Weak Laws of Large Numbers for Sequences or Arrays of Correlated Random Variables

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Abstract

In this paper we establish several weak laws of large numbers for sequences or arrays of correlated random variables based on estimates on variances of weighted sums.

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1 Introduction

Without special statements all random variables under consideration are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A sequence of random variables $\{\xi_k; k \in \mathbb{N}\}$ with finite expectations is said to obey weak law of large numbers (WLLN) in the classical (resp. modern) sense if

$$\frac{1}{n} \sum_{k=1}^{n} \xi_k - \frac{1}{n} \sum_{k=1}^{n} E\xi_k \xrightarrow{\mathbb{P}} 0 \quad (\text{resp.} \quad \frac{1}{b_n} \left( \sum_{k=1}^{n} \xi_k - a_n \right) \xrightarrow{\mathbb{P}} 0)$$

for some sequences $(a_n) \subset \mathbb{R}$ and $(b_n) \subset \mathbb{R}_+$ with $b_n \to \infty)$. Because of history reasons the laws of large numbers for sequences of independent random variables have been studied sufficiently, see [2, 4, 3, 9] and references therein. However, sequences of correlated random variables are very complex; and there
are few papers to study the laws of large numbers for them, for example [1, 7, 8]. In Section 2 we shall estimate variances of weighted sum of correlated random variables and list some lemmas. Based on them we establish several new weak laws of large numbers for sequences of correlated random variables in Section 3.

2 Preliminaries

Let $\xi_1, \cdots, \xi_n$ be random variables with finite variances $\text{Var}(\xi_i) < \infty$, $i = 1, \cdots, n$, and let $\text{Cov}(\xi_i, \xi_j)$ be the covariance of $\xi_i$ and $\xi_j$, $i, j = 1, \cdots, n$.

**Proposition 2.1** For any $(a_1, \cdots, a_n) \in \mathbb{R}^n$ it holds that

$$\text{Var}(\sum_{i=1}^{n} a_i \xi_i) \leq \left( \sum_{i=1}^{n} |a_i| \sqrt{\text{Var}(\xi_i)} \right)^2.$$  

**Proof.** Denote by $\sigma_i = \sqrt{\text{Var}(\xi_i)}$, $i = 1, \cdots, n$. By reordering for $\xi_i$, $i = 1, \cdots, n$, we may assume $\text{Var}(\xi_i) > 0$ for $i = 1, \cdots, m \leq n$, and $\text{Var}(\xi_i) = 0$ for $m < i \leq n$ if $m < n$. Recall that for any $(a_1, \cdots, a_n) \in \mathbb{R}^n$,

$$\text{Var}(\sum_{i=1}^{n} a_i \xi_i) = \sum_{i=1}^{n} a_i^2 (\text{Var}(\xi_i))^2 + \sum_{1 \leq i < j \leq n} 2a_i a_j \text{Cov}(\xi_i, \xi_j).$$

Since $|\text{Cov}(\xi_i, \xi_j)| \leq \sigma_i \sigma_j$, we have

$$\text{Var}(\sum_{i=1}^{n} a_i \xi_i) = \sum_{i=1}^{m} a_i^2 (\text{Var}(\xi_i))^2 + \sum_{1 \leq i < j \leq m} 2a_i a_j \text{Cov}(\xi_i, \xi_j). \quad (2.1)$$

Let $r_{ij}$ denote the correlation coefficient $\text{Cov}(\xi_i, \xi_j)/(\sigma_i \sigma_j)$ between $\xi_i$ and $\xi_j$ for $i, j = 1, \cdots, m$. Then $|r_{ij}| \leq 1$. From (2.1) we deduce

$$\text{Var}(\sum_{i=1}^{n} a_i \xi_i) = \left| \text{Var}(\sum_{i=1}^{n} a_i \xi_i) \right| = \sum_{i=1}^{m} \sum_{j=1}^{m} |r_{ij}| a_i a_j \sigma_i \sigma_j \leq \sum_{i=1}^{m} \sum_{j=1}^{m} |a_i a_j| \sigma_i \sigma_j = \left( \sum_{i=1}^{m} |a_i| \sigma_i \right)^2 = \left( \sum_{i=1}^{n} |a_i| \sigma_i \right)^2.$$

