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The Prime Number Double Product

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Abstract

The paper presents a new formula for the characteristic function of the prime numbers in form of a finite double product using only roots of unity.

Keywords: characteristic function of prime numbers, double product, exponential sums, logarithm, elementary group theory, roots of unity

1 Introduction

There exist formulas for the Möbius function [2], the divisor functions or the Ramanujan sums involving only roots of unity. There are also many formulas for the characteristic function of the prime numbers using the floor function and Wilson's theorem among other things [1].

However, to the best of the author's knowledge, there isn't a formula for the characteristic function of the prime numbers involving only roots of unity without using the floor function and Wilson's theorem.
In this paper, we present such a formula similar to these for the Möbius function or the divisor functions. This formula has the form of a finite double product of simple terms of roots of unity.
The only proof found in the literature uses infinite methods, namely an infinite series, even if the two products are finite and there is no proof based only on finite methods.

2 Definitions

Let $n \in N$ be a natural number.
Let $\chi_{\pi}(n)$ be the characteristic function of 1 and the prime numbers, that is:

$$\chi_\pi(n)=\begin{cases}1 & \text{if } n=1\\ 1 & \text{if } n=\text{prime}\\ 0 & \text{if } n=\text{composite}\end{cases}.$$

Define $\pi(x)=\sum_{n=2}^{x}\chi_\pi(n)$ as the prime counting function. (the number of prime numbers $\le x$.)

Let also be $p_n = n\text{-th prime number}.$

3 The Formula

Theorem: (The Prime Number Double Product)
Let $n \in N$ be a natural number. Then one has:

$$\chi_\pi(n)=\frac{1}{n^{n-1}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right).$$

This formula is interesting because it is a product of $(n-1)^2+1$ complex numbers giving as result a real number, moreover a natural number.
It is surprisingly that the natural number n^{n-1} is exactly equal to the double product term if n is a prime number.
Now it is time for the proof.

Proof:

In the proof we distinguish three cases:

1.) $n=1$

2.) $n=\text{composite}$

3.) $n=\text{prime}$

Let us start with the first.

1.) $n=1$.
In this case the formula takes the form:

$$1=\chi_\pi(1)=\frac{1}{1^0}\prod_{k=1}^{0}\prod_{l=1}^{0}\left(1-e^{2\pi ikl}\right)=1,$$

which is already true.

2.) $n=\text{composite}=k_1*l_1$ for some $k_1,l_1\in N$ and $k_1,l_1\leq n\text{-}1$.

Then we have:

$$0=\chi_\pi(n)=\frac{1}{n^{n-1}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right)=\frac{1}{n^{n-1}}\prod_{\substack{k=1\\k\neq k_1}}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right)\prod_{\substack{l^*=1\\l^*\neq l_1}}^{n-1}\left(1-e^{\frac{2\pi ik_1l^*}{n}}\right)\underbrace{\left(1-e^{\frac{2\pi i\overbrace{k_1l_1}^{=n}}{n}}\right)}_{=0},$$

which is again true, because

$$\left(1-e^{\frac{2\pi ik_1l_1}{n}}\right)=\left(1-e^{\frac{2\pi in}{n}}\right)=\left(1-e^{2\pi i}\right)=1-1=0 \text{ and } 0*x=0\ \forall x\in C.$$

3.) $n=\text{prime}$.

This is the hardest part of the proof, but also the most beautiful.
In this case we should have:

$$1=\chi_\pi(n)=\frac{1}{n^{n-1}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right).$$

Since, if n is a prime, every term of the double product is not contained in $(-\infty,0]$, we can take the logarithm on both sides to get:

$$0=\ln(1)=\ln\left(\chi_\pi(n)\right)=\ln\left(\frac{1}{n^{n-1}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right)\right)$$

$$\Leftrightarrow \quad 0 = -(n-1)\ln(n) + \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\ln\left(1-e^{\frac{2\pi ikl}{n}}\right).$$

Now it is time to use the Taylor expansion of $\ln(1-x) = -\sum_{a=1}^{\infty}\frac{x^a}{a}$

(because everything is well-defined) to get:

$$0 = -(n-1)\ln(n) - \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\sum_{a=1}^{\infty}\frac{1}{a}e^{\frac{2\pi iakl}{n}}.$$

Next we interchange the sums and obtain:

$$0 = (n-1)\ln(n) + \sum_{a=1}^{\infty}\frac{1}{a}\left(\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}e^{\frac{2\pi iakl}{n}}\right).$$

$$\Leftrightarrow \quad \sum_{a=1}^{\infty}\frac{1}{a}\left(\sum_{k=1}^{n-1}\sum_{l=1}^{n-1}e^{\frac{2\pi iakl}{n}}\right) = -(n-1)\ln(n).$$

Define now $f_n(a) = \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}e^{\frac{2\pi iakl}{n}}$.

