Symmetric bi-$f$-Derivations of Incline Algebras

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Abstract

In this paper, we introduce the concept of a symmetric bi-$f$-derivation
in incline algebras and give some properties of incline algebras. Also,
we characterize $\text{Ker}_D(K)$ and $F_a(K)$ by symmetric bi-$f$-derivations in
incline algebras.

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tone, $\text{Ker}_D(K)$

1 Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline
algebras in their book. Some authors studied incline algebras and application.
N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In
this paper, we introduce the concept of a symmetric bi-$f$-derivation in incline
algebra and give some properties of incline algebras. Also, we characterize
$\text{Ker}_D(K)$ and $F_a(K)$ by symmetric bi-$f$-derivations in incline algebras.
2 Incline algebras

An incline algebra is a set $K$ with two binary operations denoted by “+” and “∗” satisfying the following axioms:

\begin{enumerate}
\item[(K1)] $x + y = y + x$,
\item[(K2)] $x + (y + z) = (x + y) + z$,
\item[(K3)] $x ∗ (y ∗ z) = (x ∗ y) ∗ z$,
\item[(K4)] $x ∗ (y + z) = (x ∗ y) + (x ∗ z)$,
\item[(K5)] $(y + z) ∗ x = (y ∗ x) + (z ∗ x)$,
\item[(K6)] $x + x = x$,
\item[(K7)] $x + (x ∗ y) = x$,
\item[(K8)] $y + (x ∗ y) = y$
\end{enumerate}

for all $x, y, z \in K$. For convenience, we pronounce “+” (resp. “∗”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x ∗ x = x$ for all $x \in K$. Note that $x \leq y \iff x + y = y$ for all $x, y \in K$. It is easy to see that “≤” is a partial order on $K$ and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$. In an incline algebra $K$, the following properties hold.

\begin{enumerate}
\item[(K9)] $x ∗ y \leq x$ and $y ∗ x \leq x$ for all $x, y \in K$,
\item[(K10)] $y \leq z$ implies $x ∗ y \leq x ∗ z$ and $y ∗ x \leq z ∗ x$, for all $x, y, z \in K$,
\item[(K11)] If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x ∗ a \leq y ∗ b$ for all $x, y, a, b \in K$.
\end{enumerate}

Furthermore, an incline algebra $K$ is said to be commutative if $x ∗ y = y ∗ x$ for all $x, y \in K$. A map $f$ is isotone if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A subincline of an incline algebra $K$ is a non-empty subset $M$ of $K$ which is closed under the addition and multiplication. A subincline $M$ is said to be an ideal if $x \in M$ and $y \leq x$ then $y \in M$. An element “0” in an incline algebra $K$ is a zero element if $x + 0 = x = 0 + x$ and $x ∗ 0 = 0 = 0 ∗ x$ for any $x \in K$. An non-zero element “1” is called a multiplicative identity if $x ∗ 1 = 1 ∗ x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a left (resp. right) zero divisor if there exists a non-zero $b \in K$ such that $a ∗ b = 0$ (resp. $b ∗ a = 0$). A zero divisor
is an element of $K$ which is both a left zero divisor and a right zero divisor. An incline algebra $K$ with multiplicative identity $1$ and zero element $0$ is called an integral incline if it has no zero divisors. By a homomorphism of inclines, we mean a mapping $f$ from an incline algebra $K$ into an incline algebra $L$ such that $f(x + y) = f(x) + f(y)$ and $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. A mapping $f$ is isotone if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$. A subincline $I$ of an incline algebra $K$ is said to be $k$-ideal if $x + y \in I$ and $y \in I$, then $x \in I$. Let $K$ be an incline algebra. An element $a \in K$ is called a additively left cancellative if for all $a, b \in K, a + b = a + c \Rightarrow b = c$. An element $a \in K$ is called a additively right cancellative if for all $a, b \in K, b + a = c + a \Rightarrow b = c$. It is said to be additively cancellative if it is both left and right cancellative. If every element of $K$ is additively left cancellative, it is called additively left cancellative. If every element of $K$ is additively right cancellative, it is called additively right cancellative.

**Definition 2.1.** Let $K$ be an incline algebra. A mapping $D(., .) : K \times K \rightarrow K$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.

**Definition 2.2.** Let $K$ be an incline algebra and $x \in K$. A mapping $d(x) = D(x, x)$ is called trace of $D(., .)$, where $D(., .) : K \times K \rightarrow K$ is a symmetric mapping.

