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Some Further Properties for Analytic Functions with Varying Argument Defined by Hadamard Products

Shu-Hai Li, Huo Tang, Li-Na Ma and En Ao

School of Mathematics and Statistics

Chifeng University

Chifeng 024000, Inner Mongolia, China

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Abstract

The purpose of this paper is to obtain some further properties including coefficients estimates, majorization problems, distortion bounds, extreme points and radius of close-to-convexity, starlikeness and convexity for functions belonging to the class $TU_{\gamma}(\phi, \psi; \alpha, A, B)$, which are defined by Hadamard products with varying argument.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (1.1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of analytic and univalent functions. We denote by $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ ($0 \leq \beta < 1$) the class of starlike of order β in $\mathbb{U}$ and the class of convex functions of order β in $\mathbb{U}$, respectively. It is well known that $\mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(\beta) \subset \mathcal{K}(0) = \mathcal{K}$.

A function $f(z) \in \mathcal{A}$ is said to be in $US(\alpha, \beta)$, the class of α -uniformly starlike functions of order β ($0 \leq \beta < 1$), if f satisfies the condition (see [1,2])

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad \alpha \geq 0. \quad (1.2)$$

Replacing $f(z)$ in (1.2) by $zf'(z)$, we obtain

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \beta, \quad \alpha \geq 0. \quad (1.3)$$

Required for the function f to be in the subclass $UK(\alpha, \beta)$ of α -uniformly convex functions of order β .

Also, by T_γ ($\gamma \in \mathbb{R}$) we denote the class of functions $f(z) \in \mathcal{A}$ of the form (1.1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (1 - n)\gamma \quad (n = 2, 3, \dots). \quad (1.4)$$

For $\gamma = 0$ we obtain the class T_0 of functions with negative coefficients. Moreover, we define

$$T = \bigcup_{\gamma \in \mathbb{R}} T_\gamma.$$

The class T was introduced by Silverman [3] (see also [4], [5] and [31]). It is called the class of functions with varying argument of coefficients.

For two functions f and g , analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω , which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ ($z \in \mathbb{U}$), such that $f(z) = g(\omega(z))$ ($z \in \mathbb{U}$). Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence [6, p.4]:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let f and g be analytic in the open unit disk $\mathbb{U}$. We say that f is majorized by g in $\mathbb{U}$ (see [7]) and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \quad (1.5)$$

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$ such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}). \quad (1.6)$$

It may be noted here that (1.5) is closely related to the concept of quasi-subordination between analytic functions.

For arbitrary fixed real numbers A and B ($-1 \leq B < A \leq 1$), let $P(A, B)$ denote the class of functions of the form $p(z) = 1 + \sum_{j=1}^{\infty} p_j z^j$, which are analytic in $\mathbb{U}$ and satisfies the condition

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

The class $P(A, B)$ was introduced and studied by Janowski [8]. We note that a function $f(z) \in P(A, B)$ if and only if

$$\begin{cases} \left| p(z) - \frac{1+AB}{1-B^2} \right| < \frac{A-B}{1-B^2}, & (-1 < B < A \leq 1; z \in \mathbb{U}); \\ \Re\{p(z)\} > \frac{1-A}{2}, & (B = -1; z \in \mathbb{U}). \end{cases} \quad (1.7)$$

Let

$$f_i(z) = z + \sum_{j=2}^{\infty} a_{j,i} z^j \in A, \quad (i = 1, 2),$$

then $(f_1 * f_2)(z)$ be given by

$$(f_1 * f_2)(z) = z + \sum_{j=2}^{\infty} a_{j,1} a_{j,2} z^j$$

For $\alpha \geq 0$, $-1 \leq B < A \leq 1$ and for all $z \in \mathbb{U}$, Li et al. [9] defined the subclass $U(\phi, \psi, \alpha, A, B)$ of $\mathcal{A}$ which satisfies the following condition:

$$\frac{(f * \phi)(z)}{(f * \psi)(z)} - \alpha \left| \frac{(f * \phi)(z)}{(f * \psi)(z)} - 1 \right| \in P(A, B) \quad (z \in \mathbb{U}), \quad (1.8)$$

where

$$\phi(z) = z + \sum_{j=2}^{\infty} \mu_j z^j \quad \text{and} \quad \psi(z) = z + \sum_{j=2}^{\infty} \eta_j z^j$$

are analytic in $\mathbb{U}$ such that $(f * \psi)(z) \neq 0$ and $\mu_j > \eta_j \geq 0$ for $j \geq 2$.

For $\alpha = 0$, $A = 1 - 2\beta$ ($0 \leq \beta < 1$) and $B = -1$, we denote the class $U(\phi, \psi, 0, 1 - 2\beta, -1)$ by $U(\phi, \psi; \beta)$. Equivalently, $U(\phi, \psi; \beta)$ can be expressed in the form

$$U(\phi, \psi; \beta) = \left\{ f(z) \in \mathcal{A} : \Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \beta, z \in \mathbb{U} \right\}.$$

Using the results in [10, 11, 12] and (1.8), we get the following geometric interpretation.

