On Generalized $f$-Derivations of Incline Algebras

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Abstract

In this paper, we introduced the notion of generalized $f$-derivation of an incline and stated some related properties. Also, some properties of generalized $f$-derivation studied for an integral incline algebra.

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1. Introduction

The notion of incline algebras was introduced by [16] and their applications were studied by [9, 10, 11, 15]. Inclines are a special type of a semiring, also are a generalization of both Boolean and Fuzzy algebras, and they give a way to
combine algebras and ordered structures to express the degree of intensity bi-
nary relations. It has both a semiring structure and a poset structure. Inclines
can also be used to represent automata and other mathematical systems.

N. O. Alshehri [8] introduced the notion of derivation of an incline algebra
and proved some result on derivation of an incline and integral incline. Ş.
Özbay and A. Fırat [12] introduced the notion of an $f$-derivation of an incline
algebra and investigated some of its properties.

The notion of the symmetric bi-derivation was defined in [1, 2] by Maksa and
a lot of researchers studied the symmetric bi-derivation in rings, near-rings and
lattices [3, 4, 5, 6, 7]. Ş. Özbal and A. Fırat applied the notion of symmetric
bi-derivations in rings and near rings and lattices to incline algebras in [13];
gave examples and discussed some related properties. They introduced
the notion of generalized derivation in an incline algebra and investigated some of
its properties in an incline and integral incline algebra in [14].

In this paper, as a generalization of derivation of an incline algebra, the
notion of generalized $f$-derivation in an incline algebra is introduced and, some
related properties are investigated.

2. Preliminaries

**Definition 2.1.** [16] An incline algebra is a non-empty set $R$ with binary op-
erations denoted by $+$ and $\ast$ satisfying the following axioms for all $x, y, z \in R$:

(RI) $x + y = y + x,$

(RII) $x + (y + z) = (x + y) + z,$

(RIII) $x \ast (y \ast z) = (x \ast y) \ast z,$

(RIV) $x \ast (y + z) = (x \ast y) + (x \ast z),$
On generalized $f$-derivations of incline algebras

(RV) $(y + z) \ast x = (y \ast x) + (z \ast x)$,

(RVI) $x + x = x$,

(RVII) $x + (x \ast y) = x$,

(RVIII) $y + (x \ast y) = y$.

Furthermore, an incline algebra $R$ is said to be commutative if $x \ast y = y \ast x$ for all $x, y \in R$.

For convenience, we pronounce “$+$” (resp. “$\ast$”) as addition (resp. multiplication). Every distributive lattice is an incline. An incline is a distributive lattice (as a semiring) if and only if $x \ast x = x$ for all $x \in R$ ([3, Proposition (1.1.1)]). A subincline of an incline $R$ is a nonempty subset $M$ of $R$ which is closed under addition and multiplication. An ideal in an incline $R$ is a subincline $M \subseteq R$ such that if $x \in R$ and $y \leq x$ then $y \in M$. An element $0$ in an incline algebra $R$ is a zero element if $x + 0 = x = 0 + x$ and $x \ast 0 = 0 \ast x = 0$ for any $x \in X$. An element $1$ (≠ zero element) in an incline algebra $R$ is called multiplicative identity if for any $x \in R$, $x \ast 1 = 1 \ast x = x$. A non-zero element $a$ in an incline algebra $R$ with a zero element is said to be a left (resp. right) zero divisor if there exists a non-zero element $b \in R$ such that $a \ast b = 0$ (resp. $b \ast a = 0$). A zero divisor is an element of $R$ which is both a left zero divisor and a right zero divisor. An incline algebra $R$ with a multiplicative identity $1$ and a zero element $0$ is called an integral incline if it has no zero divisors.

Note that $x \leq y$ if and only if $x + y = y$ for all $x, y \in R$. It is easy to see that $\leq$ is a partial order on $R$ and that for any $x, y \in R$, the element $x + y$ is the least upper bound of $x, y$. We say that $\leq$ is induced by operation $+$. It follows that

(1) $x \ast y \leq x$ and $x \ast y \leq y$ for all $x, y \in R$. 

(2) \( y \leq z \) implies \( x \ast y \leq x \) and \( y \ast x \leq z \ast x \) for any \( x, y, z \in R \).

(3) If \( x \leq y, a \leq b \), then \( x + a \leq y + b, x \ast a \leq y \ast b \).

3. GENERALIZED \( f \)-DERIVATIONS OF INCLINE ALGEBRAS

The following definition introduces the notion of generalized \( f \)-derivation for an incline algebra.

