Symmetric bi- f -Derivations of Incline Algebras

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Abstract

In this paper, we introduce the concept of a symmetric bi- f -derivation in incline algebras and give some properties of incline algebras. Also, we characterize $Ker_D(K)$ and $F_a(K)$ by symmetric bi- f -derivations in incline algebras.

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1 Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline algebras in their book. Some authors studied incline algebras and application. N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In this paper, we introduce the concept of a symmetric bi- f -derivation in incline algebra and give some properties of incline algebras. Also, we characterize $Ker_D(K)$ and $F_a(K)$ by symmetric bi- f -derivations in incline algebras.

2 Incline algebras

An *incline algebra* is a set K with two binary operations denoted by “ $+$ ” and “ $*$ ” satisfying the following axioms:

- (K1) $x + y = y + x$,
- (K2) $x + (y + z) = (x + y) + z$,
- (K3) $x * (y * z) = (x * y) * z$,
- (K4) $x * (y + z) = (x * y) + (x * z)$,
- (K5) $(y + z) * x = (y * x) + (z * x)$,
- (K6) $x + x = x$,
- (K7) $x + (x * y) = x$,
- (K8) $y + (x * y) = y$

for all $x, y, z \in K$. For convenience, we pronounce “ $+$ ” (resp. “ $*$ ”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x * x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that “ $\leq$ ” is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K , the following properties hold.

- (K9) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$,
- (K10) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,
- (K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x * a \leq y * b$ for all $x, y, a, b \in K$.

Furthermore, an incline algebra K is said to be *commutative* if $x * y = y * x$ for all $x, y \in K$. A map f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is said to be an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element “0” in an incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 = 0 * x$ for any $x \in K$. A non-zero element “1” is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such that $a * b = 0$ (resp. $b * a = 0$). A zero divisor

is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a homomorphism of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. A mapping f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$. A subincline I of an incline algebra K is said to be *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$. Let K be an incline algebra. An element $a \in K$ is called a *additively left cancellative* if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. An element $a \in K$ is called a *additively right cancellative* if for all $a, b \in K$, $b + a = c + a \Rightarrow b = c$. It is said to be *additively cancellative* if it is both left and right cancellative. If every element of K is additively left cancellative, it is called *additively left cancellative*. If every element of K is additively right cancellative, it is called *additively right cancellative*.

Definition 2.1. Let K be an incline algebra. A mapping $D(., .) : K \times K \rightarrow K$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.

Definition 2.2. Let K be an incline algebra and $x \in K$. A mapping $d(x) = D(x, x)$ is called *trace* of $D(., .)$, where $D(., .) : K \times K \rightarrow K$ is a symmetric mapping.

Definition 2.3. Let K be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric mapping. We call D a symmetric bi-derivation on K if it satisfies the following condition

$$D(x * y, z) = (D(x, z) * y) + (x * D(y, z))$$

for all $x, y, z \in K$.

3 Symmetric bi- f -derivations of incline algebras

In what follows, let K denote an incline algebra with a zero-element unless otherwise specified.

Definition 3.1. Let K be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric mapping. We call D a symmetric bi- f -derivation on K if there exists a function $f : K \rightarrow K$ such that

$$D(x * y, z) = (D(x, z) * f(y)) + (f(x) * D(y, z))$$

for all $x, y, z \in K$.

Obviously, a symmetric bi- f -derivation D on K satisfies the relation

$$D(x, y * z) = (D(x, y) * f(z)) + (f(y) * D(x, z))$$

for all $x, y, z \in K$.

Example 3.2. Let K be a commutative incline algebra and $a \in K$. Define a mapping on K by $D(x, y) = f(x) * f(y)$ where $f : K \rightarrow K$ satisfies $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. Then we can see that D is a symmetric bi- f -derivation on K .

Example 3.3. Let K be a commutative incline algebra and $a \in K$. Define a mapping on K by $D(x, y) = (f(x) * f(y)) * a$ where $f : K \rightarrow K$ satisfies $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. Then we can see that D is a symmetric bi- f -derivation on K .

Example 3.4. Let $K = \{0, a, b, 1\}$ be a set in which “+” and “*” is defined by

+	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

*	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $D : K \times K \rightarrow K$ by

$$D(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ 0 & \text{if } (x, y) = (0, a), (a, 0) \\ 0 & \text{if } (x, y) = (0, b), (b, 0) \\ 0 & \text{if } (x, y) = (0, 1), (1, 0) \\ 0 & \text{if } (x, y) = (a, a) \\ b & \text{if } (x, y) = (b, b) \\ b & \text{if } (x, y) = (1, 1) \\ 0 & \text{if } (x, y) = (a, b) \text{ or } (b, a) \\ 0 & \text{if } (x, y) = (a, 1) \text{ or } (1, a) \\ b & \text{if } (x, y) = (b, 1) \text{ or } (1, b) \end{cases}$$

and $f : K \rightarrow K$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } x = a \\ b & \text{if } x = b \\ b & \text{if } x = 1 \end{cases}$$