Geometric interpretation. $f(z) \in U(\phi, \psi; \alpha, A, B)$ if and only if $p(z) = \frac{(f * \phi)(z)}{(f * \psi)(z)}$ takes all values in the conic domain $R_\alpha(A, B)$ which is included in the right half plane such that

$$R_\alpha(A, B) = \left\{ u + iv : u > \begin{cases} \alpha \sqrt{(u-1)^2 + v^2} + \frac{1-A}{1-B}, & -1 < B < A \leq 1; \\ \alpha \sqrt{(u-1)^2 + v^2} + \frac{1-A}{2}, & B = -1. \end{cases} \right\} \quad (1.9)$$

Denote by $\mathcal{P}(p(\alpha, A, B))$, the family of functions p , such that $p \in \mathcal{P}$, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and $p \prec p(\alpha, A, B)(z)$ in $\mathbb{U}$. The function $p(\alpha, A, B)(z)$ maps the unit disk conformally onto the domain $R_\alpha(A, B)$ such that $1 \in R_\alpha(A, B)$ and $\partial R_\alpha(A, B)$ is a curve defined by the equality

$$\partial R_\alpha(A, B) = \left\{ u + iv : u^2 = \begin{cases} \left(\alpha \sqrt{(u-1)^2 + v^2} + \frac{1-A}{1-B} \right)^2, & -1 < B < A \leq 1; \\ \left(\alpha \sqrt{(u-1)^2 + v^2} + \frac{1-A}{2} \right)^2, & B = -1. \end{cases} \right\} \quad (1.10)$$

From elementary computations, we see that (1.10) represents conic sections symmetric about the real axis. Thus $R_\alpha(A, B)$ is an elliptic domain for $\alpha > 1$, a parabolic domain for $\alpha = 1$, a hyperbolic domain for $0 < \alpha < 1$ and the right half plane

$$u > \begin{cases} (1-A)/(1-B), & -1 < B < A \leq 1; \\ (1-A)/2, & B = -1. \end{cases}$$

for $\alpha = 0$.

The functions which play the role of extremal functions for these conic regions are given as

$$p(\alpha, A, B)(z) = \begin{cases} \frac{1+(1-2\frac{1-A}{1-B})z}{1-z}, & \alpha = 0, \\ 1 + \frac{2(1-\frac{1-A}{1-B})}{\pi^2} (\log \frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & \alpha = 1, \\ \frac{1-\frac{1-A}{1-B}}{\alpha^2-1} \cos \left\{ \left(\frac{2}{\pi} \cos^{-1} \alpha \right) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{\alpha^2 - \frac{1-A}{1-B}}{\alpha^2-1}, & 0 < \alpha < 1, \\ \frac{1-\frac{1-A}{1-B}}{\alpha^2-1} \sin \left(\frac{\pi}{2h(t)} \right) \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2} \sqrt{1-t^2x^2}} dx + \frac{\alpha^2 - \frac{1-A}{1-B}}{\alpha^2-1}, & \alpha > 1, \end{cases} \quad (1.11)$$

where $-1 < B < A \leq 1$, $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1)$, every positive number α can be expressed as $\alpha = \cosh \frac{\pi h'(t)}{4h(t)}$, where $h(t)$ is the Legendre's complete elliptic integral of the first kind and $h'(t)$ is complementary integral of $h(t)$ (for details, see [10,11,12]). Also from (1.11), we get the extremum function $p(\alpha, A, -1)(z)$ ($B = -1$).

Because $p(\alpha, A, B)(z)$ is a convex univalent function, so we can write the class $U(\phi, \psi; \alpha, A, B)$ in the subordination form

$$f(z) \in U(\phi, \psi; \alpha, A, B) \iff f(z) \in \mathcal{A} \quad \text{and} \quad p(z) = \frac{(f * \phi)(z)}{(f * \psi)(z)} \prec p(\alpha, A, B)(z) \quad (z \in \mathbb{U}). \quad (1.12)$$

Let us denote

$$TU_\gamma(\phi, \psi; \alpha, A, B) = T_\gamma \cap U(\phi, \psi; \alpha, A, B),$$

$$TUS_\gamma(\alpha, \beta) = T_\gamma \cap US(\alpha, \beta), \quad TUK_\gamma(\alpha, \beta) = T_\gamma \cap UK(\alpha, \beta).$$

For suitable choices of ϕ, ψ and by specializing the parameters α, A, B involved in the class $U(\phi, \psi; \alpha, A, B)$, we also obtain the following subclasses which were studied in many earlier works:

(i) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1-2\beta, -1) = US(\alpha, \beta)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1-2\beta, -1) = UK(\alpha, \beta)$ (Shams et al. [1] and Shams and Kulkarni [2]).

