On the Extension of Properties of Cubic Measures

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Abstract

We consider the application and extension of properties of cubic measure to measurable spaces, measurable and integrable functions. The concepts of finiteness and σ -finiteness of a measure function are assumed and invoked whenever appropriate.

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1 Introduction

In the previous results, many scholars studied measurable sets generated by semirings and sigma algebras. Attention was given to developing a measure for one and two dimensional measurable spaces. A measure was not defined for measurable sets in finite-dimensional Euclidean spaces $\mathbb{R}^n$ for $n > 2$. In this article, we investigate the extension of properties of cubic measure to measurable spaces, measurable and integrable functions generated by σ -rings of subsets of three dimensional spaces, with additional properties involving monotone convergence theorem. For the notations and basic concepts used in the article reference may be made to [1, 2, 3, 4, 6]

2 The Results

Proposition 1. *Each measurable subset of $X \times Y \times Z$ is contained in some measurable cuboid, that is, if $M \subset X \times Y \times Z$ and $M \in \rho \times \tau \times \psi$, then there exists a measurable cuboid $E \times F \times H$ such that $M \subset E \times F \times H$.*

Proof. Each measurable set in $X \times Y \times Z$ is contained in a countable union of sets in $E \times F \times H$, where $E \subset X$, $F \subset Y$, and $H \subset Z$. Let $M \in \rho \times \tau \times \psi$. There exists a sequence of measurable cuboids $(E_n \times F_n \times H_n)_{n=1}^{\infty}$ such that $M \subset \bigcup_1^{\infty} E_n \times F_n \times H_n$. Defining $E = \bigcup_1^{\infty} E_n$, $F = \bigcup_1^{\infty} F_n$ and $H = \bigcup_1^{\infty} H_n$, we have $M \subset \bigcup_1^{\infty} E_n \times F_n \times H_n = E \times F \times H \Rightarrow M \subset E \times F \times H$ $\square$

Proposition 2. *Let Υ be the class of measurable subsets of X such that $G(\Upsilon) = \rho$, Φ be the class of measurable subsets of Y such that $G(\Phi) = \tau$, and Ω be the class of measurable subsets of Z such that $G(\Omega) = \psi$. If $\Delta = \{E \times F \times H : E \in \Upsilon, F \in \Phi \text{ and } H \in \Omega\}$, then $G(\Delta) = \rho \times \tau \times \psi$.*

Proof. Since Δ is measurable, it implies that $\Delta \in \rho \times \tau \times \psi$. Given that $G(\Delta)$ is the smallest σ -ring generated by Δ , it follows that $G(\Delta)$ is a subset of a σ -ring $\rho \times \tau \times \psi$ generated by subsets of $X \times Y \times Z$. Fix $E \in \Upsilon$ and $F \in \Phi$. Consider the class defined by $A = \{P \subset Z : E \times F \times P \in G(\Delta)\}$. Since $E \times F \times (\bigcup_1^{\infty} P_n) = \bigcup_1^{\infty} E \times F \times P_n \in A$ and $E \times F \times (P_1 - P_2) = \{E \times F \times P_1 - E \times F \times P_2\} \in A$, it follows that A is closed under countable unions and differences. Therefore, A is a σ -ring. Since A contains the class Ω of measurable subsets of Z and $G(\Omega) = \psi$ is the smallest σ -ring generated by Ω , we have $G(\Omega) = \psi \subset A$. Therefore if $E \in \Upsilon$, $F \in \Phi$ and $H \in \Omega$, then $E \times F \times H \in G(\Delta)$. Fix $E \in \Upsilon$ and $H \in \Omega$. Define $B = \{T \subset Y : E \times T \times H \in G(\Delta)\}$. Since $E \times (\bigcup_1^{\infty} T_n) \times H = \bigcup_1^{\infty} E \times T_n \times H \in B$ and $E \times (T_1 - T_2) \times H = E \times T_1 \times H - E \times T_2 \times H \in B$, it follows by definition that B is a σ -ring. Since B contains the class Φ of measurable subsets of Y , we have $G(\Phi) = \tau \subset B$. Therefore, if $E \in \Upsilon$, $F \in \Phi$ and $H \in \Omega$, then $E \times F \times H \in G(\Delta)$. Finally, fix $F \in \Phi$ and $H \in \Omega$. Suppose $C = \{S \subset X : S \times F \times H \in G(\Delta)\}$. Since $(\bigcup_1^{\infty} S_n) \times F \times H \in C$ and $(S_1 - S_2) \times F \times H = S_1 \times F \times H - S_2 \times F \times H \in C$, it implies that C is a σ -ring. Since C contains the class Υ of subsets of X , then $G(\Upsilon) = \rho \subset C$. If $E \in \Upsilon$, $F \in \Phi$ and $H \in \Omega$, then $E \times F \times H \in G(\Delta)$. We therefore conclude that $\rho \times \tau \times \psi \subset G(\Delta)$. Since $G(\Delta) \subset \rho \times \tau \times \psi$, we have $G(\Delta) = \rho \times \tau \times \psi$ $\square$