**Definition 2.3.** Let $K$ be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric mapping. We call $D$ a symmetric bi-derivation on $K$ if it satisfies the following condition

$$D(x \ast y, z) = (D(x, z) \ast y) + (x \ast D(y, z))$$

for all $x, y, z \in K$.

### 3 Symmetric bi-$f$-derivations of incline algebras

In what follows, let $K$ denote an incline algebra with a zero-element unless otherwise specified.

**Definition 3.1.** Let $K$ be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric mapping. We call $D$ a symmetric bi-$f$-derivation on $K$ if there exists a function $f : K \rightarrow K$ such that

$$D(x \ast y, z) = (D(x, z) \ast f(y)) + (f(x) \ast D(y, z))$$
for all $x, y, z \in K$.

Obviously, a symmetric bi-$f$-derivation $D$ on $K$ satisfies the relation

$$D(x, y \ast z) = (D(x, y) \ast f(z)) + (f(y) \ast D(x, z))$$

for all $x, y, z \in K$.

**Example 3.2.** Let $K$ be a commutative incline algebra and $a \in K$. Define a mapping on $K$ by $D(x, y) = f(x) \ast f(y)$ where $f : K \rightarrow K$ satisfies $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. Then we can see that $D$ is a symmetric bi-$f$-derivation on $K$.

**Example 3.3.** Let $K$ be a commutative incline algebra and $a \in K$. Define a mapping on $K$ by $D(x, y) = (f(x) \ast f(y)) \ast a$ where $f : K \rightarrow K$ satisfies $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in K$. Then we can see that $D$ is a symmetric bi-$f$-derivation on $K$.

**Example 3.4.** Let $K = \{0, a, b, 1\}$ be a set in which “$+$” and “$\ast$” is defined by

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Then it is easy to check that $(K, +, \ast)$ is an incline algebra. Define a map $D : K \times K \rightarrow K$ by

$$D(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0) \\
0 & \text{if } (x, y) = (0, a), (a, 0) \\
0 & \text{if } (x, y) = (0, b), (b, 0) \\
0 & \text{if } (x, y) = (0, 1), (1, 0) \\
0 & \text{if } (x, y) = (a, a) \\
b & \text{if } (x, y) = (b, b) \\
b & \text{if } (x, y) = (1, 1) \\
0 & \text{if } (x, y) = (a, b) \text{ or } (b, a) \\
0 & \text{if } (x, y) = (a, 1) \text{ or } (1, a) \\
b & \text{if } (x, y) = (b, 1) \text{ or } (1, b) 
\end{cases}$$

and $f : K \rightarrow K$ by
Proof. (i) Let $f$ be a symmetric bi-derivation on $K$. Then the following identities hold for all $x, y, z \in K$,

(i) $D(x \ast y, z) \leq f(x) + f(y)$, for all $x, y, z \in K$,

(ii) $D(x \ast y, z) \leq D(x, z) + D(y, z)$, for all $x, y, z \in K$.

Proof. (i) Let $x, y, z \in K$. By using (K9), we have $D(x, z) \ast f(y) \leq f(y)$ and $f(x) \ast D(y, z) \leq f(x) + f(y)$. Then by using (K11), we obtain $D(x, z) \ast f(y) + f(x) \ast D(y, z) \leq f(x) + f(y)$. Hence $D(x \ast y, z) \leq f(x) + f(y)$.

(ii) Let $x, y, z \in K$. By using (K9), we have $D(x, z) \ast f(y) \leq D(x, z)$ and $f(x) \ast D(y, z) \leq D(y, z)$. Then by using (K11), we obtain $D(x, z) \ast f(y) + f(x) \ast D(y \ast z) \leq D(x, z) + D(y, z)$. Hence $D(x \ast y, z) \leq D(x, z) + D(y, z)$.

Proposition 3.6. Let $K$ be an incline algebra and let $D$ be a symmetric bi-$f$-derivation on $K$. If $x \leq y$ and $f$ an isotone mapping, we have $D(x \ast y, z) \leq f(y)$.