(ii) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, A, B) = S^*(A, B)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, A, B) = K(A, B)$ (Janowski [8] and Padmanabhan et al. [13]). For example, we have $S^*(1, -1) = S^*$ and $K(1, -1) = K$.

(iii) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1, -1) = US(\alpha)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1, -1) = UK(\alpha)$ (Goodman[14], Ma and Minda [15] and Rønning [16]).

(iv) $U(z + \sum_{j=2}^{\infty} j^{n+1} z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, 1-2\beta, -1) = US_n(\alpha, \beta)$ (Bharati et al. [17] and Kuang et al. [18]).

(v) $U(z + \sum_{j=2}^{\infty} j^m z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, 1-2\beta, -1) = US_{m,n}(\alpha, \beta)$ ($0 \leq \alpha, 0 \leq \beta < 1$) (Eker and Owa [19] and Srivastava and Eker [20]).

(vi) $U(z + \sum_{j=2}^{\infty} j^m \lambda_j z^j, z + \sum_{j=2}^{\infty} j^n \tau_j z^j; \alpha, (1-\sigma a) + \sigma b, b) = E_{m,n}(\phi, \psi; a, b, \sigma, \alpha)$ ($\lambda_j > \tau_j \geq 0; -1 \leq b < a \leq 1, 0 \leq \sigma < 1$) (Srivastava et al. [21]).

(vii) $U(z + \sum_{j=2}^{\infty} j\phi_j^{\tau}(\alpha_1, \lambda, l, m)z^j, z + \sum_{j=2}^{\infty} \phi_j^{\tau}(\alpha_1, \lambda, l, m)z^j; \alpha, \beta, -1) = US_m^l(\tau, \lambda, \alpha, \beta)$ ($\alpha \geq 0, 0 \leq \beta < 1$) (Srivastava et al. [22]).

(viii) $U(z + \sum_{j=2}^{\infty} j(\frac{1+b}{j+b})^s z^j, z + \sum_{j=2}^{\infty} (\frac{1+b}{j+b})^s z^j; \alpha, 1 - \beta, -1) = UST_{s,b}(\sigma, \beta)$ ($s \in \mathbb{C}; b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}; 0 \leq \beta < 1$) (Sun et al. [23]).

(ix) $U(z + \sum_{j=2}^{\infty} j^m z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, A, B) = US_{m,n}(\alpha, A, B)$ ($0 \leq \alpha, 0 \leq \beta < 1$) (Li and Tang [24], Aouf et al. [25] and El-Ashwah et al. [26]).

In this paper, we aim to obtain some further properties, such as coefficients estimates, majorization problems, distortion bounds, extreme points and radius of close-to-convexity, star-likeness and convexity for functions belonging to the class $TU_{\gamma}(\phi, \psi; \alpha, A, B)$.

2. Preliminary results

We need the following results in our next investigation.

Lemma 2.1. *Let $f(z) \in U(\phi, \psi; \alpha, A, B)$. Then*

$$f(z) \in U(\phi, \psi; \alpha, A, B) \implies \begin{cases} f(z) \in U(\phi, \psi; \frac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)}), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ f(z) \in U(\phi, \psi; \frac{(1-A)-2\alpha}{2(1-\alpha)}), & B = -1, 2\alpha \leq 1-A. \end{cases} \quad (2.1)$$

Proof. Let $f(z) \in U(\phi, \psi; \alpha, A, B)$. Then we obtain

$$\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} \alpha \Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} - \alpha + \frac{1-A}{1-B}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \alpha \Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} - \alpha + \frac{1-A}{2}, & B = -1, 2\alpha \leq 1-A. \end{cases}$$

or, equivalently,

$$\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} \frac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1-A)-2\alpha}{2(1-\alpha)}, & B = -1, 2\alpha \leq 1-A. \end{cases}$$

If $-1 < B < A \leq 1$ and $\alpha(1-B) \leq 1-A$, then we have

$$0 \leq \frac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)} < 1$$

Also if $B = -1$ and $2\alpha \leq 1 - A$, then we obtain

$$0 \leq \frac{(1 - A) - 2\alpha}{2(1 - \alpha)} < 1$$

Thus we prove Lemma 2.1.

