Some Further Properties for Analytic Functions
with Varying Argument Defined by
Hadamard Products

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Abstract

The purpose of this paper is to obtain some further properties including coefficients estimates, majorization problems, distortion bounds, extreme points and radius of close-to-convexity, starlikeness and convexity for functions belonging to the class $TU_{\gamma}(\phi, \psi; \alpha, A, B)$, which are defined by Hadamard products with varying argument.

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1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form
\[
f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}
\]
which are analytic in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \), consisting of analytic and univalent functions. We denote by \( \mathcal{S}^*(\beta) \) and \( \mathcal{K}(\beta) \) \((0 \leq \beta < 1)\) the class of starlike of order \( \beta \) in \( U \) and the class of convex functions of order \( \beta \) in \( U \), respectively. It is well known that \( \mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^* \) and \( \mathcal{K}(\beta) \subset \mathcal{K}(0) = \mathcal{K} \).

A function \( f(z) \in \mathcal{A} \) is said to be in \( US(\alpha, \beta) \), the class of \( \alpha \)-uniformly starlike functions of order \( \beta \) \((0 \leq \beta < 1)\), if \( f \) satisfies the condition (see [1,2])
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad \alpha \geq 0. \tag{1.2}
\]

Replacing \( f(z) \) in (1.2) by \( zf'(z) \), we obtain
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \beta, \quad \alpha \geq 0. \tag{1.3}
\]

Required for the function \( f \) to be in the subclass \( UK(\alpha, \beta) \) of \( \alpha \)-uniformly convex functions of order \( \beta \).

Also, by \( T_\gamma \) \((\gamma \in \mathbb{R})\) we denote the class of functions \( f(z) \in \mathcal{A} \) of the form (1.1) for which all of non-vanishing coefficients satisfy the condition
\[
\arg(a_n) = \pi + (1 - n)\gamma \quad (n = 2, 3, \cdots). \tag{1.4}
\]

For \( \gamma = 0 \) we obtain the class \( T_0 \) of functions with negative coefficients. Moreover, we define
\[
T = \bigcup_{\gamma \in \mathbb{R}} T_\gamma.
\]

The class \( T \) was introduced by Silverman [3] (see also [4], [5] and [31]). It is called the class of functions with varying argument of coefficients.

For two functions \( f \) and \( g \), analytic in \( U \), we say that the function \( f \) is subordinate to \( g \) in \( U \), and write
\[
f(z) \prec g(z) \quad (z \in U),
\]
if there exists a Schwarz function \( \omega \), which (by definition) is analytic in \( U \) with \( \omega(0) = 0 \) and \(|\omega(z)| < 1 \ (z \in U)\), such that \( f(z) = g(\omega(z)) \ (z \in U) \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence [6, p.4]:
\[
f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]
Let \( f \) and \( g \) be analytic in the open unit disk \( \mathbb{U} \). We say that \( f \) is majorized by \( g \) in \( \mathbb{U} \) (see [7]) and write
\[
\text{if there exists a function } \varphi(z), \text{ analytic in } \mathbb{U} \text{ such that }
\]
\[
|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \mathbb{U}).
\]

It may be noted here that (1.5) is closely related to the concept of quasi-subordination between analytic functions.

For arbitrary fixed real numbers \( A \) and \( B \) \((-1 \leq B < A \leq 1)\), let \( P(A, B) \) denote the class of functions of the form \( \phi(z) = 1 + \sum_{j=1}^{\infty} p_j z^j \), which are analytic in \( \mathbb{U} \) and satisfies the condition
\[
\phi(z) \prec 1 + Az + Bz \quad (z \in \mathbb{U}).
\]

The class \( P(A, B) \) was introduced and studied by Janowski [8]. We note that a function \( f(z) \in P(A, B) \) if and only if
\[
\begin{cases}
|\phi(z) - \frac{1-AB}{1-B^2}| < \frac{A-B}{1-B^2}, & (-1 < B < A \leq 1; z \in \mathbb{U}); \\
\Re \{\phi(z)\} > \frac{1-A}{2}, & (B = -1; z \in \mathbb{U}).
\end{cases}
\]

Let
\[
f_i(z) = z + \sum_{j=2}^{\infty} a_{i,j} z^j \in A, (i = 1, 2),
\]
then \((f_1 * f_2)(z)\) be given by
\[
(f_1 * f_2)(z) = z + \sum_{j=2}^{\infty} a_{i,1} a_{i,2} z^j
\]

For \( \alpha \geq 0, -1 \leq B < A \leq 1 \) and for all \( z \in \mathbb{U} \), Li et al. [9] defined the subclass \( U(\phi, \psi, \alpha, A, B) \) of \( A \) which satisfies the following condition:
\[
\frac{(f * \phi)(z)}{(f * \psi)(z)} - \alpha \left| \frac{(f * \phi)(z)}{(f * \psi)(z)} - 1 \right| \in P(A, B) \quad (z \in \mathbb{U}),
\]
where
\[
\phi(z) = z + \sum_{j=2}^{\infty} \mu_j z^j \quad \text{and} \quad \psi(z) = z + \sum_{j=2}^{\infty} \eta_j z^j
\]
are analytic in \( \mathbb{U} \) such that \((f * \psi)(z) \neq 0 \) and \( \mu_j > \eta_j \geq 0 \) for \( j \geq 2 \).
For \( \alpha = 0, A = 1 - 2\beta \ (0 \leq \beta < 1) \) and \( B = -1 \), we denote the class \( U(\phi, \psi, 0, 1 - 2\beta, -1) \) by \( U(\phi, \psi; \beta) \). Equivalently, \( U(\phi, \psi; \beta) \) can be expressed in the form

\[
U(\phi, \psi; \beta) = \left\{ f(z) \in A : \Re \left\{ \frac{(f \ast \phi)(z)}{(f \ast \psi)(z)} \right\} > \beta, z \in \mathbb{U} \right\}.
\]

Using the results in [10, 11, 12] and (1.8), we get the following geometric interpretation.

