In this paper we give inequalities on volume growths of Euclidean bodies which are obtained by $n$-cubes through nonlinear transformations.

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1. INTRODUCTION

It is needless to say that inequalities play quite important rules in the study of many area not only in mathematics but also in other sciences. Also, it is well known that we can explain many inequalities from geometric point of view. In [3] the author showed an inequality of product type (see Proposition 1 in §2). Geometrically speaking it is concerned with growths of volumes of $n$-cubes in a Euclidean space: The growth of volumes of a big cube is greater than the growth of volumes of a small cube. Since this inequality shows also differences of volume growths of $n$-bodies through non-singular linear transformations, we are interested in volume growths of other Euclidean bodies. The aim of this paper is to give inequalities which concern with volume growths of Euclidean bodies obtained by $n$-cubes through non-linear transformations.

2. DIFFERENCE OF VOLUMES OF $n$-CUBES

In the book [3] the author showed the following result (Lemma 5.1.3, p.166).

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**Proposition 1.** For arbitrary real numbers $a_j, b_j, c_j$ ($j = 1, \ldots, n$) satisfying $0 < a_j \leq b_j$ and $c_j \geq 0$ for all $j$, we have

\[
\prod_{j=1}^{n} (b_j + c_j) - \prod_{j=1}^{n} b_j \geq \prod_{j=1}^{n} (a_j + c_j) - \prod_{j=1}^{n} a_j.
\]

Here, the equality holds if and only if one of the following cases occurs:

i) $a_j = b_j$ for $j = 1, \ldots, n$;

ii) $c_j = 0$ for $j = 1, \ldots, n$;

iii) There is $j_0$ satisfying that $a_j = b_j$ and $c_j = 0$ hold for all $j$ with $j \neq j_0$.

We can show this proposition by induction (see §3). In order to show our aim of this paper we explain a geometric meaning of this result. We take two $n$-cubes each of whose lengths of sides are $a_j$ ($j = 1, \ldots, n$) and $b_j$ ($j = 1, \ldots, n$). If we extend their sides by $c_j$, then we find that the growth of the volume of large cube is greater than that of small cube.

We can give such an explanation of (2.1) by use of other $n$-dimensional shapes in a Euclidean space. One is to use volumes of parallelohedrons. We take a linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$. We consider a parallelohedron made by $a_1 v_1, \ldots, a_n v_n$, and extend their sides by $c_j \|v_j\|$. Then the difference of their volumes is $|\det(v_1, \ldots, v_n)| \left\{ \prod_{j=1}^{n} (a_j + c_j) - \prod_{j=1}^{n} a_j \right\}$. The other is to use volumes of ellipsoid bodies. We consider an ellipsoid body whose lengths of axes are $2a_1, \ldots, 2a_n$. If we extend them by $2c_j$, then the difference of their volumes is $\kappa_n \left\{ \prod_{j=1}^{n} (a_j + c_j) - \prod_{j=1}^{n} a_j \right\}$, where $\kappa_n$ denotes the volume of an $n$-disk $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$. Thus, (2.1) shows growths of volumes of these bodies.

### 3. A Generalization of the Inequality on Volume Differences

Being inspired with Proposition 1 we are interested in volume growths of other bodies. In this section we generalize Proposition 1. As we see in §2, Proposition 1 shows the difference of Euclidean bodies which are obtained by linear diagonalizable transformation of an $n$-cube. We therefore study non-linear transformations. For this sake we first show the following elementary lemma.

**Lemma 1.** Let $f : I \to \mathbb{R}$ be a $C^1$-function. The derivative $f'$ is non-decreasing if and only if it satisfies

\[
f(b + c) - f(b) \geq f(a + c) - f(a)
\]

for arbitrary $a, b \in I$ satisfying $a \leq b$ and for an arbitrary positive $c$ with $b + c \in I$.

**Proof.** Suppose $f'$ is non-decreasing. We then have

\[
f(b + c) - f(b) = \int_{b}^{b+c} f'(t) \, dt \geq \int_{a}^{a+c} f'(t) \, dt = f(a + c) - f(a).
\]
On the other hand, we suppose (3.1) holds for arbitrary \(a, b \in I\) with \(a \leq b\) and an arbitrary positive \(c\). We then have
\[
\frac{1}{c}\{f(b + c) - f(b)\} \geq \frac{1}{c}\{f(a + c) - f(a)\}.
\]
As \(f\) is of \(C^1\), this shows \(f'(b) \geq f'(a)\). This completes the proof. \(\square\)

Here, we study when the equality holds in Lemma 1.