Proposition 2.1 and Chebyshev inequality yield

**Proposition 2.2** Under the assumptions of Proposition 2.1, for any $(a_1, \cdots, a_n) \in \mathbb{R}^n$ and for any $\varepsilon > 0$ it holds that

$$\mathbb{P} \left( \left| \sum_{i=1}^{n} a_i \xi_i - \sum_{i=1}^{n} a_i E \xi_i \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2} \text{Var}(\sum_{i=1}^{n} a_i \xi_i) \leq \frac{1}{\varepsilon^2} \left( \sum_{i=1}^{n} |a_i| \sqrt{\text{Var}(\xi_i)} \right)^2.$$
Taking \( a_i = 1/n, \ i = 1, \cdots, n \), we obtain

**Corollary 2.3** For any \( \varepsilon > 0 \) it holds that

\[
\mathbb{P}\left( \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i - \frac{1}{n} \sum_{i=1}^{n} E\xi_i \right| \geq \varepsilon \right) \leq \frac{1}{n^2 \varepsilon^2} \left( \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} \right)^2. \tag{2.2}
\]

Moreover, if \( \text{Var}(\xi_i) \in (0, \infty), \ i = 1, \cdots, n \), replacing \( \xi_i \) by \( \xi_i / \sqrt{\text{Var}(\xi_i)} \), \( i = 1, \cdots, n \), and \( \varepsilon \) by \( \varepsilon/n \) we obtain

\[
\mathbb{P}\left( \left| \sum_{i=1}^{n} \xi_i - \sum_{i=1}^{n} E\xi_i \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2}. \tag{2.3}
\]

**Lemma 2.4 (Kronecker)** For two sequences of real numbers \( (a_n) \) and \( (x_n) \), if \( 0 < a_n \uparrow \infty \) and \( \sum_{n=1}^{\infty} \frac{x_n}{a_n} \) converges, then \( \frac{1}{a_n} \sum_{i=1}^{n} x_i \to 0 \).

**Lemma 2.5** (i) *(Stolz theorem)* Let \( (x_n) \) and \( (y_n) \) be two sequences of real numbers, and let \( a \in \mathbb{R} \cup \{-\infty, +\infty\} \). If there exists \( n_0 \in \mathbb{N} \) such that \( x_n < x_{n+1} \) for every \( n > n_0 \), and \( \lim_{n \to \infty} x_n = +\infty \) and \( \lim_{n \to \infty} (y_n - y_{n-1})/(x_n - x_{n-1}) = a \), then \( \lim_{n \to \infty} y_n/x_n = a \).

(ii) *(Hardy-Landau theorem)* For a sequence \( (x_n) \subset \mathbb{R} \), if \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = b \) and there exists a constant \( K > 0 \) such that for every \( n \in \mathbb{N} \) we have either \( nx_{n+1} > -K \) or \( nx_{n+1} < K \), then \( \lim_{n \to \infty} \sum_{i=1}^{n} x_i = b \).

(i) and (ii) can be founded in Proposition 59 and Exercise 44 in [5, Chapter 1], respectively.

### 3 Weak law of large numbers

For a sequence of random variables \( \{\xi_k; k \in \mathbb{N}\} \) on \((\Omega, \mathcal{F}, \mathbb{P})\) with finite variances, Markov showed in 1913 (cf. [6, page 69]) that

\[
\lim_{n \to \infty} \frac{1}{n^2} \text{Var}\left( \sum_{i=1}^{n} \xi_i \right) = 0 \tag{3.1}
\]

is sufficient for \( \{\xi_k; k \in \mathbb{N}\} \) to obey the WLLN, i.e., for every given \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \mathbb{P}\left( \left| \frac{1}{n} \sum_{i=1}^{n} \xi_i - \frac{1}{n} \sum_{i=1}^{n} E\xi_i \right| < \varepsilon \right) = 1. \tag{3.2}
\]