**Geometric interpretation.** \( f(z) \in U(\phi, \psi; \alpha, A, B) \) if and only if \( p(z) = \frac{(f \ast \phi)(z)}{(f \ast \psi)(z)} \) takes all values in the conic domain \( R_\alpha(A, B) \) which is included in the right half plane such that

\[
R_\alpha(A, B) = \left\{ u + iv : u > \begin{cases} 
\alpha \sqrt{(u - 1)^2 + v^2 + \frac{1-A}{1-B}}, & -1 < B < A \leq 1; \\
\alpha \sqrt{(u - 1)^2 + v^2 + \frac{1-A}{2}}, & B = -1.
\end{cases} \right\}
\] (1.9)

Denote by \( \mathcal{P}(p(\alpha, A, B)) \), the family of functions \( p \), such that \( p \in \mathcal{P} \), where \( \mathcal{P} \) denotes the well-known class of Caratheodory functions and \( p < p(\alpha, A, B)(z) \) in \( \mathbb{U} \). The function \( p(\alpha, A, B)(z) \) maps the unit disk conformally onto the domain \( R_\alpha(A, B) \) such that \( 1 \in R_\alpha(A, B) \) and \( \partial R_\alpha(A, B) \) is a curve defined by the equality

\[
\partial R_\alpha(A, B) = \left\{ u + iv : u^2 = \begin{cases} 
\left( \alpha \sqrt{(u - 1)^2 + v^2 + \frac{1-A}{1-B}} \right)^2, & -1 < B < A \leq 1; \\
\left( \alpha \sqrt{(u - 1)^2 + v^2 + \frac{1-A}{2}} \right)^2, & B = -1.
\end{cases} \right\}
\] (1.10)

From elementary computations, we see that (1.10) represents conic sections symmetric about the real axis. Thus \( R_\alpha(A, B) \) is an elliptic domain for \( \alpha > 1 \), a parabolic domain for \( \alpha = 1 \), a hyperbolic domain for \( 0 < \alpha < 1 \) and the right half plane

\[
u > \begin{cases} 
(1-A)/(1-B), & -1 < B < A \leq 1; \\
(1-A)/2, & B = -1.
\end{cases}
\]

for \( \alpha = 0 \).

The functions which play the role of extremal functions for these conic regions are given as

\[
p(\alpha, A, B)(z) = \begin{cases} 
\frac{1+(1-2\frac{1-A}{1-B})z}{1-z}, & \alpha = 0, \\
1 + \frac{2(1-\frac{1-A}{1-B})}{\pi^2} \left( \log \left( \frac{1+\sqrt{2}}{1-\sqrt{2}} \right) \right)^2, & \alpha = 1, \\
\frac{1-\frac{1-A}{\alpha^2-1}}{\alpha^2-1} \cos \left\{ \left( \frac{2}{\pi} \cos^{-1} \alpha \right) \frac{\log \frac{1+\sqrt{2}}{1-\sqrt{2}}}{1-\sqrt{2}} \right\} - \frac{\alpha^2-\frac{1-A}{\alpha^2-1}}{\alpha^2-1}, & 0 < \alpha < 1, \\
\frac{1-\frac{1-A}{\alpha^2-1}}{\alpha^2-1} \sin \left( \frac{\pi}{2h(t)} \right) \int_0^{w(z)} \frac{1}{\sqrt{1-x^2}\sqrt{1-\frac{\alpha^2-1}{\alpha^2-1}}} dx + \frac{\alpha^2-\frac{1-A}{\alpha^2-1}}{\alpha^2-1}, & \alpha > 1,
\end{cases}
\] (1.11)
where \(-1 < B < A \leq 1, u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}}z, t \in (0, 1)\), every positive number \(\alpha\) can be expressed as \(\alpha = \cosh \frac{\pi h(t)}{h(t)}\), where \(h(t)\) is the Legendre's complete elliptic integral of the first kind and \(h'(t)\) is complementary integral of \(h(t)\) (for details, see [10,11,12]). Also from (1.11), we get the extremum function \(p(\alpha, A, -1)(z) (B = -1)\).

