $f^\oplus : E(G) \to 2^X$ is also injective. A bijection $f : V(G) \cup E(G) \to 2^X - \emptyset$ such that $f(uv) = f(u) \oplus f(v)$ for all $uv \in E(G)$ is called a set-sequential labeling of $G$ (see [2, 4]). A graph $G$ to be a set-sequential graph if $G$ admits a set-sequential labeling.

A graphical realisation of topologies is defined in [6] as follows. A topology $T$ on a non-empty set $X$ is said to be graphically realisable if there exists a connected graph $G(V, E)$ and a set-indexer $f : V \to 2^X$ such that $f(V) \cup f^\oplus(E) = T$. In [6], graphical realisation of finite topologies of order up to eight have been analysed.

Motivated from the above said studies, in this paper, we intend to check whether we can establish graphical realisations to the filters on a given set $X$.

## 2 Filters and Their Graphical Realisations

We first recall the definition of filters.

**Definition 2.1.** [10] A filter on a non-empty set $X$ is a non-empty family $F$ of subsets of $X$ which satisfies the following conditions.

(i) $\emptyset \notin F$

(ii) $F$ is closed under finite intersections

(iii) if $B \in F$ and $B \subset A$, then $A \in F$ for all $A, B \subset X$.

A filter on a non-empty finite set $X$ contains at most one singleton set of $X$ as it is closed under finite intersection. A filter has the maximum cardinality when it contains a singleton set. A filter may be called a maximal filter if it contains the maximum possible number of elements. Then, a filter is a maximal filter if and only if it contains a singleton set.

We now introduce the notion of graphical realisation of filters for a given non-empty finite set $X$.

**Definition 2.2.** A filter $F$ on a set $X$ is said to be graphically realisable if there exists a connected graph $G = (V, E)$ and a set-indexer $f : V \to 2^X$ such that $f(V) = F$. 

The most important and interesting question in this context is whether all filters of a given set $X$ are graphically realisable. The following theorem establishes a solution to this question.

**Theorem 2.3.** Every filter of a given non-empty finite set $X$ is graphically realisable.

**Proof.** First note that, every filter on a finite set $X$ is the set consisting of a non-empty subset of $X$ and all of its supersets in $X$. Let $\mathcal{F} = \{A_i \subset X : i = 1, 2, \ldots, n\}$ be the given filter on $X$. Assume, without loss of generality, that $A_1 \subset A_i \forall i = 2, 3, \ldots, n$. Let $G = K_{1,n-1}$ be the star whose vertices are denoted by $v_1, v_2, \ldots, v_n$ with $v_1$ as the central vertex. Assign the set $A_1$ to the central vertex $v_1$ and $A_i$ to the vertices $v_i$ for $i = 2, 3, \ldots, n$. Obviously, this assignment is an injective set assignment $f : V(G) \to 2^X$ such that $f(V) = \mathcal{F}$.

Our task will be completed if we prove that the induced edge function $f^\oplus : E(G) \to 2^X$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ is also injective. For this, if possible let $f^\oplus(v_1v_i) = f^\oplus(v_1v_j)$ for some $i \neq j$. Then, $A_1 \oplus A_i = A_1 \oplus A_j$, $\implies$ $A_i = A_j$, a contradiction. Hence, $f^\oplus(v_1v_i) = f^\oplus(v_1v_j)$ if and only if $i = j$. Therefore, $f^\oplus$ is injective. That is, $f$ is a set-indexer on $G$. Hence, $G = K_{1,n-1}$ is a graphical realisation of the filter $\mathcal{F}$. \qed

The following result is on the number of supersets in a given set $X$ containing a given subset of $X$.

**Theorem 2.4.** \cite{12} Let $|X| = n$, and $A \subset X$. If $|A| = k$, then the number of supersets of $A$ in $X$ is $2^{n-k}$.

The following result is an immediate consequence of the above theorem.

**Proposition 2.5.** If $X$ is a non empty finite set with $|X| = n$ and $\mathcal{F}$ is a filter on $X$, then the cardinality of $\mathcal{F}$ is $2^{n-k}$, where $k$ is the cardinality of the smallest set in $\mathcal{F}$.