Proof. Let $x \leq y$. Then we have $f(x) \leq f(y)$. By using (K11) and (K9), $f(x) \ast D(y, z) \leq f(y) \ast D(y, z) \leq f(y)$. Also, by using (K9), we get $D(x, z) \leq f(y) \leq f(y)$. Hence we have $D(x \ast y, z) = D(x, z) \ast f(y) + f(x) \ast D(y, z) \leq f(y) + f(y) = f(y)$.

Proposition 3.7. Let $K$ be an incline algebra. If $K$ is a distributive lattice, we have $D(x, y) \leq f(x)$ and $D(x, y) \leq f(y)$ for all $x, y \in K$.

Proof. Let $K$ be a distributive lattice. Then $D(x, y) = D(x \ast x, y) = D(x, y) \ast f(x) + f(x) \ast D(x, y)$, and so $D(x, y) + f(x) = (D(x, y) \ast f(x) + f(x) \ast D(x, y)) + f(x) = (D(x, y) \ast f(x) + D(x, y) \ast f(x)) + f(x) = (D(x, y) \ast f(x)) + f(x)$ for all $x, y \in K$. By using (K8), we get $D(x, y) + f(x) = f(x)$. Hence we obtain $D(x, y) \leq f(x)$. Similarly, we have $D(x, y) \leq f(y)$.

Proposition 3.8. Let $K$ be an incline algebra. If $K$ is a distributive lattice, we have $d(x) \leq f(x)$ for all $x \in K$. 

Proof. Let $K$ be a distributive lattice. Then we have
\[
d(x) = D(x, x) = D(x \ast x, x) = D(x, x) \ast f(x) + f(x) \ast D(x, x)
\]
\[
= D(x, x) \ast f(x) \leq f(x)
\]
for all $x \in K$.

Proposition 3.9. Let $K$ be an incline algebra and let $d$ be a trace of symmetric bi-$f$-derivation $D$ of $K$. If $f(0) = 0$, then $d(0) = 0$.

Proof. Let $x \in K$. Then we have $d(0) = D(0, 0) = D(x \ast 0, 0) = D(x, 0) \ast f(0) + f(x) \ast D(0, 0) = 0 + f(x) \ast d(0)$. If we take $x = 0$, we get $d(0) = 0 + 0 = 0$.

Theorem 3.10. Let $K$ be an integral incline and let $D$ be a symmetric bi-$f$-derivation of $K$ where $f$ is a function satisfying $f(1) = 1$ and $a \in K$. Then for all $x, y \in K$, we have
(i) $a \ast D(x, y) = 0$ implies $a = 0$ or $D = 0$,
(ii) $D(x, y) \ast a = 0$ implies $a = 0$ or $D = 0$.

Proof. (i) Let $a \ast D(x, y) = 0$ for all $x \in K$. If we replace $x$ by $x \ast z$ for $z \in K$, we obtain
\[
0 = a \ast D(x, y) = a \ast D(x \ast z, y) = a \ast (D(x, y) \ast f(z) + (f(x) \ast D(z, y))
\]
\[
= (a \ast (D(x, y) \ast f(z))) + (a \ast (f(x) \ast D(z, y)))
\]
\[
= a \ast (f(x) \ast D(z, y)).
\]
In this equation, by taking $x = 1$, we have $a \ast D(z, y) = 0$. Since $K$ is an integral incline, that is, it has no zero-divisors, we have $a = 0$ or $D(z, y) = 0$ for all $y, z \in K$. Hence we get $a = 0$ or $D = 0$. (ii) Similarly, we can prove (ii).

Theorem 3.11. Let $K$ be an incline algebra with a multiplicative identity and let $D$ be a symmetric bi-$f$-derivation of $K$ where $f$ is a function satisfying $f(1) = 1$. Then the following identities hold for all $x, y \in K$.
(i) $f(x) \ast D(1, y) \leq D(x, y)$,
(ii) If $d$ is the trace of $K$ and $d(1) = 1$, we have $f(x) \leq D(x, 1)$.

Proof. (i) Let $x, y \in K$. Then we have $D(x, y) = D(x \ast 1, y) = D(x, y) \ast f(1) + f(x) \ast D(1, y) = D(x, y) \ast 1 + f(x) \ast D(1, y) = D(x, y) + f(x) \ast D(1, x)$. Therefore, $f(x) \ast D(1, y) \leq D(x, y)$. (ii) It can be derived from (i).