Then it is easily checked that D is a symmetric bi- f -derivation of an incline algebra K . But D is not a symmetric bi-derivation since

$$0 = D(b*a, b) = D(a, b) \neq (D(b, b)*a) + (b*D(a, b)) = (b*a) + (b*0) = a + 0 = a.$$

Proposition 3.5. Let K be an incline algebra and let D be a symmetric bi- f -derivation on K . Then the following identities hold for all $x, y, z \in K$,

- (i) $D(x * y, z) \leq f(x) + f(y)$, for all $x, y, z \in K$,
- (ii) $D(x * y, z) \leq D(x, z) + D(y, z)$, for all $x, y, z \in K$.

Proof. (i) Let $x, y, z \in K$. By using (K9), we have $D(x, z) * f(y) \leq f(y)$ and $f(x) * D(y, z) \leq f(x)$. Then by using (K11), we obtain $D(x, z) * f(y) + f(x) * D(y, z) \leq f(x) + f(y)$. Hence $D(x * y, z) \leq f(x) + f(y)$.

(ii) Let $x, y, z \in K$. By using (K9), we have $D(x, z) * f(y) \leq D(x, z)$ and $f(x) * D(y, z) \leq D(y, z)$. Then by using (K11), we obtain $D(x, z) * f(y) + f(x) * D(y, z) \leq D(x, z) + D(y, z)$. Hence $D(x * y, z) \leq D(x, z) + D(y, z)$.

Proposition 3.6. Let K be an incline algebra and let D be a symmetric bi- f -derivation on K . If $x \leq y$ and f an isotone mapping, we have $D(x * y, z) \leq f(y)$.

Proof. Let $x \leq y$. Then we have $f(x) \leq f(y)$. By using (K11) and (K9), $f(x) * D(y, z) \leq f(y) * D(y, z) \leq f(y)$. Also, by using (K9), we get $D(x, z) \leq f(y) \leq f(y)$. Hence we have $D(x * y, z) = D(x, z) * f(y) + f(x) * D(y, z) \leq f(y) + f(y) = f(y)$.

Proposition 3.7. Let K be an incline algebra. If K is a distributive lattice, we have $D(x, y) \leq f(x)$ and $D(x, y) \leq f(y)$ for all $x, y \in K$.

Proof. Let K be a distributive lattice. Then $D(x, y) = D(x * x, y) = D(x, y) * f(x) + (f(x) * D(x, y))$, and so $D(x, y) + f(x) = (D(x, y) * f(x) + (f(x) * D(x, y))) + f(x) = (D(x, y) * f(x) + D(x, y) * f(x)) + f(x) = (D(x, y) * f(x)) + f(x)$ for all $x, y \in K$. By using (K8), we get $D(x, y) + f(x) = f(x)$. Hence we obtain $D(x, y) \leq f(x)$. Similarly, we have $D(x, y) \leq f(y)$.

Proposition 3.8. Let K be an incline algebra. If K is a distributive lattice, we have $d(x) \leq f(x)$ for all $x \in K$.

Proof. Let K be a distributive lattice. Then we have

$$\begin{aligned} d(x) &= D(x, x) = D(x * x, x) = D(x, x) * f(x) + f(x) * D(x, x) \\ &= D(x, x) * f(x) \leq f(x) \end{aligned}$$

for all $x \in K$.

Proposition 3.9. Let K be an incline algebra and let d be a trace of symmetric bi- f -derivation D of K . If $f(0) = 0$, then $d(0) = 0$.

Proof. Let $x \in K$. Then we have $d(0) = D(0, 0) = D(x * 0, 0) = D(x, 0) * f(0) + f(x) * D(0, 0) = 0 + f(x) * d(0)$. If we take $x = 0$, we get $d(0) = 0 + 0 = 0$.

Theorem 3.10. Let K be an integral incline and let D be a symmetric bi- f -derivation of K where f is a function satisfying $f(1) = 1$ and $a \in K$. Then for all $x, y \in K$, we have

- (i) $a * D(x, y) = 0$ implies $a = 0$ or $D = 0$,
- (ii) $D(x, y) * a = 0$ implies $a = 0$ or $D = 0$.

Proof. (i) Let $a * D(x, y) = 0$ for all $x \in K$. If we replace x by $x * z$ for $z \in K$, we obtain

$$\begin{aligned} 0 &= a * D(x, y) = a * D(x * z, y) = a * (D(x, y) * f(z) + (f(x) * D(z, y))) \\ &= (a * (D(x, y) * f(z))) + (a * (f(x) * D(z, y))) \\ &= a * (f(x) * D(z, y)). \end{aligned}$$

In this equation, by taking $x = 1$, we have $a * D(z, y) = 0$. Since K is an integral incline, that is, it has no zero-divisors, we have $a = 0$ or $D(z, y) = 0$ for all $y, z \in K$. Hence we get $a = 0$ or $D = 0$. (ii) Similarly, we can prove (ii).