**Proof.** Let $\mathcal{F}$ be a filter on a non-empty set $X$ and let $A$ be its smallest element. Then, $\mathcal{F}$ is the non-empty collection of subset of $X$ containing $A$, including $A$. Hence, if $|A| = k$, then by Theorem 2.4, $\mathcal{F}$ contains $2^{n-k}$ elements. \qed

**Remark 2.6.** In view of the above result, it can be noted that a maximal filter of a set $X$ has $2^{|X|-1}$ elements.

**Theorem 2.7.** If a $(p,q)$ graph $G$ is the graphical realisation of a filter $\mathcal{F}$ on a set $X$ with $|X| = n$, then $p = |\mathcal{F}|$ and $q = 2^{n-k} - 1$, where $k$ is the cardinality of the minimal element in $\mathcal{F}$.
Proof. If \( G(V, E) \) is a graphical realisation of \( \mathcal{F} \), then there exists an injective function \( f : V \rightarrow 2^X \) such that the induced edge function \( f^\oplus : E \rightarrow 2^X - \emptyset \) is also injective. Since \( f \) is injective, \( |\mathcal{F}| = |f(V)| = |V| \).

Let \( \mathcal{F} = \{A_1, A_2, \ldots, A_{2^n-k}\} \), where \( A_1 \) be the minimal set in \( \mathcal{F} \) cardinality \( k \). Then, \( A_1 \subseteq A_i \forall A_i \in \mathcal{F} \). We claim that the symmetric differences of elements in \( \mathcal{F} \) are the non-empty subsets of \( X - A_1 \). Since \( A_1 \subseteq A_i \), we have \( X - A_i \subseteq X - A_1 \) for all \( 1 \leq i \leq 2^n-k \). Therefore, for any subsets \( A_r \) and \( A_s \) containing \( A_1 \), we have \( A_r - A_s \subseteq A_r - A_1 \subseteq X - A_1 \) and \( A_s - A_r \subseteq A_s - A_1 \subseteq X - A_1 \). Hence, \( A_r \oplus A_s \subseteq X - A_1 \). Therefore, the set-labels of all edges of \( G \) are the non-empty subsets of \( X - A_1 \). That is, \( f^\oplus(E) \subseteq 2^{X-A_1} - \emptyset \) and hence \( |f^\oplus(E)| \leq 2^{|X-A_1|} = 2^{n-k} - 1 \). Since \( G \) is a connected graph on \( 2^{n-k} \) vertices, minimum number of edges in \( G \) is \( 2^{n-k} - 1 \).

We can observe the following theorem as an immediate consequence of Theorem 2.7.

**Theorem 2.8.** The graphical realisation of any filter is a tree.

**Proof.** If a \((p, q)\) graph \( G \) is the graphical realisation of a filter \( \mathcal{F} \) on a set \( X \) with \(|X| = n\) then by Theorem 2.5 and Theorem 2.7, \( p = 2^{n-k} \) and \( q = 2^{n-k} - 1 \) where \( k \) is the cardinality of the smallest element in \( \mathcal{F} \). That is, \( G \) is a connected graph with \(|E| = |V| - 1\). Hence, \( G \) is a tree. \(\square\)

**Corollary 2.9.** The graphical realisation of filters of cardinality 1 is \( K_1 \) and that of cardinality 2 is \( K_2 \).

**Proof.** The result follows from the fact that the tree of order 1 is \( K_1 \) and the only tree of order 2 is \( K_2 \). \(\square\)