Lemma 2.2 ([9]). *Let $f(z)$ be the function of the form (1.1). Then $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ if and only if*

$$\sum_{j=2}^{\infty} \phi_j(\mu_j, \eta_j, \alpha, A, B) |a_j| \leq A - B, \quad (2.2)$$

where

$$\begin{aligned} \phi_j(\mu_j, \eta_j, \alpha, A, B) &= (1 + (1 + |B|\alpha))(\mu_j - \eta_j) + |B\mu_j - A\eta_j| \\ (\alpha \geq 0, -1 \leq B < A \leq 1, \mu_j > \eta_j \geq 0, j \geq 2). \end{aligned} \quad (2.3)$$

Lemma 2.3 ([27, p.3]). *Let $\alpha \geq 0$ and $-1 \leq B < A \leq 1$. If $\omega(z)$ is an analytic function with $\omega(0) = 1$, then we have*

$$\omega - \alpha|\omega - 1| \prec \frac{1 + Az}{1 + Bz} \iff \omega(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1 + Az}{1 + Bz} \quad (\phi \in \mathbb{R}). \quad (2.4)$$

Lemma 2.4 ([32]). *Let $\varphi(z)$ be analytic in $\mathbb{U}$ satisfy $|\varphi(z)| \leq 1$ for $z \in \mathbb{U}$. Then*

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U}). \quad (2.5)$$

Lemma 2.5. *Let $f(z) \in U(\phi, \psi; \alpha, A, B)$. Then*

$$f(z) \in U(\phi, \psi; \alpha, A, B) \implies \begin{cases} f(z) \in U(\phi, \psi; \frac{1-A}{1-B}), & -1 < B < A \leq 1; \\ f(z) \in U(\phi, \psi; \frac{1-A}{2}), & B = -1. \end{cases} \quad (2.6)$$

Proof. By virtue of (1.8), (1.10) and the properties of the domain $R_\alpha(A, B)$, we obtain

$$\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} \frac{(1-B)\alpha+1-A}{(\alpha+1)(1-B)}, & -1 < B < A \leq 1; \\ \frac{2\alpha+1-A}{2(\alpha+1)}, & B = -1. \end{cases}$$

Thus we prove Lemma 2.5.

Lemma 2.6. If $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ and

$$\phi_j(\mu_j, \eta_j, \alpha, A, B) \geq \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B) \quad (j \geq p+1, p \geq 2), \quad (2.7)$$

then

$$\sum_{j=p+1}^{\infty} a_j \leq \frac{(A-B) - \sum_{j=2}^p \phi_j(\mu_j, \eta_j, \alpha, A, B)a_j}{\phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B)} = A_j. \quad (2.8)$$

Proof. In view of Lemma 2.2, we can write that

$$\sum_{j=p+1}^{\infty} \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B)a_j \leq (A-B) - \sum_{j=2}^p \phi_j(\mu_j, \eta_j, \alpha, A, B)a_j. \quad (2.9)$$

Then from (2.7) and (2.9), we have

$$\phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B) \sum_{j=p+1}^{\infty} a_j \leq (A-B) - \sum_{j=2}^p \phi_j(\mu_j, \eta_j, \alpha, A, B)a_j.$$

Thus we obtain

$$\sum_{j=p+1}^{\infty} a_j \leq \frac{(A-B) - \sum_{j=2}^p \phi_j(\mu_j, \eta_j, \alpha, A, B)a_j}{\phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B)}.$$

Also, we easily obtain

Lemma 2.7. If $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ and $\phi_j(\mu_j, \eta_j, \alpha, A, B)$ defined by (2.3) satisfies (2.7), then

$$\sum_{j=p+1}^{\infty} ja_j \leq \frac{(A-B) - \sum_{j=2}^p \phi_j(\mu_j, \eta_j, \alpha, A, B)a_j}{\frac{1}{p+1} \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, \alpha, A, B)} = B_j. \quad (2.10)$$

3. Main Results

Theorem 3.1. Let $f(z) \in U(\phi, \psi; \alpha, A, B)$. Then

$$|a_2| \leq \begin{cases} \frac{2(1-(1-A)/(1-B))}{(\mu_2-\eta_2)(1-\alpha)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{2(1-(1-A)/2)}{(\mu_2-\eta_2)(1-\alpha)}, & B = -1, 2\alpha \leq 1-A. \end{cases} \quad (3.1)$$

and

$$|a_j| \leq \begin{cases} \frac{2(1-(1-A)/(1-B))}{(\mu_j-\eta_j)(1-\alpha)} \prod_{k=1}^{j-2} (1 + \frac{2(1-(1-A)/(1-B))}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)}), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; j \geq 3; \\ \frac{2(1-(1-A)/2)}{(\mu_j-\eta_j)(1-\alpha)} \prod_{k=1}^{j-2} (1 + \frac{2(1-(1-A)/2)}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)}), & B = -1, 2\alpha \leq 1-A; j \geq 3. \end{cases} \quad (3.2)$$

Proof. Suppose that $f \in U(\phi, \psi; \alpha, A, B)$. Then, by Lemma 2.1, we obtain

$$\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} \frac{(1-A)-\alpha(1-B)}{(1-\alpha)(1-B)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1-A)-2\alpha}{2(1-\alpha)}, & B = -1, 2\alpha \leq 1-A. \end{cases}$$