**Definition 3.1.** Let \( R \) be an incline algebra. A function \( D : R \to R \) is called a generalized \( f \)-derivation of \( R \), if there exists a derivation \( d \) and a function \( f \) of \( R \) such that for all \( x, y \in R \)

\[
D(x \ast y) = (D(x) \ast f(y)) + (f(x) \ast d(y))
\]

**Example 3.1.** Let \( R = \{0, a, b, c, d, 1\} \), and we define the sum ” \(+\) ” and product ” \(*\) ” on \( R \) as follows:

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Then \( (R, +, \ast) \) is an incline but not a distributive lattice. We define a function \( D : R \to R \) for all \( x \) in \( R \) by

\[
D(x) = \begin{cases} 
0, & x=0 \\
d, & \text{otherwise}
\end{cases}
\]

and let \( d : R \to R \) be a derivation of \( R \) such that

\[
d(x) = \begin{cases} 
0, & x=0, a \\
d, & \text{otherwise}
\end{cases}
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Then \(D\) is not a generalized derivation of \(R\) since \(D(a \ast b) = D(0) = 0\), but \((D(a) \ast b)+(a \ast d(b)) = (d \ast b)+(a \ast d) = d+0 = d\) and thus \(D(a \ast b) \neq (D(a) \ast b)+(a \ast d(b))\).

But if we define \(f\) as a function of \(R\) as

\[
f(x) = \begin{cases} 
0, & x=b, c, d, 1 \\
a, & \text{otherwise}
\end{cases}
\]

then \(D\) satisfies the equation in Definition 3.1, i.e. \(D\) is a generalized \(f\)-derivation of \(R\).

**Proposition 3.2.** Let \(R\) be an incline algebra and \(D\) be a generalized \(f\)-derivation of \(R\) associated with a derivation \(d\) and a function \(f\) of \(R\). Then the followings hold for all \(x, y\) in \(R\):

(i) \(D(x \ast y) \leq D(x) + d(y)\),

(ii) If \(x \leq y\) and \(f\) is an order preserving mapping then \(D(x \ast y) \leq f(y)\).

(i) Let \(x, y \in R\). We know that from (1) we have \(D(x) \ast f(y) \leq D(x)\) and \(f(x) \ast d(y) \leq d(y)\). Then by using (3) we get \(D(x \ast y) = (D(x) \ast f(y)) + (f(x) \ast d(y)) \leq D(x) + d(y)\). Hence we find \(D(x \ast y) \leq D(x) + d(y)\).
(ii) Let $x \leq y$ and $f$ be an order preserving mapping. Then by using (3) and (1) we get $f(x) \ast d(y) \leq f(y) \ast d(y) \leq f(y)$. Similarly, we can get $D(x) \ast f(y) \leq f(y)$. Then we obtain, $D(x \ast y) = (D(x) \ast f(y)) + (f(x) \ast d(y)) \leq f(y) + f(y) = f(y)$. Hence we have $D(x \ast y) \leq f(y)$.

**Proposition 3.3.** Let $R$ be an incline algebra with a zero element and $D$ be a generalized $f$-derivation of $R$ associated with a derivation $d$ and a function $f$ of $R$. If $f(0) = 0$ then we have $D(0) = 0$.

Proof: Since $R$ is an incline algebra with a zero element we have $x \ast 0 = 0 \ast x = 0$ for all $x \in R$ then we can write
$$D(0) = D(x \ast 0) = (D(x) \ast f(0)) + (f(x) \ast d(0)) = f(x) \ast d(0).$$
By [8] we have $d(0) = 0$. Therefore $D(0) = 0$.

**Proposition 3.4.** Let $R$ be an incline algebra with a multiplicative identity element and $D$ be a generalized $f$-derivation of $R$ associated with a derivation $d$ and a function $f$ of $R$. If $f(1) = 1$ then the followings hold for all $x \in R$:

(i) $f(x) \ast d(1) \leq D(x)$,

(ii) If $d(1) = 1$, then $f(x) \leq D(x)$.

Proof:

(i) Since $R$ is an incline algebra with a multiplicative identity element we have $x \ast 1 = 1 \ast x = x$ for all $x \in R$, then we can write $D(x) = D(x \ast 1) = (D(x) \ast f(1)) + (f(x) \ast d(1))$. Then by using $f(1) = 1$ we have $D(x) = D(x) + (f(x) \ast d(1))$. Therefore we get, $f(x) \ast d(1) \leq D(x)$.