In general, it is not easy to check the Markov condition (3.1). For a sequence \( (b_n) \subset \mathbb{R}_+ \), since \( \text{Var}(\frac{1}{b_n} \sum_{i=1}^{n} \xi_i) \leq \left( \frac{1}{b_n} \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} \right)^2 \) by Proposition 2.1 we obtain
Theorem 3.1 Let \(\{\xi_k; k \in \mathbb{N}\}\) be a sequence of random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) with finite variances. If a sequence of positive numbers \((b_n)\) satisfies
\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} = 0, \tag{3.3}
\]
then for any \(\varepsilon > 0\) it holds that
\[
\lim_{n \to \infty} \mathbb{P}\left( \left| \frac{1}{b_n} \sum_{i=1}^{n} \xi_i - \frac{1}{b_n} \sum_{i=1}^{n} E\xi_i \right| < \varepsilon \right) = 1; \tag{3.4}
\]
in particular, if \(\sum_{n=1}^{\infty} \frac{\sqrt{\text{Var}(\xi_n)}}{b_n}\) converges then (3.4) holds by Lemma 2.4.

If \(\{\xi_k; k \in \mathbb{N}\}\) is pairwise independent, \(\text{Var}(\sum_{i=1}^{n} \xi_i) = \sum_{i=1}^{n} \text{Var}(\xi_i)\) and so the Markov condition (3.1) becomes
\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(\xi_i) = 0.
\]
This is implied in the condition (3.3) with \((b_n) = (n)\) because
\[
\left( \frac{1}{n} \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} \right)^2 = \frac{1}{n^2} \left( \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} \right)^2 \geq \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(\xi_i)
\]
by Jensen inequality. Hence Theorem 3.1 with \((b_n) = (n)\) is actually contained in the above Markov assertion. Clearly, Theorem 3.1 and Lemma 2.5(i) lead to

Corollary 3.2 Let \(\{\xi_k; k \in \mathbb{N}\}\) be a sequence of random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) with finite variances, and let \((b_n) \subset \mathbb{R}_+\) satisfy the conditions: \(\lim_{n \to \infty} b_n = +\infty\) and there exists \(n_0 \in \mathbb{N}\) such that \(b_n < b_{n+1}\) for every \(n > n_0\). If
\[
\lim_{n \to \infty} \frac{\sqrt{\text{Var}(\xi_n)}}{b_n - b_{n-1}} = 0, \tag{3.5}
\]
then (3.4) holds for every \(\varepsilon > 0\). In particular, if \(\lim_{n \to \infty} \text{Var}(\xi_n) = 0\) then (3.2) holds for any \(\varepsilon > 0\).

Remark 3.3 The following two conditions are almost equivalent,
\[
\lim_{n \to \infty} \text{Var}(\xi_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\text{Var}(\xi_i)} = 0.
\]
Indeed, by Lemma 2.5(i) the former implies the latter. Conversely, suppose that the latter holds and that there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$ we have either $n(\sqrt{\text{Var}(\xi_{n+1})} - \sqrt{\text{Var}(\xi_n)}) > -K$ or $n(\sqrt{\text{Var}(\xi_{n+1})} - \sqrt{\text{Var}(\xi_n)}) < K$. Then the former follows from these and Lemma 2.5(ii). However, it is easy to construct a sequence satisfying the latter only. Consider a sequence of random variables $\{\xi_k\}_{k=1}^{\infty}$ with variances

$$\text{Var}(\xi_k) = 0 \text{ for } k \neq 2^n, \quad \text{and} \quad \text{Var}(\xi_k) = n^2 \text{ for } k = 2^n, \quad n = 1, 2, \cdots. \quad (3.6)$$

Clearly, they do not satisfy the first condition. But it is easy to compute

$$\frac{\sqrt{\text{Var}(\xi_1)} + \cdots + \sqrt{\text{Var}(\xi_k)}}{k} = \begin{cases} 0 & \text{if } k = 1, \\ \frac{1}{2} \frac{1 + \cdots + (n-1)}{k} & \text{if } 2^{n-1} \leq k \leq 2^n. \end{cases}$$

Note that for $2^{n-1} \leq k \leq 2^n$ we have

$$\frac{n(n-1)}{2^{n+1}} \leq \frac{1 + \cdots + (n-1)}{k} \leq \frac{n(n-1)}{2^n}.$$

It follows that

$$\lim_{k \to \infty} \frac{\sqrt{\text{Var}(\xi_1)} + \cdots + \sqrt{\text{Var}(\xi_k)}}{k} = 0.$$

So Theorem 3.1 assures that the sequence $\{\xi_k; k \in \mathbb{N}\}$ satisfies the WLLN.

