Uniformly Primary Elements of Multiplicative Lattices

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Abstract

The paper must have abstract. In this study, we introduce the concept of uniformly primary elements of multiplicative lattices, which imposes a certain boundedness condition on the usual notion of primary elements of multiplicative lattices. Then we investigate some properties on uniformly primary elements of multiplicative lattices with examples.

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1 Introduction

A multiplicative lattice $L$ is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins
and has compact greatest element $1_L$ (least element $0_L$) as a multiplicative identity (zero). In this paper, we assume that all lattices are multiplicative lattices with $1_L$ compact multiplicative identity. Multiplicative lattices have been studied largely by E.W. Johnson, C. Jayaram, the current authors, and others, see, for example, [3−7].

By a $C$−lattice, we mean a (not necessarily modular) complete multiplicative lattice, with least element $0_L$ and compact greatest element $1_L$ (a multiplicative identity), which is generated under joins by multiplicatively closed subset $C$ of compact elements. The ideal lattice $L(R)$ of a ring $R$ where $R$ is commutative ring with identity is an example for $C$−lattice.

Throughout this paper, $L$ denotes a $C$−lattice and $L_*$ denotes the set of all compact elements of $L$.

An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in $L$ is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for any $a, b \in L_*$. An element $m < 1_L$ in $L$ is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. The radical of an element $a \in L$, denoted by $\sqrt{a}$, is defined by $\sqrt{a} = \sqrt{\{x \in L_* : x^n \leq a \text{ for some integer } n \}}$. An element $a$ of a multiplicative lattice $L$ is called compact if $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \bigvee b_{\alpha_2} \bigvee ... \bigvee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. Since $1_L$ is compact element, then maximal elements exist in $L$. Moreover, if $L$ is $C$−lattice, then for every $a \in L$, $\sqrt{a} = \bigvee \{p \in L : a \leq p \text{ and } p \text{ is a prime element}\}$ (See Theorem 3.6 of [7]).

An element $q < 1_L$ in $L$ is said to be primary if $ab \leq q$ implies $a \leq q$ or $b \leq \sqrt{q}$ for every pair of compact elements $a, b \in L$. An element $q \in L$ is $p$−primary if $q$ is primary and $\sqrt{q} = p$ is prime. This statement is equivalent to that an element $q < 1_L$ in $L$ is a primary if for every $a, b \in L_*$, $ab \leq q$ implies $a \leq q$ or $b^n \leq q$ for some positive integer $n$ since the radical of an element $a \in L$ is $\sqrt{a} = \bigvee \{x \in L_* : x^n \leq a \text{ for some integer } n \}$. For any maximal element $m$, $m^n$ is $m$−primary for every positive integer $n$.

For a multiplicative lattice $L$ and $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \sqrt{a}$.

J.A.Cox and A.J.Hetzel have introduced uniformly primary ideals of a commutative ring with non-zero identity in [6]. In this study, we introduce the concept of uniformly primary elements of multiplicative lattices, which imposes a certain boundedness condition on the usual notion of primary elements of multiplicative lattices. We begin with definitions of uniformly primary elements and Noether strongly primary elements and Mori strongly primary element of multiplicative lattices. Then some examples are given to explain these definitions. A proper element $q$ is said to be a uniformly primary element of $L$ if there exists a positive integer $m$ such that whenever $x, y \in L$ satisfy $xy \leq q$, $x \not\leq q$, then $y^m \leq q$. Then we investigate the relationship between uniformly primary element of $L$, Noether strongly primary element of $L$, Mori strongly primary element of $L$ and primary elements of $L$. After that, Proposition 2.13
and Theorem 2.15 satisfy properties of especial collections of uniformly primary elements of $L$ that share a common radical. Then we give an example for contrary situation. Proposition 2.19 are investigated uniformly primary elements on $L/a$ for $a \in L$. Finally, we investigate that a primary element of $L$ is a uniformly primary element of $L$ under the condition join of finitely compact element of the radical of a primary element of $L$ in Proposition 2.21.