Theorem 3.11. Let K be an incline algebra with a multiplicative identity and let D be a symmetric bi- f -derivation of K where f is a function satisfying $f(1) = 1$. Then the following identities hold for all $x, y \in K$.

- (i) $f(x) * D(1, y) \leq D(x, y)$,
- (ii) If d is the trace of K and $d(1) = 1$, we have $f(x) \leq D(x, 1)$.

Proof. (i) Let $x, y \in K$. Then we have $D(x, y) = D(x * 1, y) = D(x, y) * f(1) + f(x) * D(1, y) = D(x, y) * 1 + f(x) * D(1, y) = D(x, y) + f(x) * D(1, y)$. Therefore, $f(x) * D(1, y) \leq D(x, y)$. (ii) It can be derived from (i).

Proposition 3.12. Let K be an incline algebra. If $D(x, y) = f(x)$ and $D(w, y) = f(w)$, we have $D(x * w, y) = f(x) * f(w)$.

Proof. Let $x, y, w \in K$. Then we have

$$\begin{aligned} D(x * w, y) &= D(x, y) * f(w) + f(x) * D(w, y) = f(x) * f(w) + f(x) * f(w) \\ &= f(x) * f(w). \end{aligned}$$

Definition 3.13. Let K be an incline algebra. If $D : K \times K \rightarrow K$ be a symmetric mapping. We call D a *joinitive mapping* if it satisfies

$$D(x + y, z) = D(x, z) + D(y, z)$$

for all $x, y, z \in K$.

Proposition 3.14. Let K be an incline algebra and let d be a trace of joinitive symmetric bi- f -derivation D of K . Then the following identities hold for all $x, y \in K$,

- (i) $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$,
- (ii) $D(x * y, x) \leq d(x)$.

Proof. (i) Let $x, y \in K$. Then we have

$$\begin{aligned} d(x + y) &= D(x + y, x + y) = D(x, x + y) + D(y, x + y) \\ &= D(x, x) + D(x, y) + D(y, x) + D(y, y) \\ &= D(x, x) + D(y, y) + D(x, y). \end{aligned}$$

Hence we get $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$.

(ii) Let $x, y \in K$. It follows from (K7) that $d(x) = D(x, x) = D(x + (x * y), x) = D(x, x) + D(x * y, x)$, which implies $D(x * y, x) \leq d(x)$.

Proposition 3.15. Let K be an incline algebra and let D be a trace of joinitive symmetric bi- f -derivation D of K . Then $D(x * y, y) \leq D(x, y)$ for all $x, y \in K$.

Proof. Let $x, y \in K$. Then we have

$$D(x, y) = D(x + x * y, y) = D(x, y) + D(x * y, y),$$

which implies $D(x * y, y) \leq D(x, y)$.

Definition 3.16. Let K be an incline algebra and let D be a symmetric bi- f -derivation on K . If $x \leq w$ implies $D(x, y) \leq D(w, y)$, D is called an *isotone symmetric bi- f -derivation* for all $x, y, w \in K$.

Theorem 3.17. Let K be an incline algebra and let D be a jointive symmetric bi- f -derivation D of K . Then D is an isotone symmetric bi- f -derivation of K .

Proof. Let x and w be such that $x \leq w$. Then $x + w = w$, and so

$$D(w, y) = D(w + x, y) = D(w, y) + D(x, y)$$

This implies that $D(x, y) \leq D(w, y)$. This completes the proof.

Let D be a symmetric bi- f -derivation of K . Fix $a \in K$ and define a set $F_a(K)$ by

$$F_a(K) := \{x \in K \mid D(x, a) = f(x)\}$$

for all $x \in K$.

Proposition 3.18. Let D be a jointive symmetric bi- f -derivation and let f be an endomorphism on K . Then $F_a(K)$ is a subincline of K .

Proof. Let $x, y \in F_a(K)$. Then we have $D(x, a) = f(x)$ and $D(y, a) = f(y)$, and so

$$\begin{aligned} D(x * y, a) &= D(x, a) * f(y) + f(x) * D(y, a) \\ &= f(x) * f(y) + f(x) * f(y) \\ &= f(x) * f(y) = f(x * y). \end{aligned}$$

Hence we get $x * y \in F_a(K)$. Also, we get $D(x + y, a) = D(x, a) + D(y, a) = f(x) + f(y) = f(x + y)$, and so $x + y \in F_a(K)$. This completes the proof.