The sequence $\{\xi_k\}_{k=1}^{\infty}$ with variances as in (3.6) does not satisfy the following theorem too.

**Theorem 3.4 (Bernstein law of large numbers)** For a sequence of random variables $\{\xi_k; k \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that all $\text{Var}(\xi_n)$ are positive and uniformly bounded, and that the correlation coefficients $r_{ij} = \text{Cov}(\xi_i, \xi_j)/(\sigma_i \sigma_j)$ converge to zero as $|i - j| \to \infty$. Then it obeys the WLLN, i.e., (3.2) holds for every $\varepsilon > 0$.

The celebrated Kolmogorov-Feller WLLN for a sequence $\{\xi_k; k \in \mathbb{N}\}$ of independent and identically distributed random variables states that $\frac{1}{n} \sum_{k=1}^{n} \xi_k - E(\xi_1 I_{|\xi_1| \leq n}) \to 0$ in probability as $n \to \infty$ if and only if $n\mathbb{P}(|\xi_1| > n) \to 0$ as $n \to \infty$ (cf. [3, Theorem 4.1], [9, Theorem 9.2.4] or [2, Theorem 5.2.3]). There also exist some recent generalizations of it, see [3, 7, 1]. The following result may be considered a generalization of the sufficiency part of Kolmogorov-Feller WLLN to sequences of correlated and identically distributed random variables.

**Theorem 3.5** Let $\{\xi_k; k \in \mathbb{N}\}$ be a sequence of identically distributed random variables, and let $(b_n)$ be a sequence of positive numbers with $\sup_n \frac{n\sqrt{n}}{b_n} < \infty$. Suppose that

$$\lim_{x \to \infty} x\mathbb{P}(|\xi_1| > x) = 0. \quad (3.7)$$
Then for every $\varepsilon > 0$ it holds that
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{1}{b_n} \sum_{i=1}^{n} \xi_i - \frac{1}{b_n} \sum_{i=1}^{n} E\xi_{n,i} \right| < \varepsilon \right) = 1, \tag{3.8}
\]
where $\xi_{n,i} := \xi_i I_{\{|\xi_i| \leq n\}}$, $i = 1, \ldots, n$.

If $E|\xi_1| < \infty$, which implies (3.7), the corresponding Theorem 3.5 is a generalization of Khintchin WLLN. Though our condition that $\sup_n \frac{n \sqrt{n}}{b_n} < \infty$ is stronger than one in [7, (1)] in many cases our sequence $\{\xi_k; k \in \mathbb{N}\}$ is not required to be negatively associated (resp. independent) comparing with Theorem 1 (resp. Theorem 2) in [7].