**Lemma 2.** Let \(a, b\) be real numbers with \(a < b\) and \(c\) be a positive number. Suppose a \(C^1\)-function \(f\) has non-decreasing derivative \(f'\) on the interval \([a, b + c]\). Then \(f\) is a constant function or a polynomial of degree 1 if and only if \(f(b + c) - f(b) = f(a + c) - f(a)\) holds.

**Proof.** Since the “only if” part is trivial, we are enough to show the “if” part. As we have
\[
0 = \{f(b + c) - f(b)\} - \{f(a + c) - f(a)\} = \int_a^{a+c} \{f'(t + b - a) - f'(t)\} dt
\]
and \(f\) is of \(C^1\), we find \(f'(t + b - a) = f'(t)\) \((a \leq t \leq a + c)\). Similarly, as the equality shows also \(f(b + c) - f(a + c) = f(b) - f(a)\), we find \(f'(t + c) = f'(t)\) \((a \leq t \leq b)\). Since \(f'\) is not decreasing, these show that \(f'\) is constant on the interval \([a, b + c]\). \(\square\)

Clearly the same result holds for a \(C^1\)-function whose derivative is non-increasing. By these results, we find that in Lemma 1, the equality holds if and only if one of the following conditions holds:

i) \(a = b\),
ii) \(c = 0\),
iii) \(f\) is constant on \([a, b + c]\),
iv) \(f\) is a polynomial of degree 1 on \([a, b + c]\).

We now give a generalization of Proposition 1.

**Theorem 1.** Let \(f_j : I \to \mathbb{R} \ (j = 1, \ldots, n)\) be positive \(C^1\)-functions of an interval \(I\). Suppose that their derivatives \(f'_j\) are nonnegative and non-decreasing. We then have
\[
(3.2) \prod_{j=1}^{n} f_j(b_j + c_j) - \prod_{j=1}^{n} f_j(b_j) \geq \prod_{j=1}^{n} f_j(a_j + c_j) - \prod_{j=1}^{n} f_j(a_j)
\]
for arbitrary \(a_j, b_j \in I\) and arbitrary nonnegative \(c_j\) satisfying \(a_j \leq b_j\) and \(b_j + c_j \in I\).

**Proof.** We shall show the inequality by mathematical induction on \(n\). When \(n = 1\), the inequality follows from Lemma 1. Assume that the inequality holds for \(n - 1\). By our condition we see that every \(f_j\) is monotone non-decreasing.
Since each $f_j$ is positive, we have $\prod_{j=1}^{n-1} f_j(b_j + c_j) \geq \prod_{j=1}^{n-1} f_j(b_j) > 0$, hence we obtain
\[
\prod_{j=1}^{n} f_j(b_j + c_j) - \prod_{j=1}^{n} f_j(b_j)
= \left\{ \prod_{j=1}^{n-1} f_j(b_j + c_j) - \prod_{j=1}^{n-1} f_j(b_j) \right\} f_n(b_n + c_n)
+ \left\{ \prod_{j=1}^{n-1} f_j(b_j) \right\} \{ f_n(b_n + c_n) - f_n(b_n) \}
\geq \left\{ \prod_{j=1}^{n-1} f_j(a_j + c_j) - \prod_{j=1}^{n-1} f_j(a_j) \right\} f_n(a_n + c_n)
+ \left\{ \prod_{j=1}^{n-1} f_j(a_j) \right\} \{ f_n(a_n + c_n) - f_n(a_n) \}
= \prod_{j=1}^{n} f_j(a_j + c_j) - \prod_{j=1}^{n} f_j(a_j).
\]
Thus we obtain the inequality. \hfill \Box

From the geometric point of view, Theorem 1 shows the difference of volume growths of Euclidean bodies obtained by $n$-cubes through a non-linear transformation $(x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n))$ of $\mathbb{R}^n$, where each $f_j : \mathbb{R} \to \mathbb{R}$ is monotone increasing with $f_j(0) = 0$. We note that we may suppose $f_j(0) = 0$ with the aid of parallel translations.

Our result is closely related with convexity of functions (for convex functions, see [2] for example).