Now we study the behaviour of $f_n(a)$ for every $a \in N$ and $n = \text{prime}$.
First, let $a = n*b$ for some $b \in N$.

Then we have:

$$f_n(a) = f_n(n*b) = \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}e^{\frac{2\pi inbkl}{n}} = \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\underbrace{e^{2\pi ibkl}}_{=1} = (n-1)^2.$$

Second, let $a \neq n*b$ for all $b \in N$.
In this case we have to use two well-known facts from algebra.

1.) $\sum_{m=1}^{n} e^{\frac{2\pi i m}{n}} = 0 \ \forall n \in N_{\geq 2} \Rightarrow \sum_{m=1}^{n-1} e^{\frac{2\pi i m}{n}} = (-1) \ \forall n \in N_{\geq 2}$, because $e^{2\pi i} = 1$.

2.) $\left(Z / pZ \right)$ is a field if $p = \text{prime}$.

=> Every element $a \in \left(Z / pZ \right)$ has a unique inverse $b \in \left(Z / pZ \right)$ and two different elements have not the same inverse.

=> $a * b = n \Leftrightarrow b = n * a^{-1}$

=> $\forall a \in \left(Z / pZ \right) \forall n \in \left(Z / pZ \right) \exists! b \in \left(Z / pZ \right)$ with $a * b = n$

and for no two "a's" we have the same "b's".
$(a_1, a_2$ and $a_1 b = a_2 b = n \iff a_1 = a_2)$

Now we can compute $f_n(a)$ for $n = \text{prime}$ and $a \neq n * b$.
We have that $\left(Z / nZ \right)$ is a field and there is an isomorphism between the sets

$$\left\{ e^{\frac{2\pi i m}{n}} \right\}_{m=0}^{n-1} \text{ and } \left(Z / nZ \right)$$

Therefore, we can think of $\left\{ e^{\frac{2\pi i m}{n}} \right\}_{m=0}^{n-1}$ as a model of $\left(Z / nZ \right)$.

Furthermore, $\left(Z / nZ \right)$ has $\left| \left(Z / nZ \right)^{*} \right| = (n-1)$ elements without counting the identity (=1).

Now $f_n(a) = \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} e^{\frac{2\pi i a k l}{n}} = \sum_{k=1}^{n-1} \sum_{l=1}^{n-1} \left(e^{\frac{2\pi i a l}{n}} \right)^{k}$.

Because $a \neq n * b$ and by fact 2.), we have

$$\left\{ e^{\frac{2\pi i a l}{n}} \right\}_{l=1}^{n-1} = \left\{ e^{\frac{2\pi i m}{n}} \right\}_{m=1}^{n-1} = \left\{ e^{\frac{2\pi i k m}{n}} \right\}_{m=1}^{n-1} = \left\{ e^{\frac{2\pi i q}{n}} \right\}_{q=1}^{n-1} \text{(without the identity} = 1\text{)}.$$

Thus $f_n(a) = \sum_{k=1}^{n-1}\sum_{l=1}^{n-1}\left(e^{\frac{2\pi i a l}{n}} \right)^k = \sum_{k=1}^{n-1}\sum_{m=1}^{n-1}\left(e^{\frac{2\pi i m}{n}} \right)^k = \sum_{k=1}^{n-1}\sum_{m=1}^{n-1} e^{\frac{2\pi i k m}{n}} = \sum_{k=1}^{n-1}\sum_{q=1}^{n-1} e^{\frac{2piq}{n}}$.