**Proof of Theorem 3.5.** Since all $\xi_k, k \in \mathbb{N}$ are identically distributed, so are $\xi_{n,i}$, $i = 1, \ldots, n$. Let $\eta_n := \xi_1 + \cdots + \xi_n$ and $\eta'_n := \xi_{n,1} + \cdots + \xi_{n,n}$, $n = 1, 2, \ldots$. Note that
\[
\mathbb{P} \left( \left| \frac{1}{b_n} (\eta_n - E\eta'_n) \right| > \varepsilon \right) = \mathbb{P} \left( \left| \frac{1}{b_n} (\eta'_n - E\eta'_n) \right| > \varepsilon \right) + \mathbb{P}(\eta_n \neq \eta'_n) \tag{3.9}
\]
for every $\varepsilon > 0$. For the second term on the left hand we have
\[
\mathbb{P}(\eta_n \neq \eta'_n) \leq \mathbb{P}(\cup_{i=1}^{n} \{\xi_i \neq \xi_{n,i}\}) \leq \sum_{i=1}^{n} \mathbb{P}(\xi_i \neq \xi_{n,i})
\quad = \sum_{i=1}^{n} \mathbb{P}(|\xi_i| > n) = n\mathbb{P}(|\xi_1| > n) \to 0 \quad \text{as } n \to \infty \tag{3.10}
\]
by (3.7). Moreover, using Chebyshev inequality and Proposition 2.2 we may estimate the first term on the left hand of (3.9) as follows:
\[
\mathbb{P} \left( \left| \frac{1}{b_n} (\eta'_n - E\eta'_n) \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \text{Var} \left( \frac{\eta'_n}{b_n} \right) \leq \frac{1}{\varepsilon^2} \left( \frac{1}{b_n} \sum_{i=1}^{n} \text{Var}(\xi_{n,i}) \right)
\quad = \frac{1}{\varepsilon^2} \left( \frac{n}{b_n} \right)^2 \text{Var}(\xi_{n,1}) \leq \frac{1}{\varepsilon^2} \left( \frac{n}{b_n} \right)^2 E(\xi_{n,1})^2. \tag{3.11}
\]
From [4, Lemma 2.2.8] (see also [9, Lemma 6.2.11]) it follows that
\[
E(\xi_{n,1})^2 = 2 \int_{0}^{\infty} y\mathbb{P}(|\xi_{n,1}| > y)dy = 2 \int_{0}^{n} y\mathbb{P}(|\xi_{1}| > y)dy
\quad = 2 \int_{0}^{n} y(\mathbb{P}(|\xi_{1}| > y) - \mathbb{P}(|\xi_{1}| > n))dy \leq 2 \int_{0}^{n} y\mathbb{P}(|\xi_{1}| > y)dy.
\]
This and (3.11) yield
\[
\mathbb{P} \left( \left| \frac{1}{b_n} (\eta'_n - E\eta'_n) \right| > \varepsilon \right) \leq \frac{2}{\varepsilon^2} \left( \frac{n}{b_n} \right)^2 \int_{0}^{n} y\mathbb{P}(|\xi_{1}| > y)dy. \tag{3.12}
\]
For any $\epsilon > 0$, by (3.7) there exists $y_0 > 0$ such that $y \mathbb{P}(|X_1| > y) < \epsilon$ for every $y \geq y_0$. So for any $n > y_0$ we have

$$\left(\frac{n}{b_n}\right)^2 \int_0^n y^2 \mathbb{P}(|X_1| > y)dy \leq \left(\frac{n}{b_n}\right)^2 \int_0^{y_0} y^2 \mathbb{P}(|X_1| > y)dy + \frac{2\epsilon}{\epsilon^2} \int_n^{y_0} y^2 \mathbb{P}(|X_1| > y)dy$$

$$\leq \left(\frac{n}{b_n}\right)^2 \int_0^{y_0} y^2 \mathbb{P}(|X_1| > y)dy + \frac{2\epsilon}{\epsilon^2} \int_0^{y_0} y^2 \mathbb{P}(|X_1| > y)dy + \frac{n\epsilon}{\epsilon^2} \leq \left(\frac{n}{b_n}\right)^2 y_0^2 + \left(\frac{n\sqrt{n}}{b_n}\right)^2 \epsilon.$$

Since $\sup_n \frac{n\sqrt{n}}{b_n} < \infty$ implies $n/b_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n \to \infty} \mathbb{P}\left(\left|\frac{1}{b_n} \sum_{i=1}^{k_n} X_{n,i} - E\sum_{i=1}^{k_n} X_{n,i}\right| > \epsilon\right) \leq \lim_{n \to \infty} \frac{2\epsilon}{\epsilon^2} \left(\frac{n}{b_n}\right)^2 \int_0^{y_0} y^2 \mathbb{P}(|X_1| > y)dy \leq \frac{2}{\epsilon^2} \left(\sup_n \frac{n\sqrt{n}}{b_n}\right)^2 \epsilon.$$

Let $\epsilon \to 0$. We obtain $\lim_{n \to \infty} \mathbb{P}\left(|\frac{1}{b_n} \sum_{i=1}^{k_n} X_{n,i} - E\sum_{i=1}^{k_n} X_{n,i}| > \epsilon\right) = 0$. This and (3.9)-(3.10) lead to the expected (3.8).