Because \(p(\alpha, A, B)(z)\) is a convex univalent function, so we can write the class \(U(\phi, \psi; \alpha, A, B)\) in the subordination form

\[
f(z) \in U(\phi, \psi; \alpha, A, B) \iff f(z) \in A \quad \text{and} \quad p(z) = \frac{(f \ast \phi)(z)}{(f \ast \psi)(z)} < p(\alpha, A, B)(z) \quad (z \in U).
\]

Let us denote

\[
TU_\gamma(\phi, \psi; \alpha, A, B) = T_\gamma \cap U(\phi, \psi; \alpha, A, B),
\]

\[
TU*S_\gamma(\alpha, \beta) = T_\gamma \cap US(\alpha, \beta), \quad TU*K_\gamma(\alpha, \beta) = T_\gamma \cap UK(\alpha, \beta).
\]

For suitable choices of \(\phi, \psi\) and by specializing the parameters \(\alpha, A, B\) involved in the class \(U(\phi, \psi; \alpha, A, B)\), we also obtain the following subclasses which were studied in many earlier works:

(i) \(U\left(\frac{z}{(1-z)^2}, \frac{z^2}{(1-z)^3}; \alpha, 1 - 2\beta, -1\right) = US(\alpha, \beta)\) and \(U\left(\frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1 - 2\beta, -1\right) = UK(\alpha, \beta)\) (Shams et al. [1] and Shams and Kulkarni [2]).

(ii) \(U\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, A, B\right) = S^*(A, B)\) and \(U\left(\frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, A, B\right) = K(A, B)\) (Janowski [8] and Padmanabhan et al. [13]). For example, we have \(S^*(1, -1) = S^*\) and \(K(1, -1) = K\).

(iii) \(U\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1, -1\right) = US(\alpha)\) and \(U\left(\frac{z + z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1, -1\right) = UK(\alpha)\) (Goodman[14], Ma and Minda [15] and Rønning [16]).

(iv) \(U(z + \sum_{j=2}^{\infty} j^{n+1} z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, 1 - 2\beta, -1) = US_n(\alpha, \beta)\) (Bharati et al. [17] and Kuang et al. [18]).

(v) \(U(z + \sum_{j=2}^{\infty} j^m z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, 1 - 2\beta, -1) = US_{m,n}(\alpha, \beta)\) (Eker and Owa [19] and Srivastava and Eker [20]).

(vi) \(U(z + \sum_{j=2}^{\infty} j^m \lambda_j z^j, z + \sum_{j=2}^{\infty} j^n \tau_j z^j; \alpha, (1 - \sigma a) + \sigma b, b) = E_{m,n}(\phi, \psi; a, b, \sigma, \alpha)\) (\(\lambda_j > \tau_j \geq 0; -1 \leq b < a \leq 1, 0 \leq \sigma < 1\)) (Srivastava et al. [21]).
(vii) \( U(z + \sum_{j=2}^{\infty} j \phi_j^*(\alpha_1, \lambda, l, m)z^j, z + \sum_{j=2}^{\infty} j \phi_j^*(\alpha_1, \lambda, l, m)z^j; \alpha, \beta, -1) = US_m^l(\tau, \lambda, \alpha, \beta) \) (\( \alpha \geq 0, 0 \leq \beta < 1 \)) (Srivastava et al. [22]).

(viii) \( U(z + \sum_{j=2}^{\infty} j \frac{1+b}{j+b} s z^j, z + \sum_{j=2}^{\infty} j \frac{1+b}{j+b} s z^j; \alpha, 1-\beta, -1) = UST_{s,b}(\sigma, \beta) \) (\( s \in \mathbb{C}; b \in \mathbb{C}\setminus\{0, -1, -2, \cdots\}; 0 \leq \beta < 1 \)) (Sun et al. [23]).

(ix) \( U(z + \sum_{j=2}^{\infty} j^m z^j, z + \sum_{j=2}^{\infty} j^n z^j; \alpha, A, B) = US_{m,n}(\alpha, A, B) \) (\( 0 \leq \alpha, 0 \leq \beta < 1 \)) (Li and Tang [24], Aouf et al. [25] and El-Ashwah et al. [26]).

In this paper, we aim to obtain some further properties, such as coefficients estimates, majorization problems, distortion bounds, extreme points and radius of close-to-convexity, starlikeness and convexity for functions belonging to the class \( TU_\gamma(\phi, \psi; \alpha, A, B) \).

2. Preliminary results

We need the following results in our next investigation.

**Lemma 2.1.** Let \( f(z) \in U(\phi, \psi; \alpha, A, B) \). Then

\[
\begin{align*}
\text{if} & \quad f(z) \in U(\phi, \psi; (1-A)\alpha, (1-B)\beta), \quad -1 < B < A \leq 1, \quad \alpha(1-B) \leq 1 - A; \\
\text{then} & \quad f(z) \in U(\phi, \psi; (1-A)\alpha - 2\alpha B, \quad B = -1, \quad 2\alpha \leq 1 - A. \\
& \quad (2.1)
\end{align*}
\]

**Proof.** Let \( f(z) \in U(\phi, \psi; \alpha, A, B) \). Then we obtain

\[
\Re \left\{ \frac{(f*\phi)(z)(f*\psi)(z)}{(f*\psi)(z)} \right\} > \begin{cases} \\
\alpha \Re \left\{ \frac{(f*\phi)(z)}{(f*\psi)(z)} \right\} - \alpha + \frac{1-A}{1-B}, \quad -1 < B < A \leq 1, \quad \alpha(1-B) \leq 1 - A; \\
\alpha \Re \left\{ \frac{(f*\phi)(z)}{(f*\psi)(z)} \right\} - \alpha + \frac{1-A}{2}, \quad B = -1, \quad 2\alpha \leq 1 - A.
\end{cases}
\]

or, equivalently,

\[
\Re \left\{ \frac{(f*\phi)(z)}{(f*\psi)(z)} \right\} > \begin{cases} \\
\frac{(1-A)\alpha(1-B)}{(1-\alpha)(1-B)}, \quad -1 < B < A \leq 1, \quad \alpha(1-B) \leq 1 - A; \\
\frac{(1-A)2\alpha}{2(1-\alpha)}, \quad B = -1, \quad 2\alpha \leq 1 - A.
\end{cases}
\]