Let us define the function $p(z)$ by

$$p(z) = \begin{cases} \frac{(1-\alpha) \frac{(f * \phi)(z)}{(f * \psi)(z)} - \frac{(1-A)-\alpha(1-B)}{(1-B)}}{\frac{A-B}{1-B}}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1-\alpha) \frac{(f * \phi)(z)}{(f * \psi)(z)} - \frac{(1-A)-2\alpha}{2}}{\frac{1+A}{2}}, & B = -1, 2\alpha \leq 1-A. \end{cases}$$

Hence $p(z)$ is analytic in U with $p(0) = 1$ and $\Re p(z) > 0$ ($z \in U$). Let

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

So we get

$$\frac{(f * \phi)(z)}{(f * \psi)(z)} = \begin{cases} 1 + \frac{A-B}{(1-\alpha)(1-B)}(c_1 z + c_2 z^2 + \dots), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ 1 + \frac{1+A}{2(1-\alpha)}(c_1 z + c_2 z^2 + \dots), & B = -1, 2\alpha \leq 1-A. \end{cases}$$

or, equivalently,

$$(f * \phi)(z) - (f * \psi)(z) = \begin{cases} \frac{A-B}{(1-\alpha)(1-B)}((f * \psi)(z))(c_1 z + c_2 z^2 + \dots), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{1+A}{2(1-\alpha)}((f * \psi)(z))(c_1 z + c_2 z^2 + \dots), & B = -1, 2\alpha \leq 1-A, \end{cases}$$

which implies that

$$(\mu_j - \eta_j) a_j = \begin{cases} \frac{A-B}{(1-\alpha)(1-B)}(c_{j-1} + a_2 c_{j-2} + \dots + a_{j-1} c_1), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{1+A}{2(1-\alpha)}(c_{j-1} + a_2 c_{j-2} + \dots + a_{j-1} c_1), & B = -1, 2\alpha \leq 1-A. \end{cases}$$

Applying the coefficient estimates $|c_j| \leq 2$ ($j \geq 1$) (see [33]), we obtain

$$|a_j| \leq \begin{cases} \frac{2(A-B)}{(\mu_j - \eta_j)(1-\alpha)(1-B)}(1 + a_2 + \dots + a_{j-1}) & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{(1+A)}{(\mu_j - \eta_j)(1-\alpha)}(1 + a_2 + \dots + a_{j-1}), & B = -1, 2\alpha \leq 1-A. \end{cases} \quad (3.3)$$

For $j = 2$,

$$|a_2| \leq \begin{cases} \frac{2(A-B)}{(\mu_2-\eta_2)(1-\alpha)(1-B)}, & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{1+A}{(\mu_2-\eta_2)(1-\alpha)}, & B = -1, 2\alpha \leq 1-A. \end{cases}$$

which proves (3.1).

For $j = 3$,

$$|a_3| \leq \begin{cases} \frac{2(A-B)}{(\mu_3-\eta_3)(1-\alpha)(1-B)} \left(1 + \frac{2(A-B)}{(\mu_2-\eta_2)(1-\alpha)(1-B)}\right), & -1 < B < A \leq 1, \alpha(1-B) \leq 1-A; \\ \frac{1+A}{(\mu_2-\eta_2)(1-\alpha)} \left(1 + \frac{1+A}{(\mu_2-\eta_2)(1-\alpha)}\right), & B = -1, 2\alpha \leq 1-A. \end{cases}$$

Therefore (3.2) holds true for $j = 3$. Assume that (3.3) is true for $j = m$.

If $-1 < B < A \leq 1$ and $\alpha(1-B) \leq 1-A$, then we have

$$\begin{aligned} |a_{m+1}| &\leq \frac{2(A-B)}{(\mu_{m+1}-\eta_{m+1})(1-\alpha)(1-B)} \left\{ 1 + \frac{2(A-B)}{(\mu_2-\eta_2)(1-\alpha)(1-B)} \right. \\ &\quad + \frac{2(A-B)}{(\mu_3-\eta_3)(1-\alpha)(1-B)} \left(1 + \frac{2(A-B)}{(\mu_2-\eta_2)(1-\alpha)(1-B)}\right) + \dots \\ &\quad \left. + \frac{2(A-B)}{(\mu_{m-1}-\eta_{m-1})(1-\alpha)(1-B)} \prod_{k=1}^{m-2} \left(1 + \frac{2(A-B)}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)(1-B)}\right) \right\}. \\ &= \frac{2(A-B)}{(\mu_{m+1}-\eta_{m+1})(1-\alpha)(1-B)} \prod_{k=1}^{m-1} \left(1 + \frac{2(A-B)}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)(1-B)}\right) \end{aligned}$$

Also, if $B = -1$ and $2\alpha \leq 1-A$, then we have

$$|a_{m+1}| \leq \frac{1+A}{(\mu_{m+1}-\eta_{m+1})(1-\alpha)} \prod_{k=1}^{m-1} \left(1 + \frac{1+A}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)}\right).$$

So (3.2) is true for $j = m+1$. Consequently, using the mathematical induction, we get that (3.2) holds true for all $j \geq 3$.