There exists a more general version of the Kolmogorov-Feller WLLN for triangular arrays ([9, Theorem 9.2.5]), which includes ([4, Theorem 2.2.6]).

**Theorem 3.6** Consider a triangular array sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, $X_{n,i}, i = 1, \ldots, k_n, n = 1, 2, \ldots$. Suppose that any two of $\{X_{n,i}\}_{i=1}^{k_n}$ are independent for every $n \in \mathbb{N}$ and that $(b_n)$ is a sequence of positive numbers such that $b_n \to \infty$ and

(a) $\sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > b_n) \to 0$,

(b) $\frac{1}{b_n^2} \sum_{i=1}^{k_n} E(\xi_{n,i})^2 \to 0$, where $\xi_{n,i} := X_{n,i}I_{(|X_{n,i}| \leq b_n)}$.

Then for $\eta_n := X_{n,1} + \cdots + X_{n,k_n}$ and $a_n := \sum_{i=1}^{k_n} E\xi_{n,i}$ it holds that

$$\lim_{n \to \infty} \frac{\eta_n - a_n}{b_n} \mathbb{P} \to 0. \quad (3.13)$$

It was showed in [3, page 274, Theorem 3.3] that if $k_n = n$ and $X_{n,i} = X_i$, $i = 1, \ldots, n$ the condition (b) in Theorem 3.6 can be replaced by slightly weak

(b') $\frac{1}{b_n} \sum_{i=1}^{k_n} \text{Var}(\xi_{n,i}) \to 0$;

and conversely (3.13) implies (a) and (b').

If the independence condition for random variables in Theorem 3.6 is not satisfied we can obtain
Theorem 3.7 For a triangular array sequence of random variables, $\xi_{n,i}, i = 1, \ldots, k_n, n = 1, 2, \ldots$, suppose that $(b_n)$ is a sequence of positive numbers such that $b_n \to \infty$ and

(a) $\sum_{i=1}^{k_n} P(|\xi_{n,i}| > b_n) \to 0$,

(b) $\frac{1}{b_n} \sum_{i=1}^{k_n} \sqrt{\text{Var}(\xi_{n,i})} \to 0$, where $\bar{\xi}_{n,i} := \xi_{n,i} I_{\{|\xi_{n,i}| \leq b_n\}}$.

Then for $\eta_n := \sum_{i=1}^{k_n} \xi_{n,i}$ and $a_n := \sum_{i=1}^{k_n} E\bar{\xi}_{n,i}$ it holds that

$$\lim_{n \to \infty} \frac{\eta_n - a_n}{b_n} \overset{p}{\to} 0.$$  \hspace{1cm} (3.14)

Proof. Put $\bar{\eta}_n := \sum_{i=1}^{k_n} \bar{\xi}_{n,i}$. For every given $\varepsilon > 0$ we have

$$P\left(\frac{\eta_n - a_n}{b_n} > \varepsilon\right) \leq P(\eta_n \neq \bar{\eta}_n) + P\left(\left|\frac{\eta_n - a_n}{b_n}\right| > \varepsilon\right).$$  \hspace{1cm} (3.15)

Obverse that the condition (a) implies

$$P(\eta_n \neq \bar{\eta}_n) \leq P\left(\bigcup_{i=1}^{k_n} \{\xi_{n,i} \neq \bar{\xi}_{n,i}\}\right) \leq \sum_{i=1}^{k_n} P(|\xi_{n,i}| > b_n) \to 0.$$  \hspace{1cm} (3.16)

Moreover, using Chebyshev inequality, Proposition 2.2 and the condition (b) we deduce

$$P\left(\left|\frac{\eta_n - a_n}{b_n}\right| > \varepsilon\right) \leq \frac{1}{\varepsilon^2} \text{Var} \left(\frac{\bar{\eta}_n}{b_n}\right) \leq \frac{1}{\varepsilon^2} \left(\frac{1}{b_n} \sum_{i=1}^{k_n} \sqrt{\text{Var}(\xi_{n,i})}\right)^2 \to 0.$$  \hspace{1cm}

This and (3.15)-(3.16) lead to the desired (3.14). \hspace{1cm} $\Box$

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References


Weak laws of large numbers for sequences


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