**Theorem 2.** Let $f_j : I \to \mathbb{R}$ ($j = 1, \ldots, n$) be positive $C^1$-functions of an interval $I$. Suppose that their derivative $f'_j$ are nonpositive and non-decreasing. We then have
\[
(3.3) \quad \prod_{j=1}^{n} f_j(b_j + c_j) - \prod_{j=1}^{n} f_j(b_j) \geq \prod_{j=1}^{n} f_j(a_j + c_j) - \prod_{j=1}^{n} f_j(a_j)
\]
for arbitrary $a_j, b_j \in I$ and arbitrary nonnegative $c_j$ satisfying $a_j \leq b_j$ and $b_j + c_j \in I$.

**Proof.** We show the inequality by mathematical induction on $n$. When $n = 1$, the inequality follows from Lemma 1. We assume the inequality holds for $n-1$. By our condition we see that every $f_j$ is monotone non-increasing. Since each $f_j$ is positive, we have $0 < \prod_{j=1}^{n-1} f_j(b_j + c_j) \leq \prod_{j=1}^{n-1} f_j(b_j)$ and $0 < \prod_{j=1}^{n-1} f_j(a_j + c_j) \leq \prod_{j=1}^{n-1} f_j(a_j)$. Taking account that $f_n(a_n + c_n) - f_n(a_n)$ and $f(b_n + c_n) - f(b_n)$ are nonpositive, we get the inequality by the same argument as in the proof of Theorem 1. \hfill \Box
We here discuss when the equality holds in (3.2) or in (3.3).

**Proposition 2.** Let $f_j : I \to \mathbb{R}$ ($j = 1, \ldots, n$) be positive $C^1$-functions of an interval $I$. Suppose that their derivatives $f_j'$ are nonnegative and non-decreasing. For $a_j, b_j \in I$ and $c_j \geq 0$ satisfying $a_j \leq b_j$ and $b_j + c_j \in I$ the equality

$$\prod_{j=1}^{n} f_j(b_j + c_j) - \prod_{j=1}^{n} f_j(b_j) = \prod_{j=1}^{n} f_j(a_j + c_j) - \prod_{j=1}^{n} f_j(a_j)$$

hold if and only if one of the following conditions holds:

i) There is a subset $J$ of $\{1, \ldots, n\}$ satisfying that $f_j$ is constant on $[a_j, b_j + c_j]$ for $j \in J$ and that $a_j = b_j$ for $j \not\in J$;

ii) There is a subset $J$ of $\{1, \ldots, n\}$ satisfying that $f_j$ is constant on $[a_j, b_j + c_j]$ for $j \in J$ and that $c_j = 0$ for $j \not\in J$;

iii) there is $j_0$ satisfying the following conditions;

(a) for $j \neq j_0$ either $a_j = b_j$ and $c_j = 0$ hold, or $f_j$ is constant,

(b) $a_{j_0} = b_{j_0}$ holds, or $c_{j_0} = 0$ holds, or $f_{j_0}$ is a polynomial of degree 1

on $[a_{j_0}, b_{j_0} + c_{j_0}]$.

In the conditions i) and ii), $J$ might be an empty set or might coincide with $\{1, \ldots, n\}$.

**Proof.** We shall show our result by mathematical induction on $n$. When $n = 1$, it is the case that the equality in Lemma 1 holds. Hence we find our assertion holds in this case.

We suppose our assertion holds when $n - 1$. By the equality we have

$$\left\{\prod_{j=1}^{n-1} f_j(b_j + c_j) - \prod_{j=1}^{n-1} f_j(b_j)\right\} f_n(b_n + c_n) - \left\{\prod_{j=1}^{n-1} f_j(a_j + c_j) - \prod_{j=1}^{n-1} f_j(a_j)\right\} f_n(a_n + c_n)$$

$$+ \prod_{j=1}^{n-1} f_j(b_j)\left\{f_n(b_n + c_n) - f_n(b_n)\right\} - \prod_{j=1}^{n-1} f_j(a_j)\left\{f_n(a_n + c_n) - f_n(a_n)\right\} = 0.$$ 

Since $f_j$ and $f_j'$ are non-decreasing and since we have an inequality (3.2) for $n - 1$, we obtain that both of the following two equalities hold:

$$\prod_{j=1}^{n-1} f_j(b_j + c_j) - \prod_{j=1}^{n-1} f_j(b_j)\right\} f_n(b_n + c_n)$$

$$= \left\{\prod_{j=1}^{n-1} f_j(a_j + c_j) - \prod_{j=1}^{n-1} f_j(a_j)\right\} f_n(a_n + c_n),$$

$$\prod_{j=1}^{n-1} f_j(b_j)\left\{f_n(b_n + c_n) - f_n(b_n)\right\} = \prod_{j=1}^{n-1} f_j(a_j)\left\{f_n(a_n + c_n) - f_n(a_n)\right\}.$$
As $f$ is a positive function, (3.4) shows that one of the following conditions holds:

i) $\prod_{j=1}^{n-1} f_j(b_j + c_j) = \prod_{j=1}^{n-1} f_j(b_j)$ and $\prod_{j=1}^{n-1} f_j(a_j + c_j) = \prod_{j=1}^{n-1} f_j(a_j),$

ii) $f_n(a_n + c_n) = f_n(b_n + c_n)$ and $\prod_{j=1}^{n-1} f_j(b_j + c_j) - \prod_{j=1}^{n-1} f_j(b_j) = \prod_{j=1}^{n-1} f_j(a_j + c_j) - \prod_{j=1}^{n-1} f_j(a_j).

Also, (3.5) shows that one of the following conditions holds:

i) $f_n(b_n + c_n) = f_n(b_n)$ and $f_n(a_n + c_n) = f_n(a_n),$

ii) $f_n(b_n + c_n) - f_n(b_n) = f_n(a_n + c_n) - f_n(a_n)$ and $\prod_{j=1}^{n-1} f_j(a_j) = \prod_{j=1}^{n-1} f_j(b_j).

Therefore, we find that one of the following cases occurs:

C-1) $f_n(a_n) = f_n(b_n) = f_n(a_n + c_n) = f_n(b_n + c_n)$ and $\prod_{j=1}^{n-1} f_j(b_j + c_j) - \prod_{j=1}^{n-1} f_j(b_j) = \prod_{j=1}^{n-1} f_j(a_j + c_j) - \prod_{j=1}^{n-1} f_j(a_j)$ hold:

C-2) $f_n(a_n) = f_n(a_n + c_n)$, $f_n(b_n) = f_n(b_n + c_n)$, $\prod_{j=1}^{n-1} f_j(b_j + c_j) = \prod_{j=1}^{n-1} f_j(b_j)$ and $\prod_{j=1}^{n-1} f_j(a_j + c_j) = \prod_{j=1}^{n-1} f_j(a_j)$ hold:

C-3) $f_n(a_n) = f_n(b_n)$, $f_n(a_n + c_n) = f_n(b_n + c_n)$, $\prod_{j=1}^{n-1} f_j(a_j) = \prod_{j=1}^{n-1} f_j(b_j)$ and $\prod_{j=1}^{n-1} f_j(a_j + c_j) = \prod_{j=1}^{n-1} f_j(b_j + c_j)$

C-4) $f_n(b_n + c_n) - f_n(b_n) = f_n(a_n + c_n) - f_n(a_n)$ and $\prod_{j=1}^{n-1} f_j(b_j + c_j) = \prod_{j=1}^{n-1} f_j(b_j) = \prod_{j=1}^{n-1} f_j(a_j + c_j) = \prod_{j=1}^{n-1} f_j(a_j)$ hold.

We study the above cases one by one. In the case C-1), we find that either $f_n$ is constant on $[a_n, b_n + c_n]$ or $a_n = b_n$ and $c_n = 0$ hold. With the assumption of the induction, we get our assertion in this case. In the case C-4), we find that either $f_j$ is constant on $[a_j, b_j + c_j]$ or $a_j = b_j$ and $c_j = 0$ hold for all $j$ with $j \leq n - 1$, and that one of the following conditions holds: $a_n = b_n$, $c_n = 0$, $f_n$ is constant on $[a_n, b_n + c_n]$, $f_n$ is a polynomial of degree 1 on $[a_n, b_n + c_n].$

Hence we get our assertion also in this case.

We next study the case C-2). Since each $f_j$ is non-decreasing, we find $f_j(a_j) = f_j(a_j + c_j)$ and $f(b_j) = f(b_j + c_j)$ hold for all $j$. For each $j$, these two equalities lead us to that either $c_j = 0$ holds or $f_j$ is constant on $[a_j, a_j + c_j] \cup [b_j, b_j + c_j]$. We consider the latter case. When $b_j \leq a_j + c_j$, it is clear that $f_j$ is constant on $[a_j, b_j + c_j]$. When $a_j + c_j < b_j$, as $f_j'$ is in non-decreasing, we find that $f_j$ is constant on $[a_j, b_j + c_j]$. Thus, in the case C-2) we find that there is a subset $J$ of $\{1, \ldots, n\}$ satisfying that $f_j$ is constant on $[a_j, b_j + c_j]$ for $j \in J$ and $c_j = 0$ for $j \notin J$.