**Corollary 2.10.** The graphical realisation of filters of cardinality \( 2^2 \) is isomorphic to \( K_{1,3} \).

**Proof.** Let \( \mathcal{F} = \{A, B, C, X\} \) be the filter of cardinality 4 on \( X \) such that \( A \) is the smallest subset of \( X \) in \( \mathcal{F} \) and let \( G \) be its graphical realisation. Then, by Theorem 2.7, \( G \) must be a tree of order 4. Hence \( G \) is isomorphic either to \( K_{1,3} \) or to \( P_4 \). Then, it is done if we show that \( G \) is not isomorphic to \( P_4 \). Let \( A \) be the minimal set in \( \mathcal{F} \). Then \( A \) is a subset of all other elements \( B, C \) and \( X \). First, consider the graph \( P_4 : v_1v_2v_3v_4 \). The vertices of \( P_4 \) can be labeled by the sets \( A, B, C \) and \( X \) in an injective manner in \( 4! = 24 \) ways. But some of these labelings are isomorphic labelings of \( P_4 \) in terms of the adjacency between the vertices concerned. For example, \( (A, B, C, X) \) and \( (X, C, B, A) \) are isomorphic labelings of \( P_4 \). Then, the desired labeling can be done in any one of the following 12 different ways. \( (X, B, C, A), (X, C, A, B), (X, C, B, A), (B, C, A, X), (B, C, X, A), \)
Consider the labeling \((X, B, C, A)\). Then, since \(A \subset B\) and \(C \subset X\) and if possible let \(B \not\subset C\). Then, \(B \cap C = A\) and \(B \cup C = X\). Hence, \(X \oplus B = X - B, B \oplus C = X - A, C \oplus A = C - A\). But \(X - B = B \cup C - B = (B \cup C) \cap B^c = (B \cap B^c) \cup (C \cap B^c) = C \cap B^c = C - B\) and \(C - A = C \cap A^c = C \cap (B \cap C)^c = C \cap (B^c \cup C^c) = (C \cap B^c) \cup (C \cap C^c) = C \cap B^c = C - B\). Therefore, \(X \oplus B = C \oplus A\). This indicates that the symmetric difference between any pair of elements in the labelings written above is not injective. Hence \(G\) is not isomorphic to \(P_4\).

In view of Theorem 2.5, Theorem 2.7 and Proposition 2.5, we now establish the following result.

**Theorem 2.11.** No non-trivial tree on odd number of vertices can be a graphical realisation of a filter of a non-empty finite set.

**Proof.** Let \(T\) be a tree on odd number of vertices. If possible, let \(T\) be a realisation of a given filter \(F\) on a non-empty set \(X\). Then, by Proposition 2.5, \(F\) has the cardinality \(2^{|X|} - |A|\), where \(A\) is the minimal set in \(F\). Then, by Theorem 2.7, \(|V| = |F| = 2^{|X|} - |A|\), a contradiction to the fact that \(T\) has odd number of vertices. Therefore, \(T\) can not be a realisation of any filter \(F\).

**Theorem 2.12.** If a \((p, q)\) graph \(G\) is a graphical realisation of a filter \(F\), then \(G\) is a set-sequential graph.

**Proof.** Suppose that \(G\) is the graphical realisation of a filter \(F\). Let the cardinality of \(F\) be \(2^n\) for some \(n \in \mathbb{N}\). Then \(p = 2^n\) and \(q = 2^{n-1}\). Let \(X\) be any set of \(n+1\) elements. Let \(F'\) be the maximal filter on \(X\). Then, \(|F'| = 2^{|X|} - 1\). Now, assign the vertices of \(G\) by elements in \(F'\) in an injective manner such that the edge valuations are distinct. As explained in Theorem 2.7, the edge valuations are non-empty subsets of \(X - A\), where \(A\) is the only singleton set in \(F'\). Therefore, \(f(V)\) contains \(2^n\) sets and \(f \oplus (E)\) contains a total of \(2^n - 1\) sets. Therefore, the total number of vertex and edge assignments are \(2^n + 2^n - 1 = 2^{n+1} - 1\), which is equal to the number non-empty distinct subsets of \(X\). Hence \(G\) is set sequential.

**Remark 2.13.** The converse of the above theorem is not true. That is, every set sequential graphs need not be the graphical realisations of some filters.

From the study we have made so far, we infer that no path \(P_n\) is a graphical realisation of a filter on a non-empty finite set \(X\). Hence, we strongly believe that the following problem hold.

**Problem 2.14.** For \(n \geq 2\), no path \(P_{2^n}\) can be a realisation of a filter defined on a non-empty finite set \(X\).
3 Conclusion

In this paper, our aim is to find graphical realisation of a filter defined on a given non-empty finite set $X$. Certain problems in this area are still open. We have characterised the graphical realisations of filters having cardinality up to $2^2$.

Problem 3.1. Find the non isomorphic graphical realisations of filters of cardinality $2^n$ for $n \geq 3$?

We have found that trees on odd number of vertices are not the realisations of filters. Also, it is not necessary that all trees having even number of vertices are graphical realisations of some filters on $X$. Then,

Problem 3.2. Determine the conditions required for a tree $T$ to be a graphical realisation of a given filter on $X$.

References


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