(ii) It can be derived from (i).
Proposition 3.5. Let $R$ be an integral incline and $D$ be a generalized $f$-derivation of $R$ associated with a derivation $d$ and a function $f$ of $R$ and $a$ be an element of $R$. If $f(1) = 1$ then for all $x \in R$ we have:

(i) $a \ast D(x) = 0$ implies that $a = 0$ or $d = 0$,

(ii) $d(x) \ast a = 0$ and $D(x) \ast a = 0$ imply that $a = 0$ or $D = 0$.

Proof:

(i) Let $a \ast D(x) = 0$ for all $x \in R$. If we replace $x$ by $x \ast y$ for $y \in R$ we get

$$
0 = a \ast D(x) = a \ast D(x \ast y) = a \ast [(D(x) \ast f(y)) + (f(x) \ast d(y))] \\
= (a \ast (D(x) \ast f(y))) + (a \ast (f(x) \ast d(y))) \\
= a \ast (f(x) \ast d(y))
$$

In this equation by taking $x = 1$ we get $a \ast d(y) = 0$. Since $R$ is an integral incline we have $a = 0$ or $d = 0$.

(ii) Let $d(x) \ast a = 0$ and $D(x) \ast a = 0$ for all $x \in R$. If we replace $x$ by $x \ast y$ for $y \in R$ in $D(x) \ast a = 0$ we get

$$
0 = D(x) \ast a = D(x \ast y) \ast a = [(D(x) \ast f(y)) + (f(x) \ast d(y))] \ast a \\
= ((D(x) \ast f(y)) \ast a) + ((f(x) \ast d(y)) \ast a) \\
= (D(x) \ast f(y)) \ast a
$$

In this equation by taking $y = 1$ we get $D(x) \ast a = 0$. Since $R$ is an integral incline we have $a = 0$ or $D = 0$.

Theorem 3.6. Let $M$ be a nonzero ideal of an integral incline $R$. If $D$ is a nonzero generalized $f$-derivation of $R$ associated with a nonzero derivation $d$ and a function $f$ of $R$, then $D$ is nonzero on $M$. 

Proof: Assume that $D$ is a nonzero generalized $f$-derivation of $R$ associated with a nonzero derivation $d$ and a function $f$ of $R$ but $D$ is zero generalized $f$-derivation on $M$. Let $x \in M$. Then we have $D(x) = 0$. Let $y \in R$. By (1) $x \ast y \leq x$ and since $M$ is an ideal of $R$, $x \ast y \in M$, we have $D(x \ast y) = 0$, so we can write that

$$0 = D(x \ast y) = (D(x) \ast f(y)) + (f(x) \ast d(y)) = f(x) \ast d(y).$$

We know by our assumption that $R$ has no zero divisors, so we have $f(x) = 0$ for all $x \in M$ or $d(y) = 0$ for all $y \in R$. Since $f$ is a nonzero function of $R$, we get that $d(y) = 0$ for all $y \in R$. This contradicts with our assumption that is $d$ is a nonzero derivation on $R$. Hence, $D$ is nonzero on $M$.

**Theorem 3.7.** Let $D$ be a nonzero generalized $f$-derivation of an integral incline $R$ associated with a nonzero derivation $d$ and a function $f$ on $M$. If $M$ is a nonzero ideal of $R$, and $a \in R$ such that $a \ast D(M) = 0$, then $a = 0$.

Proof: By Theorem 4.6 we know that there is an element $m$ in $M$ such that $D(m) \neq 0$. Let $M$ be a nonzero ideal of $R$, and $a \in R$ such that $a \ast d(M) = 0$. Then for $m, n \in M$ we can write

$$0 = a \ast D(m \ast n) = a \ast (D(m) \ast f(n) + f(m) \ast d(n)) = a \ast D(m) \ast f(n) + a \ast f(m) \ast d(n) = a \ast f(m) \ast d(n).$$

Since $R$ is an integral incline, $d$ is a nonzero derivation on $M$ and $f$ is a nonzero function on $M$ we have $a = 0$.

**Proposition 3.8.** Let $R$ be an incline algebra with a multiplicative identity element and $D$ be a generalized $f$-derivation of $R$ associated with a derivation $d$ and a function $f$ of $R$. If $f(1) = 1$ then for all $x \in R$ we have $D(x) = (D(1) \ast f(x)) + d(x)$. 
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Proof: Since $D(x) = D(1 \ast x) = (D(1) \ast f(x)) + (f(1) \ast d(x))$ and by using $f(1) = 1$ we have $D(x) = (D(1) \ast f(x)) + d(x)$.

References


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