If \( -1 < B < A \leq 1 \) and \( \alpha(1-B) \leq 1 - A \), then we have

\[
0 \leq \frac{(1-A)\alpha(1-B)}{(1-\alpha)(1-B)} < 1
\]
Also if $B = -1$ and $2\alpha \leq 1 - A$, then we obtain
\[
0 \leq \frac{(1 - A) - 2\alpha}{2(1 - \alpha)} < 1
\]
Thus we prove Lemma 2.1.

**Lemma 2.2 ([9]).** Let $f(z)$ be the function of the form (1.1). Then $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ if and only if
\[
\sum_{j=2}^{\infty} \phi_j(\mu_j , \eta_j , \alpha , A , B) |a_j| \leq A - B, \tag{2.2}
\]
where
\[
\phi_j(\mu_j , \eta_j , \alpha , A , B) = (1 + (1 + |B|\alpha))(\mu_j - \eta_j) + |B|\mu_j - A\eta_j \tag{2.3}
\]
\[
(\alpha \geq 0 , -1 \leq B < A \leq 1 , \mu_j > \eta_j \geq 0 , j \geq 2).
\]

**Lemma 2.3 ([27, p.3]).** Let $\alpha \geq 0$ and $-1 \leq B < A \leq 1$. If $\omega(z)$ is an analytic function with $\omega(0) = 1$, then we have
\[
\omega - \alpha|\omega - 1| < \frac{1 + Az}{1 + Bz} \iff \omega(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} < \frac{1 + Az}{1 + Bz} \quad (\phi \in \mathbb{R}). \tag{2.4}
\]

**Lemma 2.4 ([32]).** Let $\varphi(z)$ be analytic in $U$ satisfy $|\varphi(z)| \leq 1$ for $z \in U$. Then
\[
|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in U). \tag{2.5}
\]

**Lemma 2.5.** Let $f(z) \in U(\phi, \psi; \alpha, A, B)$. Then
\[
f(z) \in U(\phi, \psi; \alpha, A, B) \implies \begin{cases} f(z) \in U(\phi, \psi; \frac{1-A}{1-B}), & -1 < B < A \leq 1; \\ f(z) \in U(\phi, \psi; \frac{1-A}{2}), & B = -1. \end{cases} \tag{2.6}
\]

**Proof.** By virtue of (1.8), (1.10) and the properties of the domain $R_\alpha(A, B)$, we obtain
\[
\Re \left\{ \frac{(f \ast \phi)(z)}{(f \ast \psi)(z)} \right\} > \begin{cases} \frac{(1-B)\alpha+1-A}{(\alpha+1)(1-B)}, & -1 < B < A \leq 1; \\ \frac{2\alpha+1-A}{2(\alpha+1)}, & B = -1. \end{cases}
\]
Thus we prove Lemma 2.5.
Lemma 2.6. If $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ and
\[ \phi_j(\mu_j, \eta_j, A, B) \geq \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B) \quad (j \geq p + 1, p \geq 2), \] (2.7)
then
\[ \sum_{j=p+1}^{\infty} a_j \leq \frac{(A-B)- \sum_{j=2}^{p} \phi_j(\mu_j, \eta_j, A, B)a_j}{\phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B)} = A_j. \] (2.8)

Proof. In view of Lemma 2.2, we can write that
\[ \sum_{j=p+1}^{\infty} \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B)a_j \leq (A-B) - \sum_{j=2}^{p} \phi_j(\mu_j, \eta_j, A, B)a_j. \] (2.9)
Then from (2.7) and (2.9), we have
\[ \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B) \sum_{j=p+1}^{\infty} a_j \leq (A-B) - \sum_{j=2}^{p} \phi_j(\mu_j, \eta_j, A, B)a_j. \]
Thus we obtain
\[ \sum_{j=p+1}^{\infty} a_j \leq \frac{(A-B) - \sum_{j=2}^{p} \phi_j(\mu_j, \eta_j, A, B)a_j}{\phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B)}. \]
Also, we easily obtain

Lemma 2.7. If $f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B)$ and $\phi_j(\mu_j, \eta_j, A, B)$ defined by (2.3) satisfies (2.7), then
\[ \sum_{j=p+1}^{\infty} j a_j \leq \frac{(A-B) - \sum_{j=2}^{p} \phi_j(\mu_j, \eta_j, A, B)a_j}{j^{p+1} \phi_{p+1}(\mu_{p+1}, \eta_{p+1}, A, B)} = B_j. \] (2.10)