We finally study the case C-3). Since each $f_j$ is non-decreasing, we find $f_j(a_j) = f_j(b_j)$ and $f(a_j + c_j) = f(b_j + c_j)$ hold for all $j$. For each $j$, these two equalities lead us to that either $a_j = b_j$ holds or $f_j$ is constant on $[a_j + c_j, b_j + c_j]$. When $a_j + c_j < b_j$, it is clear that $f_j$ is constant on $[a_j, b_j + c_j]$. When $b_j \leq a_j + c_j$, as $f_j'$ is not decreasing, we find that $f_j$ is constant on $[a_j, b_j + c_j]$. Thus, in the case C-2) we find that there is a subset $J$ of $\{1, \ldots, n\}$ satisfying that $f_j$ is constant on $[a_j, b_j + c_j]$ for $j \in J$ and $a_j = b_j$ for $j \notin J$. Thus we get our conclusion. □
We here give some examples. We take \( f_j(x) = x^{\alpha_j} \) \((x \geq 0)\) with \( \alpha_j \geq 1 \). When all \( \alpha_j \) are integers, the following is a consequence of Proposition 1.

**Example 1.** Let \( \alpha \) be a real number with \( \alpha \geq 1 \). For arbitrary \( a_j, b_j, c_j \) satisfying \( 0 < a_j \leq b_j, \ c_j \geq 0 \), we have

\[
\prod_{j=1}^{n} (b_j + c_j)^{\alpha_j} - \prod_{j=1}^{n} b_j^{\alpha_j} \geq \prod_{j=1}^{n} (a_j + c_j)^{\alpha_j} - \prod_{j=1}^{n} a_j^{\alpha_j},
\]

where equality holds if and only if \( a_j = b_j \) for \( j = 1, \ldots, n \) or \( c_j = 0 \) for \( j = 1, \ldots, n \).

If we take \( f(x) = \sinh x \), as \( f'(x) = \cosh x \) is a positive increasing function on \( x \geq 0 \), we have the following.

**Example 2.** For arbitrary \( a_j, b_j, c_j \) satisfying \( 0 < a_j \leq b_j, \ c_j \geq 0 \), we have

\[
\prod_{j=1}^{n} \sinh(b_j + c_j) - \prod_{j=1}^{n} \sinh b_j \geq \prod_{j=1}^{n} \sinh(a_j + c_j) - \prod_{j=1}^{n} \sinh a_j,
\]

where equality holds if and only if \( a_j = b_j \) for \( j = 1, \ldots, n \) or \( c_j = 0 \) for \( j = 1, \ldots, n \).

We can explain Theorem 2 geometrically. We consider volumes of \( n \)-subcubes. This time we measure lengths sides from the diagonal point. That is, we take \( f_j(x) = R_j - x(0 \leq x \leq R_j) \).

**Example 3.** For arbitrary \( a_j, b_j, c_j \) satisfying \( 0 < a_j \leq b_j < R_j, \ c_j \geq 0 \), we have

\[
\prod_{j=1}^{n} \{R_j - (b_j + c_j)\} - \prod_{j=1}^{n} \{R_j - b_j\} \geq \prod_{j=1}^{n} \{R_j - (a_j + c_j)\} - \prod_{j=1}^{n} \{R_j - a_j\},
\]

where equality holds if and only if \( a_j = b_j \) for \( j = 1, \ldots, n \) or \( c_j = 0 \) for \( j = 1, \ldots, n \).

If we take \( f(x) = 1 - \sin x \), as \( f'(x) = -\cos x \) is a negative increasing function on the interval \( 0 \leq x \leq \pi/2 \), we have the following.

**Example 4.** For arbitrary \( a_j, b_j, c_j \) satisfying \( 0 < a_j \leq b_j < \pi/2, \ c_j \geq 0 \), we have

\[
\prod_{j=1}^{n} \{1 - \sin(b_j + c_j)\} - \prod_{j=1}^{n} \{1 - \sin b_j\} \geq \prod_{j=1}^{n} \{1 - \sin(a_j + c_j)\} - \prod_{j=1}^{n} \{1 - \sin a_j\},
\]

where equality holds if and only if \( a_j = b_j \) for \( j = 1, \ldots, n \) or \( c_j = 0 \) for \( j = 1, \ldots, n \).

These examples are closely related with volume elements of geodesic spheres of real space forms, which are standard spheres, Euclidean space, and real hyperbolic spaces. For volumes of geodesic balls it is better for the reader to refer [1] for example.
REFERENCES


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