Proposition 3. *If $f : Y \times Z \rightarrow X \times Y \times Z$ is a function defined on $Y \times Z$ and $\rho \times \tau \times \psi$ is σ -ring of subsets of $X \times Y \times Z$, then the class of all subsets of the form $f^{-1}(M)$ where M is in $\rho \times \tau \times \psi$, is a σ -ring of subsets of $Y \times Z$.*

Proof. Rectangles are used to generate σ -rings of subsets of $X \times Y$. Based on these findings, we generate σ -rings of subsets $E \times F \times H$ of $X \times Y \times Z$

and show that the inverse image of the σ -ring of subsets of $X \times Y \times Z$ is a σ -ring of subsets of $Y \times Z$. Let M_1 and M_2 be both in $\rho \times \tau \times \psi$ and K be the class of subsets of $Y \times Z$. Since $\rho \times \tau \times \psi$ is a σ -ring, it is closed under set differences and countable unions. If $M_1 - M_2 \in \rho \times \tau \times \psi$, then $f^{-1}(M_1 - M_2) = f^{-1}(M_1) - f^{-1}(M_2)$. But $f^{-1}(M_1) - f^{-1}(M_2) \in K \subset Y \times Z$. Therefore, K is closed under set differences. Let $(E_i \times F_i \times H_i)_{i=1}^{\infty}$ be a sequence of subsets in $\rho \times \tau \times \psi$. Since $\rho \times \tau \times \psi$ is a σ -ring, then $\cup_{i=1}^{\infty} E_i \times F_i \times H_i \in \rho \times \tau \times \psi$. Therefore, $f^{-1}(\cup_{i=1}^{\infty} E_i \times F_i \times H_i) = \cup_{i=1}^{\infty} f^{-1}(E_i \times F_i \times H_i) \in K \subset Y \times Z$. From these facts, it is clear that K is closed under countable unions. Hence, K is a σ -ring of subsets of $Y \times Z$. $\square$

Proposition 4. *If h is a measurable function with respect to $\rho \times \tau \times \psi$, then for $x \in X$, the function h_x is measurable with respect to $\tau \times \psi$.*

Proof. Fix $x \in X$ and let $N(h_x) = (N(h))_x$. Now, $N(h_x) = \{(y, z) \in Y \times Z : h(x, y, z) \neq 0\}$. If $(y, z) \in N(h_x)$, $\Leftrightarrow h_x(y, z) \neq 0 \Leftrightarrow h(x, y, z) \neq 0 \Leftrightarrow (x, y, z) \in N(h) \Leftrightarrow (y, z) \in (N(h))_x \Rightarrow N(h_x) = (N(h))_x$. If B is a Borel set of real numbers, then $h_x^{-1}(B) = (h^{-1}(B))_x$. The following relations are equivalent. Suppose $(y, z) \in h_x^{-1}(B) \Leftrightarrow h_x(y, z) \in h_x(h_x^{-1}(B)) \Leftrightarrow h(x, y, z) \in B \Leftrightarrow h^{-1}(h(x, y, z)) \in h^{-1}(B) \Leftrightarrow (x, y, z) \in h^{-1}(B) \Leftrightarrow (y, z) \in (h^{-1}(B))_x \Rightarrow h_x^{-1}(B) = (h^{-1}(B))_x$. Since $N(h) \cap h^{-1}(B)$ is by definition measurable, the set $(N(h) \cap h^{-1}(B))_x = (N(h))_x \cap (h^{-1}(B))_x = N(h_x) \cap h_x^{-1}(B)$ is measurable. Therefore, $h_x \in \tau \times \psi$. $\square$

Proposition 5. *If (X, ρ, μ) , (Y, τ, ν) and (Z, ψ, ω) are finite measure spaces, then for every measurable set M in $X \times Y \times Z$, the functions f_M , g^M and h^M are integrable and $\int f_M \delta_\mu = \int g^M \delta_\nu = \int h^M \delta_\omega$.*

Proof. Since M is a set in $\rho \times \tau \times \psi$, the functions f_M , g^M and h^M are bounded. Also the functions f_M , g^M and h^M are measurable. Since (X, ρ, μ) , (Y, ν, τ) and (Z, ψ, ω) are finite measure spaces and the functions f_M , g^M and h^M are bounded and measurable, they are integrable. Let $E \times F \times H$ be any measurable cuboid, such that $\mu(E) < \infty$, $\nu(F) < \infty$ and $\omega(H) < \infty$. For finite measure spaces (X, ρ, μ) , (Y, τ, ν) and (Z, ψ, ω) and integrable functions f , g and h , it is established that $\int f_{E \times F \times H} \delta_\mu = \int g^{E \times F \times H} \delta_\nu = \int h_{E \times F \times H} \delta_\omega$. If $M = E \times F \times H$, then $\int f_M \delta_\mu = \int g^M \delta_\nu = \int h^M \delta_\omega$. If M belongs to the ring R generated by $E \times F \times H$, then M can be written as a finite union of measurable cuboids. Therefore M satisfies $\int f_M \delta_\mu = \int g^M \delta_\nu = \int h^M \delta_\omega$. Let us define real valued set functions ι , κ and λ on $\rho \times \tau \times \psi$ by $\iota(M) = \int f_M \delta_\mu$, $\kappa(M) = \int g^M \delta_\nu$ and $\lambda(M) = \int h^M \delta_\omega$. Now $\iota = \kappa = \lambda$ on R . We need to show that $\iota = \kappa = \lambda$ on a σ -ring $G(R)$ generated by R . This implies that ι , κ and λ are (finite) measures on $\rho \times \tau \times \psi$. A measure λ defined on a measurable space $(X \times Y \times Z, \rho \times \tau \times \psi)$ is said to be finite if it obeys the