3. Main Results

Theorem 3.1. Let $f(z) \in U_\gamma(\phi, \psi; A, B)$. Then
\[ |a_2| \leq \begin{cases} \frac{2(1-(1-A)/(1-B))}{(\mu_2-\eta_2)(1-\alpha)}, & -1 < B < A \leq 1, \ \alpha(1-B) \leq 1 - A; \\ \frac{2(1-(1-A)/2)}{(\mu_2-\eta_2)(1-\alpha)}, & B = -1, \ 2\alpha \leq 1 - A. \end{cases} \] (3.1)
and
\[ |a_j| \leq \begin{cases} \frac{2(1-(1-A)/(1-B))}{(\mu_j-\eta_j)(1-\alpha)} \prod_{k=1}^{j-2} (1 + \frac{2(1-(1-A)/(1-B))}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)}), & -1 < B < A \leq 1, \ \alpha(1-B) \leq 1 - A; j \geq 3; \\ \frac{2(1-(1-A)/2)}{(\mu_j-\eta_j)(1-\alpha)} \prod_{k=1}^{j-2} (1 + \frac{2(1-(1-A)/2)}{(\mu_{k+1}-\eta_{k+1})(1-\alpha)}), & B = -1, \ 2\alpha \leq 1 - A; j \geq 3. \end{cases} \] (3.2)
**Proof.** Suppose that \( f \in U(\phi, \psi; \alpha, A, B) \). Then, by Lemma 2.1, we obtain

\[
\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} 
(1-A)-\alpha(1-B) \over (1-\alpha)(1-B), & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
\frac{(1-A) - 2\alpha}{2(1-\alpha)}, & B = -1, \; 2\alpha \leq 1-A.
\end{cases}
\]

Let us define the function \( p(z) \) by

\[
p(z) = \begin{cases} 
\frac{(1-\alpha)(f*\phi)(z) - (1-A)-\alpha(1-B)}{1-B}, & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
\frac{(1-\alpha)(f*\phi)(z) - (1-A) - 2\alpha}{2}, & B = -1, \; 2\alpha \leq 1-A.
\end{cases}
\]

Hence \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \) and \( \Re p(z) > 0 \; (z \in U) \). Let

\[
p(z) = 1 + c_1z + c_2z^2 + \ldots.
\]

So we get

\[
\frac{(f * \phi)(z)}{(f * \psi)(z)} = \begin{cases} 
1 + \frac{A-B}{(1-\alpha)(1-B)}(c_1z + c_2z^2 + \ldots), & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
1 + \frac{1+A}{2(1-\alpha)}(c_1z + c_2z^2 + \ldots), & B = -1, \; 2\alpha \leq 1-A.
\end{cases}
\]

or, equivalently,

\[
(f*\phi)(z) - (f*\psi)(z) = \begin{cases} 
\frac{A-B}{(1-\alpha)(1-B)}((f*\psi)(z))(c_1z + c_2z^2 + \ldots), & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
\frac{1+A}{2(1-\alpha)}((f*\psi)(z))(c_1z + c_2z^2 + \ldots), & B = -1, \; 2\alpha \leq 1-A,
\end{cases}
\]

which implies that

\[
(\mu_j - \eta_j)a_j = \begin{cases} 
\frac{A-B}{(1-\alpha)(1-B)}(c_{j-1} + a_2c_{j-2} + \ldots + a_{j-1}c_1), & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
\frac{1+A}{2(1-\alpha)}(c_{j-1} + a_2c_{j-2} + \ldots + a_{j-1}c_1), & B = -1, \; 2\alpha \leq 1-A.
\end{cases}
\]

Applying the coefficient estimates \( |c_j| \leq 2 \; (j \geq 1) \) (see [33]), we obtain

\[
|a_j| \leq \begin{cases} 
\frac{2(A-B)}{(\mu_j - \eta_j)(1-\alpha)(1-B)}(1 + a_2 + \ldots + a_{j-1}), & -1 < B < A \leq 1, \; \alpha(1-B) \leq 1-A; \\
\frac{(1+A)}{(\mu_j - \eta_j)(1-\alpha)}(1 + a_2 + \ldots + a_{j-1}), & B = -1, \; 2\alpha \leq 1-A.
\end{cases}
\]

(3.3)
For $j = 2$,

$$|a_2| \leq \left\{ \begin{array}{ll}
\frac{2(A-B)}{\mu_2 - \eta_2}(1-B), & -1 < B < A \leq 1, \; \alpha(1 - B) \leq 1 - A; \\
\frac{1 + A}{\mu_2 - \eta_2}(1 - \alpha), & B = -1, \; 2\alpha \leq 1 - A.
\end{array} \right.$$

which proves (3.1).

For $j = 3$,

$$|a_3| \leq \left\{ \begin{array}{ll}
\frac{2(A-B)}{(\mu_3 - \eta_3)(1-\alpha)(1-B)} \left(1 + \frac{2(A-B)}{(\mu_2 - \eta_2)(1-\alpha)(1-B)} \right), & -1 < B < A \leq 1, \; \alpha(1 - B) \leq 1 - A; \\
\frac{1 + A}{(\mu_3 - \eta_3)(1-\alpha)}, & B = -1, \; 2\alpha \leq 1 - A.
\end{array} \right.$$

Therefore (3.2) holds true for $j = 3$. Assume that (3.3) is true for $j = m$.