Proposition 3.19. Let K be additively right cancellative and let D be a jointive symmetric bi- f -derivation and let f be an endomorphism on K . Then $F_a(K)$ is a k -ideal of K .

Proof. Let $x + y \in F_a(K)$ and $y \in F_a(K)$. Then we have $f(x) + f(y) = f(x + y) = D(x + y, a) = D(x, a) + D(y, a) = D(x, a) + f(y)$. Since K is additively right cancellative, we have $f(x) = D(x, a)$, which implies $x \in F_a(K)$. This completes the proof.

Definition 3.20. Let K be an incline algebra. If $D : K \times K \rightarrow K$ be a symmetric mapping. Define a set $Ker_D(K)$ by

$$Ker_D(K) = \{x \in K \mid D(0, x) = 0\}.$$

Proposition 3.21. Let K be an incline algebra and let D be a jointive symmetric bi- f -derivation D . If $x \leq y$ and $y \in \text{Ker}_D(K)$, then we have $x \in \text{Ker}_D(K)$.

Proof. Let $x \leq y$ and $y \in \text{Ker}_D(K)$. Then we get $x + y = y$ and $D(0, y) = 0$. Hence we get

$$\begin{aligned} 0 &= D(0, y) = D(0, x + y) \\ &= D(0, x) + D(0, y) \\ &= D(0, x) + 0 = D(0, x), \end{aligned}$$

which implies $x \in \text{Ker}_D(K)$. This completes the proof.

Proposition 3.22. Let K be an incline algebra and let D be a jointive symmetric bi- f -derivation of K . Then $\text{Ker}_D(K)$ is a subincline of K .

Proof. Let $x, y \in \text{Ker}_D(K)$. Then $D(0, x) = D(0, y) = 0$, and so

$$\begin{aligned} D(0, x * y) &= D(x * y, 0) = D(x, 0) * f(y) + f(x) * D(y, 0) \\ &= 0 * f(y) + f(x) * 0 = 0 + 0 \\ &= 0, \end{aligned}$$

which implies $x * y \in \text{Ker}_D(K)$. Now $D(x + y, 0) = D(x, 0) + D(y, 0) = 0 + 0 = 0$. Hence $x + y \in \text{Ker}_D(K)$. This completes the proof.

Theorem 3.23. Let D be a jointive symmetric bi- f -derivation of K . Then $\text{Ker}_D(K)$ is an ideal of K .

Proof. By Proposition 3.23, $\text{Ker}_D(K)$ is a subincline of K . Let $x \leq y$ and $y \in \text{Ker}_D(K)$. Then $x + y = y$ and $D(0, y) = 0$. Thus

$$\begin{aligned} 0 &= D(0, y) = D(0, x + y) = D(0, x) + D(0, y) \\ &= D(0, x) + 0 = D(0, x), \end{aligned}$$

which implies $x \in \text{Ker}_D(K)$.

Proposition 3.24. Let K be an incline algebra and let D be a symmetric bi- f -derivation of K . If there exists $a \in K$ such that $D(x, z) * a = 0$ for all $x, z \in K$, then $z \in \text{Ker}_D(K)$.

Proof. Let D be a symmetric bi- f -derivation of K and $a \in K$. Then we have for all $x, z \in K$,

$$\begin{aligned} D(0, z) &= D(0 * a, z) = D(0, z) * a + 0 * D(a, z) \\ &= 0 + 0 = 0, \end{aligned}$$

which implies $z \in \text{Ker}_D(K)$.

Corollary 3.25. Let K be an incline algebra and let D be a symmetric bi- f -derivation of K . If there exists $a \in K$ such that $D(x, z) * a = 0$ for all $x, z \in K$, then $d(0) = D(0, 0) = 0$.

Theorem 3.26. Let K be a commutative integral incline algebra. Suppose that there exist two jointive symmetric bi- f -derivations D_1 and D_2 such that $D_1(d_2(x), x) = 0$ for all $x \in K$ where d_1, d_2 denote the trace of D_1 and D_2 , respectively. If f is a nonzero function on K , then $d_1 = 0$.

Proof. Let $D_1(d_2(x), x) = 0$ where d_1, d_2 denote the trace of symmetric bi-derivations D_1 and D_2 , respectively. Then by using (K7), we have

$$\begin{aligned} D_1(d_2(x) + (d_2(x) * x), x) &= D_1(d_2(x), x) + D_1(d_2(x) * x, x) \\ &= D_1(d_2(x), x) * f(x) + f(d_2(x)) * D_1(x, x) \\ &= f(d_2(x)) * d_1(x). \end{aligned}$$

Since K has no zero divisors, we have $f(d_2(x)) = 0$ or $d_1(x) = 0$. But since f is a nonzero function on K , we get $d_1(x) = 0$ for all $x \in K$.

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