Remark 3.1. Taking $\alpha = 0, \mu_j = j, \eta_j = 1, B = -1$ and $A = 1 - 2\beta$ ($0 \leq \beta < 1$) in Theorem 3.1, we obtain the results of Robertson [34].

Theorem 3.2. Let the function $f \in \mathcal{A}$ and suppose that $f \in TU_\gamma(\phi, \psi; \alpha, A, B)$ ($0 \leq \alpha \neq 1$). If $(f * \phi)(z)$ is majorized by $(f * \psi)(z)$ and $|(f * \phi)(z)| \leq |z(f * \psi)'(z)|$, then

$$|(f * \phi)'(z)| \leq |(f * \psi)'(z)| \quad (|z| \leq r_0), \quad (3.4)$$

where $r_0 = r_0(\alpha, A, B)$ is the smallest positive root of the equation

$$\left[\frac{A-B}{|1-\alpha|} + |B| \right] r^3 - [1 + 2|B|]r^2 - \left[\frac{A-B}{|1-\alpha|} + |B| + 2 \right] r + 1 = 0 \quad (3.5)$$

$$(z \in \mathbb{U}; -1 \leq B < A \leq 1; 0 \leq \delta \leq r_0; [\frac{A-B}{|1-\alpha|} + |B|]\delta \leq 1).$$

Proof. Suppose that $f \in TU_\gamma(\phi, \psi; \alpha, A, B)$. Then, by Lemma 2.3, we obtain

$$\frac{f(z) * \phi(z)}{f(z) * \psi(z)}(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1 + Az}{1 + Bz},$$

or, equivalently,

$$\frac{f(z) * \phi(z)}{f(z) * \psi(z)} \prec \frac{1 + (\frac{A - \alpha B e^{-i\phi}}{1 - \alpha e^{-i\phi}})z}{1 + Bz} \quad (3.6)$$

which holds true for all $z \in \mathbb{U}$.

We find from (3.6) that

$$\frac{f(z) * \phi(z)}{f(z) * \psi(z)} = \frac{1 + (\frac{A - \alpha B e^{-i\phi}}{1 - \alpha e^{-i\phi}})\omega(z)}{1 + B\omega(z)}, \quad (3.7)$$

where $\omega(z) = c_1 z + c_2 z^2 + \dots \in \mathcal{W}$, $\mathcal{W}$ denotes the well known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the conditions:

$$\omega(0) = 0, \quad |\omega(z)| \leq |z| \quad (z \in \mathbb{U}).$$

From (3.7), we get

$$|(f * \psi)(z)| \leq \frac{1 + |B||z|}{1 - (\frac{A-B}{|1-\alpha|} + |B|)|z|} |(f * \phi)(z)|. \quad (3.8)$$

Next, since $(f * \phi)(z)$ is majorized by $(f * \psi)(z)$ in $\mathbb{U}$, from (1.6), we have

$$(f * \phi)(z) = \varphi(z)(f * \psi)(z).$$

Differentiating it with respect to z and multiplying by z , we get

$$(f * \phi)'(z) = \varphi'(z)(f * \psi)(z) + \varphi(z)(f * \psi)'(z). \quad (3.9)$$

Thus, by Lemma 2.4, (3.8) and (3.9), we get

$$|(f * \phi)'(z)| \leq \left[|\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \cdot \frac{(1 + |B||z|)|z|}{[1 - (\frac{A-B}{|1-\alpha|} + |B|)|z|]} \right] |(f * \psi)'(z)|, \quad (3.10)$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$|(f * \phi)'(z)| \leq \left[\frac{\psi(\rho)}{(1-r^2)[1 - (\frac{A-B}{|1-\alpha|} + |B|)r]} \right] |(f * \psi)'(z)|, \quad (3.11)$$

where

$$\psi(\rho) = -r(1 + |B|r)\rho^2 + (1-r^2) \left[1 - \left(\frac{A-B}{|1-\alpha|} + |B| \right) r \right] \rho + r(1 + |B|r) \quad (3.12)$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(\alpha, A, B)$, where $r_0 = r_0(\alpha, A, B)$ is the smallest positive root of (3.5). Furthermore, if $0 \leq \delta \leq r_0(\alpha, A, B)$, then the function $\psi(\rho)$ defined by

$$\psi(\rho) = -\delta(1 + |B|\delta)\rho^2 + (1-\delta^2) \left[1 - \left(\frac{A-B}{|1-\alpha|} + |B| \right) \delta \right] \rho + (1 + |B|\delta)\delta \quad (3.13)$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\psi(\rho) \leq \psi(1) = (1-\delta^2) \left[1 - \left(\frac{A-B}{|1-\alpha|} + |B| \right) \delta \right] \quad (0 \leq \rho \leq 1; 0 \leq \delta \leq r_0(\alpha, A, B)). \quad (3.14)$$

Hence, upon setting $\rho = 1$ in (3.13), we conclude that (3.4) of Theorem 3.2 holds true for $|z| \leq r_0 = r_0(\alpha, A, B)$, which completes the proof of Theorem 3.2.