If $-1 < B < A \leq 1$ and $\alpha(1 - B) \leq 1 - A$, then we have

$$|a_{m+1}| \leq \frac{2(A-B)}{(\mu_{m+1} - \eta_{m+1})(1-\alpha)(1-B)} \left(1 + \frac{2(A-B)}{(\mu_2 - \eta_2)(1-\alpha)(1-B)} \right) + \ldots + \frac{2(A-B)}{(\mu_{m-1} - \eta_{m-1})(1-\alpha)(1-B)} \prod_{k=1}^{m-2} \left(1 + \frac{2(A-B)}{(\mu_{k+1} - \eta_{k+1})(1-\alpha)(1-B)} \right).$$

Also, if $B = -1$ and $2\alpha \leq 1 - A$, then we have

$$|a_{m+1}| \leq \frac{1 + A}{(\mu_{m+1} - \eta_{m+1})(1-\alpha)} \prod_{k=1}^{m-1} \left(1 + \frac{1 + A}{(\mu_{k+1} - \eta_{k+1})(1-\alpha)} \right).$$

So (3.2) is true for $j = m + 1$. Consequently, using the mathematical induction, we get that (3.2) holds true for all $j \geq 3$.

**Remark 3.1.** Taking $\alpha = 0, \mu_j = j, \eta_j = 1, B = -1$ and $A = 1 - 2\beta$ $(0 \leq \beta < 1)$ in Theorem 3.1, we obtain the results of Robertson [34].

**Theorem 3.2.** Let the function $f \in A$ and suppose that $f \in TU_\gamma(\phi, \psi; \alpha, A, B)$ $(0 \leq \alpha \neq 1)$. If $(f * \phi)(z)$ is majorized by $(f * \psi)(z)$ and $|(f * \phi)(z)| \leq |z(f * \psi)'(z)|$, then

$$|(f * \phi)'(z)| \leq |(f * \psi)'(z)| \; (|z| \leq r_0),$$

(3.4)
where \( r_0 = r_0(\alpha, A, B) \) is the smallest positive root of the equation

\[
\left[ \frac{A - B}{1 - \alpha} + |B| \right] r^3 - [1 + 2|B||r^2 - \left[ \frac{A - B}{1 - \alpha} + |B| + 2 \right] r + 1 = 0 \tag{3.5}
\]

\((z \in U; -1 \leq B < A \leq 1; 0 \leq \delta \leq r_0; \left| \frac{A - B}{1 - \alpha} + |B| \right| \delta \leq 1)\).

**Proof.** Suppose that \( f \in TU_\gamma(\phi, \psi; \alpha, A, B) \). Then, by Lemma 2.3, we obtain

\[
\frac{f(z) \ast \phi(z)}{f(z) \ast \psi(z)} (1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} < \frac{1 + Az}{1 + Bz},
\]

or, equivalently,

\[
\frac{f(z) \ast \phi(z)}{f(z) \ast \psi(z)} < \frac{1 + \left( \frac{A - \alpha Be^{-i\phi}}{1 - \alpha e^{-i\phi}} \right) z}{1 + Bz}, \tag{3.6}
\]

which holds true for all \( z \in U \).

We find from (3.6) that

\[
\frac{f(z) \ast \phi(z)}{f(z) \ast \psi(z)} = \frac{1 + \left( \frac{A - \alpha Be^{-i\phi}}{1 - \alpha e^{-i\phi}} \right) \omega(z)}{1 + B\omega(z)}, \tag{3.7}
\]

where \( \omega(z) = c_1 z + c_2 z^2 + \cdots \in W, W \) denotes the well known class of the bounded analytic functions in \( U \) and satisfies the conditions:

\[
\omega(0) = 0, \quad |\omega(z)| \leq |z| \quad (z \in U).
\]

From (3.7), we get

\[
|(f \ast \psi)(z)| \leq \frac{1 + |B||z|}{1 - \left( \frac{A - B}{1 - \alpha} + |B| \right) |z|} |(f \ast \phi)(z)|. \tag{3.8}
\]

Next, since \((f \ast \phi)(z)\) is majorized by \((f \ast \psi)(z)\) in \( U \), from (1.6), we have

\[
(f \ast \phi)(z) = \varphi(z)(f \ast \psi)(z).
\]

Differentiating it with respect to \( z \) and multiplying by \( z \), we get

\[
(f \ast \phi)'(z) = \varphi'(z)(f \ast \psi)(z) + \varphi(z)(f \ast \psi)'(z). \tag{3.9}
\]

Thus, by Lemma 2.4, (3.8) and (3.9), we get

\[
|(f \ast \phi)'(z)| \leq \left[ |\varphi(z)| + \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \cdot \frac{1 + |B||z|}{1 - \left( \frac{A - B}{1 - \alpha} + |B| \right) |z|} \right] |(f \ast \psi)'(z)|, \tag{3.10}
\]

