On the Finiteness and Sigma Finiteness of a Cubic Measure Function

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Abstract

The theory of measures has attracted a great deal of research in the recent past. The concept of measures generalises notions of length, area and volume. In this paper, we have formulated a three dimensional structure of measurable spaces and proved that a cubic measure function is finite if it is positive, additive, monotone, continuous from below and its values are real numbers.

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1 Introduction

The notations used in this paper are standard and reference may be made to [1, 2, 3]. The σ -finiteness of a measurable space $(X \times Y, S \times T)$ is generated by covering each measurable set $E \subseteq X \times Y$ by countably many measurable rectangles of finite measure. The σ -finite measure function $\lambda = \mu \times \nu$ is applied on measurable subsets of $X \times Y$. However, the geometric properties of this function cannot be applied in analysis and the solutions to problems that occur in abstract situations in finite dimensional Euclidean space $\mathbf{R}^n, n > 2$, see e.g.[3, 4]. It is on the basis of these limitations that this study has extended the existing knowledge on measurable spaces by investigating the finiteness of

the cubic measure functions of measurable spaces. The approach used involves decomposing subsets of $X \times Y \times Z$ into finite disjoint sets and defining a measure on the measurable space $(X \times Y \times Z, \rho \times \tau \times \psi)$, where $\rho \times \tau \times \psi$ is the σ -ring generated by subsets of $X \times Y \times Z$.

2 The Results

Lemma 1. *Let $R_1 \times R_2 \times R_3$ be a ring of subsets of $X \times Y \times Z$. If R is a ring generated by the class of all cuboids $A \times B \times C$ where $A \times B \times C \in R_1 \times R_2 \times R_3$, then R coincides with a class of all finite disjoint unions $K = \cup_1^n A_i \times B_i \times C_i$ where $(A_i \times B_i \times C_i)_1^n \in R_1 \times R_2 \times R_3$.*

2.1 Properties of cubic measure function

a. *Positivity:* The cubic measure function $\mu \times v \times \omega$ is said to be positive if $\mu \times v \times \omega(A \times B \times C) \geq 0$, for every measurable set $A \times B \times C$.

b. *The measure of an empty set is zero:* The cubic measure of an empty set is zero, that is, $\mu \times v \times \omega(A \times B \times C) = 0$, if

- (i) $A \times B \times C$ is an empty set,
- (ii) any of the sets A , B or C is an empty set,
- (iii) any of the products $A \times B$, $A \times C$ or $B \times C$ is an empty set.

c. *Additive:* If $A \times B \times C$ and $E \times F \times H$ are disjoint sets in $\rho \times \tau \times \psi$ whose union is also in $\rho \times \tau \times \psi$, then the measure function $\mu \times v \times \omega$ is said to be additive if $\mu \times v \times \omega[(A \times B \times C) \cup (E \times F \times H)] = \mu \times v \times \omega(A \times B \times C) + \mu \times v \times \omega(E \times F \times H) = \mu(A)v(B)\omega(C) + \mu(E)v(F)\omega(H)$.

d. *Continuous from below:* If $A \times B \times C$ is a measurable set and $A_n \times B_n \times C_n$ ($n = 1, 2, 3, \dots$) is a sequence of sets in $\rho \times \tau \times \psi$, such that $A_n \times B_n \times C_n \uparrow A \times B \times C$ and $\Pi(A_n \times B_n \times C_n) \uparrow \Pi(A \times B \times C)$, then the measure function $\Pi = \mu \times v \times \omega$ is said to be continuous from below.

e. *Monotone:* If $A_1 \times B_1 \times C_1$ and $A_2 \times B_2 \times C_2$ are measurable sets in $X \times Y \times Z$ such that $A_1 \times B_1 \times C_1 \subset A_2 \times B_2 \times C_2$, the set function $T^{A \times B \times C}$ is monotone if $T^{A_1 \times B_1 \times C_1} \leq T^{A_2 \times B_2 \times C_2}$.

Theorem 1. *A cubic measure function $T^{A \times B \times C}$ is finite if it is positive, additive, monotone, continuous from below and its values are real numbers i.e. for each finite cuboid $A \times B \times C$, $T^{A \times B \times C}(E \times F \times H) = \mu(A \cap E)v(B \cap F)\omega(C \cap H) < \infty$ for every measurable cuboid $E \times F \times H$.*

Proof. Let (X, ρ, μ_A) , (Y, τ, v_B) and (Z, ψ, ω_C) be finite measure spaces. Define the function $T^{A \times B \times C} : X \times Y \times Z \rightarrow \mathfrak{R}$ by the formula $T^{A \times B \times C}(M) = \int f_M \delta_{\mu_A}$, where M is a measurable subset of $A \times B \times C$ and f_M is a real valued function defined on X . To show that the function $T^{A \times B \times C}$ is additive, let