Proposition 3.12. Let $K$ be an incline algebra. If $D(x, y) = f(x)$ and $D(w, y) = f(w)$, we have $D(x \ast w, y) = f(x) \ast f(w)$. 
Proof. Let $x, y, w \in K$. Then we have
\[
D(x \ast w, y) = D(x, y) \ast f(w) + f(x) \ast D(w, y) = f(x) \ast f(w) + f(x) \ast f(w) = f(x) \ast f(w).
\]

Definition 3.13. Let $K$ be an incline algebra. If $D : K \times K \rightarrow K$ be a symmetric mapping. We call $D$ a \textit{joinitive mapping} if it satisfies
\[
D(x + y, z) = D(x, z) + D(y, z)
\]
for all $x, y, z \in K$.

Proposition 3.14. Let $K$ be an incline algebra and let $d$ be a trace of joinitive symmetric bi-f-derivation $D$ of $K$. Then the following identities hold for all $x, y \in K$,
\begin{enumerate}[(i)]
  
  \item $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$,
  
  \item $D(x \ast y, x) \leq d(x)$.
\end{enumerate}

Proof. (i) Let $x, y \in K$. Then we have
\[
d(x + y) = D(x + y, x + y) = D(x, x + y) + D(y, x + y)
= D(x, x) + D(x, y) + D(y, x) + D(y, y)
= D(x, x) + D(y, y) + D(x, y).
\]
Hence we get $d(x + y) = d(x) + d(y) + D(x, y)$ and $d(x) + d(y) \leq d(x + y)$.

(ii) Let $x, y \in K$. It follows from (K7) that $d(x) = D(x, x) = D(x + (x \ast y), x) = D(x, x) + D(x \ast y, x)$, which implies $D(x \ast y, x) \leq d(x)$.

Proposition 3.15. Let $K$ be an incline algebra and let $D$ be a trace of jointive symmetric bi-f-derivation $D$ of $K$. Then $D(x \ast y, y) \leq D(x, y)$ for all $x, y \in K$.

Proof. Let $x, y \in K$. Then we have
\[
D(x, y) = D(x + x \ast y, y) = D(x, y) + D(x \ast y, y),
\]
which implies $D(x \ast y, y) \leq D(x, y)$.

Definition 3.16. Let $K$ be an incline algebra and let $D$ be a symmetric bi-f-derivation on $K$. If $x \leq w$ implies $D(x, y) \leq D(w, y)$, $D$ is called an \textit{isotone symmetric bi-f-derivation} for all $x, y, w \in K$. 
Theorem 3.17. Let $K$ be an incline algebra and let $D$ be a joinitive symmetric bi-$f$-derivation $D$ of $K$. Then $D$ is an isotone symmetric bi-$f$-derivation of $K$.

**Proof.** Let $x$ and $w$ be such that $x \leq w$. Then $x + w = w$, and so
\[ D(w, y) = D(w + x, y) = D(w, y) + D(x, y) \]
This implies that $D(x, y) \leq D(w, y)$. This completes the proof.

Let $D$ be a symmetric bi-$f$-derivation of $K$. Fix $a \in K$ and define a set $F_a(K)$ by
\[ F_a(K) := \{ x \in K \mid D(x, a) = f(x) \} \]
for all $x \in K$.

**Proposition 3.18.** Let $D$ be a joinitive symmetric bi-$f$-derivation and let $f$ be an endomorphism on $K$. Then $F_a(K)$ is a subincline of $K$.

**Proof.** Let $x, y \in F_a(K)$. Then we have $D(x, a) = f(x)$ and $D(y, a) = f(y)$, and so
\[ D(x * y, a) = D(x, a) * f(y) + f(x) * D(y, a) \]
\[ = f(x) * f(y) + f(x) * f(y) \]
\[ = f(x) * f(y) = f(x * y). \]
Hence we get $x * y \in F_a(K)$. Also, we get $D(x + y, a) = D(x, a) + D(y, a) = f(x) + f(y) = f(x + y)$, and so $x + y \in F_a(K)$. This completes the proof.

**Proposition 3.19.** Let $K$ be additively right cancellative and let $D$ be a joinitive symmetric bi-$f$-derivation and let $f$ be an endomorphism on $K$. Then $F_a(K)$ is a $k$-ideal of $K$.

**Proof.** Let $x + y \in F_a(K)$ and $y \in F_a(K)$. Then we have $f(x) + f(y) = f(x + y) = D(x + y, a) = D(x, a) + D(y, a) = D(x, a) + f(y)$. Since $K$ is additively right cancellative, we have $f(x) = D(x, a)$, which implies $x \in F_a(K)$. This completes the proof.