properties of positivity, additivity, monotonicity, continuity and $\lambda(M) < \infty$ for a measurable subset M of $X \times Y \times Z$. If M and N are disjoint measurable sets with respect to $\rho \times \tau \times \psi$, then $f_{M \cup N} = f_M + f_N$. Integrating term by term with respect to μ , we obtain $\int f_{M \cup N} \delta_\mu = \int f_M \delta_\mu + \int f_N \delta_\mu$. But $\iota(M) = \int f_M \delta_\mu$, $\iota(N) = \int f_N \delta_\mu$ and $\iota(M \cup N) = \int f_{M \cup N} \delta_\mu$. By the additive property of the measure function ι becomes $\iota(M \cup N) = \iota(M) + \iota(N)$. Following the same procedure, we conclude that the set functions κ and λ are additive. The set function ι is monotone. That is, if M and N are sets in $\rho \times \tau \times \psi$ such that $M \subset N$, then $f_M \leq f_N \Rightarrow \int f_M \delta_\mu \leq \int f_N \delta_\mu \Rightarrow \iota(M) \leq \iota(N)$. If M is a null set i.e $M = \phi$, then $\iota(M) = \iota(\phi) = 0$. Let M_n be a sequence of sets in $\rho \times \tau \times \psi$ such that $M_n \uparrow M$, then $f_{M_n} \uparrow f_M$. By monotone convergence theorem, we have $\int f_{M_n} \delta_\mu \uparrow \int f_M \delta_\mu \Rightarrow \iota(M_n) \uparrow \iota(M)$. Therefore, we conclude that ι is a finite measure. From the same reasoning, the set functions κ and λ are finite measures. Since $\iota = \kappa = \lambda$ on R , it follows from unique extension theorem that $\iota = \kappa = \lambda$ on σ -ring $G(R)$. Hence $\iota(M) = \kappa(M) = \lambda(M)$. $\Rightarrow \int f_M \delta_\mu = \int g^M \delta_v = \int h^M \delta_\omega$. $\square$

Proposition 6. If (X, ρ, μ) , (Y, τ, ν) and (Z, ψ, ω) are finite measure spaces, there exists a unique measure λ on $\rho \times \tau \times \psi$ such that $\lambda(E \times F \times H) = \mu(E)\nu(F)\omega(H)$ for every measurable cuboid $E \times F \times H$. Moreover λ is finite and $\lambda(M) = \int f_M \delta_\mu = \int g^M \delta_v = \int h^M \delta_\omega$ for every measurable set M in $X \times Y \times Z$.

Proof. Let us define a set function λ on $\rho \times \tau \times \psi$ by the formula $\lambda(M) = \int f_M \delta_\mu$. Now λ is a finite measure on $\rho \times \tau \times \psi$ and $\lambda(M) = \int f_M \delta_\mu = \int g^M \delta_v = \int h^M \delta_\omega$. If $E \times F \times H$ is a measurable cuboid, then $\int f_{E \times F \times H} \delta_\mu = \int g^{E \times F \times H} \delta_v = \int h^{E \times F \times H} \delta_\omega = \mu(E)\nu(F)\omega(H)$. If $M = E \times F \times H$, then $\lambda(M) = \lambda(E \times F \times H) = \int f_M \delta_\mu = \int g^M \delta_v = \int h^M \delta_\omega$. If $E \times F \times H$ is a measurable cuboid, then $\lambda(E \times F \times H) = \mu(E)\nu(F)\omega(H)$. Suppose κ is any measure on $\rho \times \tau \times \psi$ such that $\kappa(E \times F \times H) = \mu(E)\nu(F)\omega(H)$ for every measurable cuboid $E \times F \times H$. It follows that $\lambda = \kappa$ on the ring R generated by the measurable cuboids. By unique extension theorem, we have $\lambda = \kappa$ on $G(R)$. Suppose $\lambda = \mu \times \nu \times \omega$ is a finite measure on $\rho \times \tau \times \psi$, then $(\mu \times \nu \times \omega)(E \times F \times H) = \mu(E)\nu(F)\omega(H)$ for every measurable cuboid $E \times F \times H$. $\square$

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