M and N be disjoint measurable sets in $X \times Y \times Z$. Then $f_{M \cup N} = f_M + f_N \Rightarrow \int f_{M \cup N} \delta_{\mu_A} = \int f_M \delta_{\mu_A} + \int f_N \delta_{\mu_A} \Rightarrow T^{A \times B \times C}(M \cup N) = T^{A \times B \times C}(M) + T^{A \times B \times C}(N)$. To show that $T^{A \times B \times C}$ is monotone, let $M \subset N$, then $\int f_M \delta_{\mu_A} \leq \int f_N \delta_{\mu_A} \Rightarrow T^{A \times B \times C}(M) \leq T^{A \times B \times C}(N)$. If M is a null set, then $T^{A \times B \times C}(M) = 0$. If M_n is a sequence of measurable sets in $X \times Y \times Z$ such that $M_n \uparrow M$ then, $f_{M_n} \uparrow f_M$. By monotone convergence theorem, we have $\int f_{M_n} \delta_{\mu_A} \uparrow \int f_M \delta_{\mu_A}$. By continuity of a measure function $T^{A \times B \times C}$ applied on measurable sets, we obtain $T^{A \times B \times C}(M_n) \uparrow T^{A \times B \times C}(M)$. Therefore, $T^{A \times B \times C}$ is a finite measure. Let $S^{A \times B \times C}$ and $Q^{A \times B \times C}$ be set functions on $\rho \times \tau \times \psi$ defined by $S^{A \times B \times C}(M) = \int g^M \delta_{v_B}$ and $Q^{A \times B \times C}(M) = \int h^M \delta_{\omega_C}$ respectively. Following the same procedure as in the above equations, we conclude that $S^{A \times B \times C}$ and $Q^{A \times B \times C}$ are finite measure functions on σ -rings generated by subsets of $X \times Y \times Z$. Since $T^{A \times B \times C} = S^{A \times B \times C} = Q^{A \times B \times C}$ on a ring R , it follows from unique extension theorem (see [3]), that $S = T = Q$ on a σ -ring $G(R)$ generated by R . Therefore, $\int f_M \delta_{\mu_A} = \int g^M \delta_{v_B} = \int h^M \delta_{\omega_C}$. Let $M = E \times F \times H$, where $E \times F \times H$ is a measurable cuboid. For finite measure spaces (X, ρ, μ_A) , (Y, τ, v_B) and (Z, ψ, ω_C) , a measurable function is equal to an integrable function (see[4]). Therefore, $\int f_{E \times F \times H} \delta_{\mu_A} = \int g^{E \times F \times H} \delta_{v_B} = \int h_{E \times F \times H} \delta_{\omega_C} = \mu_A(E) v_B(F) \omega_C(H) \Rightarrow T^{A \times B \times C}(E \times F \times H) = \int f_{E \times F \times H} \delta_{\mu_A} = \mu_A(E) v_B(F) \omega_C(H) \Rightarrow T^{A \times B \times C} = \mu_A \times v_B \times \omega_C$. $\square$

Remark: It follows that the finite measure functions $(T^{A_2 \times B_2 \times C_2})_{A_1 \times B_1 \times C_1}$ and $T^{A_1 \times B_1 \times C_1}$ are equivalent on the ring R generated by measurable cuboids. Hence by unique extension theorem, they are equivalent on σ -ring $G(R)$ generated by R .

Theorem 2. Suppose $\Pi = LUB T^{A \times B \times C}$ is a cubic measure function defined by $\Pi(M) = LUB \{T^{A \times B \times C}(M)\}$, where $A \times B \times C$ is a measurable cuboid of finite measure and M is a measurable subset of $A \times B \times C$. For each finite cuboid $A \times B \times C$, the contraction $\Pi_{A \times B \times C}$ is equal to the cubic measure function $T^{A \times B \times C}$.

Proof. Suppose $M \subset A \times B \times C$ and $T^{A \times B \times C}(M) < \infty$. If $A \times B \times C \subset P \times Q \times R$, then $(T^{P \times Q \times R})_{A \times B \times C}(M) = T^{P \times Q \times R}\{(A \times B \times C) \cap M\}$. We have $(T^{P \times Q \times R})_{A \times B \times C} = T^{A \times B \times C}$. Hence, $T^{P \times Q \times R}\{(A \times B \times C) \cap M\} = (T^{P \times Q \times R})_{A \times B \times C}(M) = T^{A \times B \times C}(M)$. Taking the LUB over all such cuboids $P \times Q \times R$, we have $\Pi = LUB T^{P \times Q \times R} \cdot \Pi_{A \times B \times C}(M) = T^{A \times B \times C}(M)$. Therefore, $\Pi_{A \times B \times C} = T^{A \times B \times C}$. $\square$

Theorem 3. If $E \times F \times H$ is a measurable cuboid with σ -finite sides, then $(\mu \times \nu \times \omega)(E \times F \times H) = \mu(E) \nu(F) \omega(H)$.

Proof. Results on product measure, show that a measure μ is σ -finite if there exists a measurable space $(X, \mathcal{B})$ and a sequence $\langle x_n \rangle$ of measurable

sets with respect to σ – algebra $\mathcal{B}$ generated by X such that $X = \cup_1^\infty x_n$ and $\mu(x_n) < \infty$. Suppose $E \times F \times H$ is a measurable cuboid with σ – finite sides. If we consider the σ – rings generated by subsets of $X \times Y \times Z$, then each of the sets E , F and H can be decomposed into countable union of sequences of measurable sets of finite measure. If P_n is a sequence of measurable sets of finite measure such that $P_n \uparrow E$, then $\mu(E) = LUB \mu(P_n)$. Similarly, let Q_n be a sequence of sets in τ_θ such that $Q_n \uparrow F$, then $v(F) = LUB v(Q_n)$. Finally, if R_n be a sequence of sets in ψ_θ such that $R_n \uparrow H$, then $\omega(H) = LUB \omega(R_n)$. Therefore if $P_n \times Q_n \times R_n \uparrow E \times F \times H$, then $(\mu \times v \times \omega)(E \times F \times H) = LUB (\mu \times v \times \omega)(P_n \times Q_n \times R_n) = LUB \mu(P_n)v(Q_n)\omega(R_n) = \mu(E)v(F)\omega(H)$. $\square$

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