**Definition 3.20.** Let $K$ be an incline algebra. If $D : K \times K \to K$ be a symmetric mapping. Define a set $\text{Ker}_D(K)$ by
\[ \text{Ker}_D(K) = \{ x \in K \mid D(0, x) = 0 \}. \]
Proposition 3.21. Let $K$ be an incline algebra and let $D$ be a joinitive symmetric bi-$f$-derivation $D$. If $x \leq y$ and $y \in Ker_D(K)$, then we have $x \in Ker_D(K)$.

Proof. Let $x \leq y$ and $y \in Ker_D(K)$. Then we get $x + y = y$ and $D(0, y) = 0$. Hence we get

$$0 = D(0, y) = D(0, x + y) = D(0, x) + D(0, y) = D(0, x) + 0 = D(0, x),$$

which implies $x \in Ker_D(K)$. This completes the proof.

Proposition 3.22. Let $K$ be an incline algebra and let $D$ be a joinitive symmetric bi-$f$-derivation of $K$. Then $Ker_D(K)$ is a subincline of $K$.

Proof. Let $x, y \in Ker_D(K)$. Then $D(0, x) = D(0, y) = 0$, and so

$$D(0, x \ast y) = D(x \ast y, 0) = D(x, 0) \ast f(y) + f(x) \ast D(y, 0) = 0 \ast f(y) + f(x) \ast 0 = 0 + 0 = 0,$$

which implies $x \ast y \in Ker_D(K)$. Now $D(x+y, 0) = D(x, 0)+D(y, 0) = 0+0 = 0$. Hence $x + y \in Ker_D(K)$. This completes the proof.

Theorem 3.23. Let $D$ be a joinitive symmetric bi-$f$-derivation of $K$. Then $Ker_D(K)$ is an ideal of $K$.

Proof. By Proposition 3.23, $Ker_D(K)$ is a subincline of $K$. Let $x \leq y$ and $y \in Ker_D(K)$. Then $x + y = y$ and $D(0, y) = 0$. Thus

$$0 = D(0, y) = D(0, x + y) = D(0, x) + D(0, y) = D(0, x) + 0 = D(0, x),$$

which implies $x \in Ker_D(K)$.

Proposition 3.24. Let $K$ be an incline algebra and let $D$ be a symmetric bi-$f$-derivation of $K$. If there exists $a \in K$ such that $D(x, z) \ast a = 0$ for all $x, z \in K$, then $z \in Ker_D(K)$.
Proof. Let $D$ be a symmetric bi-$f$-derivation of $K$ and $a \in K$. Then we have for all $x, z \in K$,

$$D(0, z) = D(0 \ast a, z) = D(0, z) \ast a + 0 \ast D(a, z) = 0 + 0 = 0,$$

which implies $z \in \text{Ker}_D(K)$.

Corollary 3.25. Let $K$ be an incline algebra and let $D$ be a symmetric bi-$f$-derivation of $K$. If there exists $a \in K$ such that $D(x, z) \ast a = 0$ for all $x, z \in K$, then $d(0) = D(0,0) = 0$.

Theorem 3.26. Let $K$ be a commutative integral incline algebra. Suppose that there exist two joinitive symmetric bi-$f$-derivations $D_1$ and $D_2$ such that $D_1(d_2(x), x) = 0$ for all $x \in K$ where $d_1, d_2$ denote the trace of $D_1$ and $D_2$, respectively. If $f$ is a nonzero function on $K$, then $d_1 = 0$.

Proof. Let $D_1(d_2(x), x) = 0$ where $d_1, d_2$ denote the trace of symmetric bi-derivations $D_1$ and $D_2$, respectively. Then by using (K7), we have

$$D_1(d_2(x) + (d_2(x) \ast x), x) = D_1(d_2(x), x) + D_1(d_2(x) \ast x, x) = D_1(d_2(x), x) \ast f(x) + f(d_2(x)) \ast D_1(x, x) = f(d_2(x)) \ast d_1(x).$$

Since $K$ has no zero divisors, we have $f(d_2(x)) = 0$ or $d_1(x) = 0$. But since $f$ is a nonzero function on $K$, we get $d_1(x) = 0$ for all $x \in K$.

References


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