By fact 1.) we have:

$$f_n(a) = \sum_{k=1}^{n-1}\underbrace{\left(\sum_{q=1}^{n-1} e^{\frac{2\pi i q}{n}} \right)}_{=(-1)} = \sum_{k=1}^{n-1}(-1) = -\sum_{k=1}^{n-1}1 = -(n-1).$$

In total, we get for $n = \text{prime}$:

$$f_n(a) = \begin{cases} -(n-1) & \text{if } a \neq n * b \text{ for all } b \in N \\ (n-1)^2 & \text{if } a = n * b \text{ for some } b \in N \end{cases}.$$

To finish the proof, we have to show that:

$$\sum_{a=1}^{\infty}\frac{1}{a}\left(\sum_{k=1}^{n-1}\sum_{l=1}^{n-1} e^{\frac{2\pi i a k l}{n}} \right) = \sum_{a=1}^{\infty}\frac{1}{a} f_n(a) = -(n-1)\ln(n).$$

For this, define now $g_n(a) = \begin{cases} 1 & \text{if } a \neq n * b \text{ for all } b \in N \\ -(n-1) & \text{if } a = n * b \text{ for some } b \in N \end{cases}.$

=> $f_n(a) = -(n-1) g_n(a)$.

=> $\sum_{a=1}^{\infty}\frac{f_n(a)}{a} = -(n-1)\sum_{a=1}^{\infty}\frac{g_n(a)}{a} = -(n-1)\ln(n)$

$$\Leftrightarrow \quad \sum_{a=1}^{\infty} \frac{g_n(a)}{a} = \ln(n).$$

In fact, it turns out that this formula is not only valid if n is a prime, moreover it holds true for all $n \in N$.
The first few cases of this formula are:

$$\ln(2) = \sum_{a=1}^{\infty} \frac{(-1)^{a+1}}{a} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - ....$$

$$\ln(3) = \sum_{a=1}^{\infty} \frac{g_3(a)}{a} = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + + - ....$$

$$\ln(4) = \sum_{a=1}^{\infty} \frac{g_4(a)}{a} = 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + + + - ....$$

$$\ln(n) = \sum_{a=1}^{\infty} \frac{g_n(a)}{a} = \lim_{k \to \infty} \left(\sum_{a=1}^{n*k} \frac{g_n(a)}{a} \right)$$

The last general formula has an easy proof.

$$\sum_{a=1}^{\infty} \frac{g_n(a)}{a} = \lim_{k \to \infty} \left(\sum_{a=1}^{n*k} \frac{g_n(a)}{a} \right) = \lim_{k \to \infty} \left(\sum_{a=1}^{n*k} \frac{1}{a} - \sum_{b=1}^{k} \frac{n}{n*b} \right) = \lim_{k \to \infty} \left(\sum_{a=1}^{n*k} \frac{1}{a} - \sum_{b=1}^{k} \frac{1}{b} \right)$$

$$= \lim_{k \to \infty} \left(\left[\ln(nk) + \gamma + o\left(\tfrac{1}{nk}\right) \right] - \left[\ln(k) + \gamma + o\left(\tfrac{1}{k}\right) \right] \right)$$

$$= \lim_{k \to \infty} \left(\ln(n) + \ln(k) + \gamma + o\left(\tfrac{1}{nk}\right) - \ln(k) - \gamma - o\left(\tfrac{1}{k}\right) \right)$$

$$= \lim_{k \to \infty} \left(\ln(n) + o\left(\tfrac{1}{k}\right) \right) = \ln(n).$$

This proves the Prime Number Double Product.

□

Corollary 1: (The real version of the Prime Number Double Product)
Let $n \in N$ be a natural number. Then one has:

$$\chi_\pi(n)=\left(\frac{2^{n-1}}{n^2}\right)^{n-1}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-\cos\left(\frac{2\pi kl}{n}\right)\right).$$

Proof:

We must only take the square of the absolute value of the Prime Number Double Product, to get its real version.
Here we use

$$|z|^2=z*\bar{z}\;,\;|z_1*z_2|=|z_1|*|z_2|\;\forall z,z_1,z_2\in C \text{ and Euler's theorem } e^{ix}=\cos(x)+i\sin(x).$$