Theorem 3.3. *Let the function $f \in \mathcal{A}$ and suppose that $f \in TU_\gamma(\phi, \psi; 1, A, B)$ ($\alpha = 1$). If $(f * \phi)(z)$ is majorized by $(f * \psi)(z)$ and $|(f * \phi)(z)| \leq |z(f * \psi)'(z)|$, then*

$$|(f * \phi)'(z)| \leq |(f * \psi)'(z)| \quad (3.15)$$

where $r_0 = r_0(A, B)$ is the smallest positive root of the equation

$$\chi(A, B) = \begin{cases} \left(\frac{1-A}{1-B} \right) r^2 - \left(3 + \frac{1-A}{1-B} \right) r + 1 = 0, & -1 < B < A \leq 1; \\ \left(\frac{1-A}{2} \right) r^2 - \left(3 + \frac{1-A}{2} \right) r + 1 = 0, & B = -1. \end{cases} \quad (3.16)$$

$$(z \in \mathbb{U}; -1 \leq B < A \leq 1; r_0 \geq 0).$$

Proof. Suppose that $f \in TU_\gamma(\phi, \psi; 1, A, B)$. Then, by Lemma 2.5, we obtain

$$\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} \frac{(1-B)+1-A}{2(1-B)}, & -1 < B < A \leq 1; \\ \frac{3-A}{4}, & B = -1, \end{cases}$$

or, equivalently,

$$\frac{(f * \phi)(z)}{(f * \psi)(z)} \prec \begin{cases} \frac{1+(1-\frac{(1-B)+1-A}{2(1-B)})z}{1-z}, & -1 < B < A \leq 1; \\ \frac{1+(1-\frac{3-A}{2})z}{1-z}, & B = -1, \end{cases} \quad (3.17)$$

which holds true for all $z \in \mathbb{U}$.

We find from (3.17) that

$$\frac{(f * \phi)(z)}{(f * \psi)(z)} = \begin{cases} \frac{1 + (1 - \frac{(1-B)+1-A}{(1-B)})\omega(z)}{1 - \omega(z)}, & -1 < B < A \leq 1; \\ \frac{1 + (1 - \frac{3-A}{2})\omega(z)}{1 - \omega(z)}, & B = -1, \end{cases}$$

where $\omega(z) = c_1 z + c_2 z^2 + \dots \in \mathcal{W}$. The remainder of Theorem 3.3 is analogous to the proof of Theorem 3.2, so we omit the details involved.

Theorem 3.4. If $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ and $\phi_j(\mu_j, \eta_j, \alpha, A, B)$ defined by (2.3) satisfies (2.7), then for $|z| = r < 1$

$$r - r^2 \sum_{j=2}^p |a_j| - |A_j| r^{p+1} \leq |f(z)| \leq r + r^2 \sum_{j=2}^p |a_j| + |A_j| r^{p+1} \quad (3.18)$$

and

$$1 - r \sum_{j=2}^p |a_j| - |B_j| r^p \leq |f'(z)| \leq 1 + r \sum_{j=2}^p |a_j| + |B_j| r^p. \quad (3.19)$$

where A_j and B_j are given by (2.8) and (2.10), respectively.

Proof. Let $f(z)$ be given by (1.1). For $|z| = r < 1$, by using Lemma 2.6, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{j=2}^p |a_j| |z|^j + \sum_{j=p+1}^{\infty} |a_j| |z|^j \\ &\leq |z| + |z|^2 \sum_{j=2}^p |a_j| + |z|^{p+1} \sum_{j=p+1}^{\infty} |a_j| \\ &\leq r + r^2 \sum_{j=2}^p |a_j| + |A_j| r^{p+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{j=2}^p |a_j| |z|^j - \sum_{j=p+1}^{\infty} |a_j| |z|^j \\ &\geq |z| - |z|^2 \sum_{j=2}^p |a_j| - |z|^{p+1} \sum_{j=p+1}^{\infty} |a_j| \\ &\geq r - r^2 \sum_{j=2}^p |a_j| - |A_j| r^{p+1} \end{aligned}$$