which upon setting

\[
|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)
\]
leads us to the inequality
\[
|(f * \phi)'(z)| \leq \left[ \frac{\psi(\rho)}{(1 - r^2)(1 - (\frac{A - B}{1 - \alpha}) + |B|r)} \right] |(f * \psi)'(z)|,
\]
where
\[
\psi(\rho) = -r(1 + |B|r\rho^2 + (1 - r^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|r \right] + r + (1 + |B|r\rho^2 + (1 - r^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|r \right] ) \rho + r(1 + |B|r)^2
\]
takes its maximum value at \( \rho = 1 \), with \( r_0 = r_0(\alpha, A, B) \), where \( r_0 = r_0(\alpha, A, B) \) is the smallest positive root of (3.5). Furthermore, if \( 0 \leq \delta \leq r_0(\alpha, A, B) \), then the function \( \psi(\rho) \) defined by
\[
\psi(\rho) = -\delta(1 + |B|\delta\rho^2 + (1 - \delta^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|\delta \right] ) + (1 + |B|\delta\rho^2 + (1 - \delta^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|\delta \right] ) \rho + (1 + |B|\delta\rho^2 + (1 - \delta^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|\delta \right] ) \rho + (1 + |B|\delta)^2
\]
is an increasing function on the interval \( 0 \leq \rho \leq 1 \), so that
\[
\psi(\rho) \leq \psi(1) = (1 - \delta^2) \left[ 1 - (\frac{A - B}{1 - \alpha}) + |B|\delta \right] \quad (0 \leq \rho \leq 1; 0 \leq \delta \leq r_0(\alpha, A, B)).
\]
Hence, upon setting \( \rho = 1 \) in (3.13), we conclude that (3.4) of Theorem 3.2 holds true for \( |z| \leq r_0 = r_0(\alpha, A, B) \), which completes the proof of Theorem 3.2.

**Theorem 3.3.** Let the function \( f \in A \) and suppose that \( f \in TU(\phi, \psi; 1, A, B) \) (\( \alpha = 1 \)). If
\[
(f * \phi)(z) \text{ is majorized by } (f * \psi)(z) \text{ and } |(f * \phi)(z)| \leq |(f * \psi)'(z)|,
\]
then
\[
|(f * \phi)'(z)| \leq |(f * \psi)'(z)|
\]
where \( r_0 = r_0(A, B) \) is the smallest positive root of the equation
\[
\chi(A, B) = \begin{cases} 
\frac{(1 - A)}{2}r^2 - (3 + \frac{1 - A}{2})r + 1 = 0, & -1 < B < A \leq 1; \\
\frac{(1 - A)}{2}r^2 - (3 + \frac{1 - A}{2})r + 1 = 0, & B = -1.
\end{cases}
\]
\[
(z \in \mathbb{U}; -1 \leq B < A \leq 1; r_0 \geq 0).
\]

**Proof.** Suppose that \( f \in TU(\phi, \psi; 1, A, B) \). Then, by Lemma 2.5, we obtain
\[
\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \begin{cases} 
\frac{(1 - B + 1 - A)}{2(1 - B)}, & -1 < B < A \leq 1; \\
\frac{3 - A}{4}, & B = -1,
\end{cases}
\]
or, equivalently,
\[
\frac{(f * \phi)(z)}{(f * \psi)(z)} > \begin{cases} 
\frac{1 + (\frac{(1 - B) + 1 - A)}{1 - z}}{1 - z}, & -1 < B < A \leq 1; \\
\frac{1 + (\frac{3 - A)}{1 - z}}{1 - z}, & B = -1,
\end{cases}
\]
(3.17)
which holds true for all \( z \in \mathbb{U} \).

We find from (3.17) that

\[
\frac{(f * \phi)(z)}{(f * \psi)(z)} = \begin{cases} 
\frac{1 + (1 - B) + 1 - A + \omega(z)}{1 - \omega(z)}, & -1 < B < A \leq 1; \\
\frac{1 + 1 - \omega(z)}{1 - \omega(z)}, & B = -1,
\end{cases}
\]

where \( \omega(z) = c_1 z + c_2 z^2 + \cdots \in \mathcal{W} \). The remainder of Theorem 3.3 is analogous to the proof of Theorem 3.2, so we omit the details involved.

**Theorem 3.4.** If \( f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B) \) and \( \phi_j(\mu_j, \eta_j, \alpha, A, B) \) defined by (2.3) satisfies (2.7), then for \(|z| = r < 1\)

\[
r - r^2 \sum_{j=2}^{p} |a_j| - |A_j|r^{p+1} \leq |f(z)| \leq r + r^2 \sum_{j=2}^{p} |a_j| + |A_j|r^{p+1}
\]

and

\[
1 - r \sum_{j=2}^{p} |a_j| - |B_j|r^p \leq |f'(z)| \leq 1 + r \sum_{j=2}^{p} |a_j| + |B_j|r^p.
\]

where \( A_j \) and \( B_j \) are given by (2.8) and (2.10), respectively.

**Proof.** Let \( f(z) \) be given by (1.1). For \(|z| = r < 1\), by using Lemma 2.6, we have

\[
|f(z)| \leq |z| + \sum_{j=2}^{p} |a_j||z|^j + \sum_{j=p+1}^{\infty} |a_j||z|^j
\]

\[
\leq |z| + |z|^2 \sum_{j=2}^{p} |a_j| + |z|^p \sum_{j=p+1}^{\infty} |a_j|
\]

\[
\leq r + r^2 \sum_{j=2}^{p} |a_j| + |A_j|r^{p+1}
\]

and

\[
|f(z)| \geq |z| - \sum_{j=2}^{p} |a_j||z|^j - \sum_{j=p+1}^{\infty} |a_j||z|^j
\]

\[
\geq |z| - |z|^2 \sum_{j=2}^{p} |a_j| - |z|^p \sum_{j=p+1}^{\infty} |a_j|
\]

\[
\geq r - r^2 \sum_{j=2}^{p} |a_j| - |A_j|r^{p+1}
\]