Thus

$$\begin{aligned}\left|1-e^{\frac{2\pi ikl}{n}}\right|^2&=\left|1-\left[\cos\left(\frac{2\pi kl}{n}\right)+i\sin\left(\frac{2\pi kl}{n}\right)\right]\right|^2=\left|1-\cos\left(\frac{2\pi kl}{n}\right)-i\sin\left(\frac{2\pi kl}{n}\right)\right|^2\\&=\left(1-\cos\left(\frac{2\pi kl}{n}\right)-i\sin\left(\frac{2\pi kl}{n}\right)\right)\left(1-\cos\left(\frac{2\pi kl}{n}\right)+i\sin\left(\frac{2\pi kl}{n}\right)\right)\\&=1-\cos\left(\frac{2\pi kl}{n}\right)+i\sin\left(\frac{2\pi kl}{n}\right)-\cos\left(\frac{2\pi kl}{n}\right)+\cos^2\left(\frac{2\pi kl}{n}\right)-i\sin\left(\frac{2\pi kl}{n}\right)\cos\left(\frac{2\pi kl}{n}\right)\\&\quad-i\sin\left(\frac{2\pi kl}{n}\right)+i\sin\left(\frac{2\pi kl}{n}\right)\cos\left(\frac{2\pi kl}{n}\right)+\sin^2\left(\frac{2\pi kl}{n}\right)\\&=1-2\cos\left(\frac{2\pi kl}{n}\right)+\underbrace{\left[\cos^2\left(\frac{2\pi kl}{n}\right)+\sin^2\left(\frac{2\pi kl}{n}\right)\right]}_{=1}=2-2\cos\left(\frac{2\pi kl}{n}\right)=2\left(1-\cos\left(\frac{2\pi kl}{n}\right)\right).\end{aligned}$$

Thus we obtain:

$$\begin{aligned}\chi_\pi(n)=|\chi_\pi(n)|^2&=\left|\frac{1}{n^{n-1}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right)\right|^2=\frac{1}{n^{2(n-1)}}\left|\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-e^{\frac{2\pi ikl}{n}}\right)\right|^2=\frac{1}{n^{2(n-1)}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left|1-e^{\frac{2\pi ikl}{n}}\right|^2\\&=\frac{1}{n^{2(n-1)}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}2\left(1-\cos\left(\frac{2\pi kl}{n}\right)\right)=\frac{2^{(n-1)^2}}{n^{2(n-1)}}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-\cos\left(\frac{2\pi kl}{n}\right)\right)=\left(\frac{2^{n-1}}{n^2}\right)^{n-1}\prod_{k=1}^{n-1}\prod_{l=1}^{n-1}\left(1-\cos\left(\frac{2\pi kl}{n}\right)\right).\end{aligned}$$

This is the claimed formula.

□

In the next step, we can use the two Prime Number Double Product formulas to get two formulas for $\pi(x)$ and p_n.

Corollary 2: (Two formulas for $\pi(x)$)
Let $x \in R$ be a real number. Then one has:

$$1.)\ \pi(x) = \sum_{n=2}^{x} \frac{1}{n^{n-1}} \prod_{k=1}^{n-1} \prod_{l=1}^{n-1} \left(1 - e^{\frac{2\pi ikl}{n}}\right).$$

$$2.)\ \pi(x) = \sum_{n=2}^{x} \left(\frac{2^{n-1}}{n^2}\right)^{n-1} \prod_{k=1}^{n-1} \prod_{l=1}^{n-1} \left(1 - \cos\left(\frac{2\pi kl}{n}\right)\right).$$

Proof:

Just use $\pi(x) = \sum_{n=2}^{x} \chi_\pi(n)$ and insert our two formulas for $\chi_\pi(n)$.

□

Corollary 3: (Two formulas for p_n)
Let $n \in N$ be a natural number. Then one has:

$$1.)\ p_n = \frac{1}{4} \sum_{t=1}^{n^2+1} t * \left(1 - (-1)^{2^{\left(n - \sum_{m=2}^{t} \frac{1}{m^{m-1}} \prod_{k=1}^{m-1} \prod_{l=1}^{m-1} \left(1 - e^{\frac{2\pi ikl}{m}}\right)\right)^2}}\right)\left(1 + (-1)^{2^{\left(n - \sum_{m=2}^{t-1} \frac{1}{m^{m-1}} \prod_{k=1}^{m-1} \prod_{l=1}^{m-1} \left(1 - e^{\frac{2\pi ikl}{m}}\right)\right)^2}}\right).$$