Furthermore for $|z| = r < 1$, by using Lemma 2.7, we obtain

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{j=2}^p j |a_j| |z|^{j-1} + \sum_{j=p+1}^{\infty} j |a_j| |z|^{j-1} \\ &\leq 1 + |z| \sum_{j=2}^p j |a_j| + |z|^p \sum_{j=p+1}^{\infty} j |a_j| \\ &\leq 1 + r \sum_{j=2}^p |a_j| + |B_j| r^p \end{aligned}$$

and

$$\begin{aligned}
|f'(z)| &\geq 1 - \sum_{j=2}^p j|a_j||z|^{j-1} - \sum_{j=p+1}^{\infty} j|a_j||z|^{j-1} \\
&\geq 1 - |z| \sum_{j=2}^p j|a_j| - |z|^p \sum_{j=p+1}^{\infty} j|a_j| \\
&\geq 1 - r \sum_{j=2}^p |a_j| - |B_j|r^p,
\end{aligned}$$

thus we have (3.19).

Theorem 3.5. Let the function $f(z)$ defined by (1.1) satisfy (1.4). We define

$$f_1(z) = z, \quad f_j(z) = z - \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma} z^j \quad (j = 2, 3, \dots), \quad (3.20)$$

where $\phi_j(\mu_j, \eta_j, \alpha, A, B)$ is given by (2.3). Then $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z), \quad (3.21)$$

where $\lambda_j > 0$ and $\sum_{j=1}^{\infty} \lambda_j = 1$.

Proof. Suppose that

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = z - \sum_{j=1}^{\infty} \lambda_j \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma} z^j.$$

Then

$$\begin{aligned}
\sum_{j=1}^{\infty} \phi_j(\mu_j, \eta_j, \alpha, A, B) |\lambda_j \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma}| &= (A-B) \sum_{j=2}^{\infty} \lambda_j \\
&= (A-B)(1 - \lambda_1) \\
&< A-B.
\end{aligned}$$

By Lemma 2.2, we have $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$.

Conversely, suppose that $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$. Since

$$|a_j| \leq \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} \quad (j = 2, 3, \dots),$$

we may set

$$\lambda_j = \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{(A-B)|e^{i(1-j)\gamma}|} |a_j| \quad (j = 2, 3, \dots)$$

and

$$\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j.$$

Then

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z).$$

This completes the proof of Theorem 3.5.

Theorem 3.6. Let the function $f(z)$ defined by (1.1) be in the class $TU(\phi, \psi; \alpha, A, B)$, and $\phi_j(\mu_j, \eta_j, \alpha, A, B)$ be given by (2.3). Then, we have

(1) The function $f(z)$ is close-to-convex of μ ($0 \leq \mu < 1$) in $|z| < r_1$ where

$$r_1 = \inf_j \left\{ \frac{(1-\mu)\phi_j(\mu_j, \eta_j, \alpha, A, B)}{j(A-B)} \right\}^{\frac{1}{j-1}} (j \geq 2) \quad (3.22)$$

(2) The function $f(z)$ is starlike of η ($0 \leq \eta < 1$) in $|z| < r_2$ where

$$r_2 = \inf_j \left\{ \frac{(1-\eta)\phi_j(\mu_j, \eta_j, \alpha, A, B)}{(j-\eta)(A-B)} \right\}^{\frac{1}{j-1}} (j \geq 2) \quad (3.23)$$

(3) The function $f(z)$ is convex of μ ($0 \leq \xi < 1$) in $|z| < r_3$ where

$$r_3 = \inf_j \left\{ \frac{(1-\xi)\phi_j(\mu_j, \eta_j, \alpha, A, B)}{j(j-\xi)(A-B)} \right\}^{\frac{1}{j-1}} (j \geq 2) \quad (3.24)$$

Proof. (1) We must show that $|f'(z) - 1| < 1 - \mu$ for $|z| < r_1$. We have

$$|f'(z) - 1| \leq \sum_{j=2}^{\infty} j|a_j||z|^{j-1}.$$

Thus $|f'(z) - 1| < 1 - \mu$ if

$$\sum_{j=2}^{\infty} \frac{j}{1-\mu} |a_j||z|^{j-1} \leq 1. \quad (3.25)$$

By Lemma 2.2, we have

$$\sum_{j=2}^{\infty} \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{A-B} |a_j| \leq 1 \quad (3.26)$$

Hence (3.25) will be true if

$$\frac{j|z|^{j-1}}{1-\mu} \leq \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{A-B},$$

or, if

$$|z| \leq \left\{ \frac{(1-\mu)\phi_j(\mu_j, \eta_j, \alpha, A, B)}{j(A-B)} \right\}^{\frac{1}{j-1}} (j \geq 2), \quad (3.27)$$

which follows from (3.22). Similarly, we can prove (2) and (3). This completes the proof of Theorem 3.6.

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