Furthermore for \(|z| = r < 1\), by using Lemma 2.7, we obtain

\[
|f'(z)| \leq 1 + \sum_{j=2}^{p} j|a_j||z|^{j-1} + \sum_{j=p+1}^{\infty} j|a_j||z|^{j-1}
\]

\[
\leq 1 + |z| \sum_{j=2}^{p} j|a_j| + |z|^p \sum_{j=p+1}^{\infty} j|a_j|
\]

\[
\leq 1 + r \sum_{j=2}^{p} |a_j| + |B_j|r^p
\]

and
\[
|f'(z)| \geq 1 - \sum_{j=2}^{p} j|a_j||z|^{j-1} - \sum_{j=p+1}^{\infty} j|a_j||z|^{j-1}
\geq 1 - |z| \sum_{j=2}^{p} j|a_j| - |z|^p \sum_{j=p+1}^{\infty} j|a_j|
\geq 1 - r \sum_{j=2}^{p} |a_j| - |B_j|r^p,
\]
thus we have (3.19).

**Theorem 3.5.** Let the function \( f(z) \) defined by (1.1) satisfy (1.4). We define

\[
f_1(z) = z, \quad f_j(z) = z - \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma} z^j \quad (j = 2, 3, \cdots),
\]
where \( \phi_j(\mu_j, \eta_j, \alpha, A, B) \) is given by (2.3). Then \( f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z),
\]
where \( \lambda_j > 0 \) and \( \sum_{j=1}^{\infty} \lambda_j = 1 \).

**Proof.** Suppose that

\[
f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = z - \sum_{j=1}^{\infty} \lambda_j \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma} z^j.
\]

Then

\[
\sum_{j=1}^{\infty} \phi_j(\mu_j, \eta_j, \alpha, A, B) |\lambda_j| \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} e^{i(1-j)\gamma} = (A - B) \sum_{j=2}^{\infty} \lambda_j
\]

\[
= (A - B)(1 - \lambda_1)
\]

\[
< A - B.
\]

By Lemma 2.2, we have \( f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B) \).

Conversely, suppose that \( f(z) \in TU_\gamma(\phi, \psi; \alpha, A, B) \). Since

\[
|a_j| \leq \frac{A-B}{\phi_j(\mu_j, \eta_j, \alpha, A, B)} \quad (j = 2, 3, \cdots),
\]
we may set

\[
\lambda_j = \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{(A-B)|e^{i(1-j)\gamma}|} |a_j| \quad (j = 2, 3, \cdots)
\]

and

\[
\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j.
\]

Then

\[
f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z).
\]
This completes the proof of Theorem 3.5.

**Theorem 3.6.** Let the function $f(z)$ defined by (1.1) be in the class $TU(\phi, \psi; \alpha, A, B)$, and $\phi_j(\mu_j, \eta_j, \alpha, A, B)$ be given by (2.3). Then, we have

1. The function $f(z)$ is close-to-convex of $\mu$ ($0 \leq \mu < 1$) in $|z| < r_1$ where
   
   $$r_1 = \inf_j \left\{ \left(1 - \mu\right) \phi_j(\mu_j, \eta_j, \alpha, A, B) \right\}^{\frac{1}{j-1}} (j \geq 2) \tag{3.22}$$

2. The function $f(z)$ is starlike of $\eta$ ($0 \leq \eta < 1$) in $|z| < r_2$ where
   
   $$r_2 = \inf_j \left\{ \left(1 - \eta\right) \phi_j(\mu_j, \eta_j, \alpha, A, B) \right\}^{\frac{1}{j-1}} (j \geq 2) \tag{3.23}$$

3. The function $f(z)$ is convex of $\mu$ ($0 \leq \xi < 1$) in $|z| < r_3$ where
   
   $$r_3 = \inf_j \left\{ \left(1 - \xi\right) \phi_j(\mu_j, \eta_j, \alpha, A, B) \right\}^{\frac{1}{j-1}} (j \geq 2) \tag{3.24}$$

**Proof.** (1) We must show that $|f'(z) - 1| < 1 - \mu$ for $|z| < r_1$. We have

$$|f'(z) - 1| \leq \sum_{j=2}^{\infty} j|a_j||z|^{j-1}.$$ 

Thus $|f'(z) - 1| < 1 - \mu$ if

$$\sum_{j=2}^{\infty} \frac{j}{1-\mu} |a_j||z|^{j-1} \leq 1. \tag{3.25}$$

By Lemma 2.2, we have

$$\sum_{j=2}^{\infty} \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{A - B} |a_j| \leq 1 \tag{3.26}$$

Hence (3.25) will be true if

$$\frac{j|z|^{j-1}}{1-\mu} \leq \frac{\phi_j(\mu_j, \eta_j, \alpha, A, B)}{A - B},$$

or, if

$$|z| \leq \left\{ \frac{(1-\mu)\phi_j(\mu_j, \eta_j, \alpha, A, B)}{j(A - B)} \right\}^{\frac{1}{j-1}} (j \geq 2), \tag{3.27}$$

which follows from (3.22). Similarly, we can prove (2) and (3). This completes the proof of Theorem 3.6.

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References


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