$$2.)\ p_n = \frac{1}{4} \sum_{t=1}^{n^2+1} t * \left(1 - (-1)^{2^{\left(n - \sum_{m=2}^{t} \left(\frac{2^{m-1}}{m^2}\right)^{m-1} \prod_{k=1}^{m-1} \prod_{l=1}^{m-1} \left(1 - \cos\left(\frac{2\pi kl}{m}\right)\right)\right)^2}}\right)\left(1 + (-1)^{2^{\left(n - \sum_{m=2}^{t-1} \left(\frac{2^{m-1}}{m^2}\right)^{m-1} \prod_{k=1}^{m-1} \prod_{l=1}^{m-1} \left(1 - \cos\left(\frac{2\pi kl}{m}\right)\right)\right)^2}}\right).$$

Proof:
It is well-known that $p_n \le n^2 + 1 \ \forall n \in N$.
(because $p_n \le n\ln(n) + n\ln(\ln(n)) \ \forall n \ge 6$ [3])
Let now be $n \in N$ and $t \in N$. Note that $\pi(p_m) = m$.

Observe that $\delta_{n,\pi(t)} = \begin{cases} 0 & \text{if } t < p_n \\ 1 & \text{if } p_n \le t < p_{n+1} \\ 0 & \text{if } t \ge p_{n+1} \end{cases}$ and that $\delta_{n,\pi(k-1)} = \begin{cases} 0 & \text{if } t < p_n + 1 \\ 1 & \text{if } p_n + 1 \le t < p_{n+1} + 1 \\ 0 & \text{if } t \ge p_{n+1} + 1 \end{cases}$

such that $\delta_{p_n,t} = \delta_{n,\pi(t)}\left(1-\delta_{n,\pi(t-1)}\right) = \begin{cases} 0 & \text{if } t < p_n \\ 1 & \text{if } t = p_n \\ 0 & \text{if } p_n + 1 \le t < p_{n+1} + 1 \\ 0 & \text{if } t \ge p_{n+1} + 1 \end{cases} = \begin{cases} 1 \text{ if } t = p_n \\ 0 \text{ if } t \ne p_n \end{cases}.$

We also have the following formula for $\delta_{i,j}$:

$$\delta_{i,j} = \frac{1}{2}\left(1-(-1)^{2^{(i-j)^2}}\right) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j \end{cases}.$$

Therefore, we have that:

$$\begin{aligned} p_n &= \sum_{t=1}^{n^2+1} t * \delta_{p_n,t} = \sum_{t=1}^{n^2+1} t * \delta_{n,\pi(t)}\left(1-\delta_{n,\pi(t-1)}\right) \\ &= \sum_{t=1}^{n^2+1} t * \frac{1}{2}\left(1-(-1)^{2^{(n-\pi(t))^2}}\right)\left(1-\frac{1}{2}\left(1-(-1)^{2^{(n-\pi(t-1))^2}}\right)\right) \\ &= \frac{1}{4}\sum_{t=1}^{n^2+1} t * \left(1-(-1)^{2^{(n-\pi(t))^2}}\right)\left(2-\left(1-(-1)^{2^{(n-\pi(t-1))^2}}\right)\right) \\ &= \frac{1}{4}\sum_{t=1}^{n^2+1} t * \left(1-(-1)^{2^{(n-\pi(t))^2}}\right)\left(2-1+(-1)^{2^{(n-\pi(t-1))^2}}\right) \\ &= \frac{1}{4}\sum_{t=1}^{n^2+1} t * \left(1-(-1)^{2^{(n-\pi(t))^2}}\right)\left(1+(-1)^{2^{(n-\pi(t-1))^2}}\right) \end{aligned}$$

Inserting now the two formulas from Corollary 2 for $\pi(x)$ into this final expression, we get the two formulas from Corollary 3 for p_n.

□

4 Conclusion

With this, I end my article over the Prime Number Double Product by noting that this formula, even if it is useless for primality testing, is only true, because three different areas of mathematics work perfectly together, namely, number theory, analysis and algebra. For me it was unexpected that a finite region in the complex plane (complex version) or the real line (real version) determines the distribution of the prime numbers in a nontrivial way.

References

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