2 Properties of Uniformly Primary Elements of a Multiplicative Lattices

Definition 2.1 An element $q < 1_L$ in $L$ is called a uniformly primary element in $L$ if there exists a positive integer $m$ such that whenever $x, y \in L_*$ satisfy $xy \leq q$, $x \not\leq q$, then $y^m \leq q$. We call that $m$ is order of $q$ if $m$ is the smallest positive integer for which above property holds and it is denoted by $\text{ord}_L(q) = m$ or $\text{ord}(q) = m$.

Example 2.2 Every prime element in $L$ is a uniformly primary element of order 1.

Definition 2.3 Let $q$ be a $p$–primary element of $L$.

1. $q$ is a Noether strongly primary element of $L$ if $p^n \leq q$ for some positive integer $n$. We say that $n$ is the exponent of $q$ if $n$ is the smallest positive integer for which above property holds and it is denoted by $e(q) = n$.

2. $q$ is a Mori strongly primary element of $L$ if there is a compact element $a \not\leq q$ such that $ap \leq q$.

Example 2.4 Let $L = I(R)$ where $R$ is a commutative ring and let $I(R)$ be the set of ideals of $R$. Then $q = (8)$ is a $p = (2)$–primary element of $L$. Also $q$ is a Mori strongly primary element of $L$ since $(4)p \subseteq q$ for $(4) \not\subseteq (8)$. Moreover $q$ is a Noether strongly primary element of $L$.

Proposition 2.5 If $q$ is a Noether strongly primary element of $L$, then $q$ is a Mori strongly primary element of $L$.

Proof 2.6 Let $q$ be a Noether strongly primary element of $L$ and $e(q) = n$. Then $p^n \leq q$ and $p^{n-1} \not\leq q$. So there exists $a \in L_*$ such that $a \leq p^{n-1}$ and $a \not\leq q$. Then $ap \leq p^{n-1}p = p^n \leq q$. Hence $q$ is a Mori strongly primary element of $L$.

Proposition 2.7 If $q$ is a Noether strongly $p$–primary element of $L$, then $q$ is a uniformly $p$–primary element of $L$ and $\text{ord}(q) \leq e(q)$.
Proof 2.8 Let $q$ be a Noether strongly $p$–primary element of $L$. Now, let $x, y \in L_*$ such that $xy \leq q$, $x \nleq q$. Then $y \leq p$ since $q$ is a $p$–primary element of $L$. Thus $y^{\ell(a)} \leq p^{\ell(q)} \leq q$. Therefore, $q$ is a uniformly $p$–primary element of $L$ and also $\ord(q) \leq e(q)$.

Proposition 2.9 Let $p$ be a prime element of $L$. If $p^n$ is a primary element of $L$ for every positive integer $n$, then $p^n$ is a uniformly primary element of $L$ of order at most $n$. Particularly, if $p$ is a maximal element of $L$, then $p^n$ is a uniformly primary element of $L$ of order at most $n$ where $n$ is a positive integer.

Proof 2.10 It is clear that $p = \sqrt{p^n}$ for every positive integer $n$. Assume that $x, y \in L_*$ such that $xy \leq p^n$, $x \nleq p^n$. Then $y \leq \sqrt{p^n} = p$. Hence $y^n \leq p^n$, that is, $p^n$ is a uniformly primary element of $L$ and so $\ord(p^n) \leq n$. Let $p$ be a maximal element of $L$. Since any exponent of $p$ is a $p$–primary element of $L$, then $p^n$ is uniformly $p$–primary element of $L$.

Proposition 2.11 An element $q$ of $L$ is a uniformly $p$–primary if and only if the following statements hold:

1. $q$ is a $p$–primary element of $L$.
2. There is a positive integer $n$ such that $p = \sqrt{\{x \in L_* \mid x^n \leq q\}}$.

Furthermore, $\ord(q) = m$ if and only if $m$ is the smallest positive integer holding statement (2).

Proof 2.12 ($\Rightarrow$) : Let $q$ be a uniformly primary element of $L$. It is clear that $q$ is a $p$–primary element of $L$. Assume that $x \leq p$ for any $x \in L_*$. Then there is some positive integer $m$ such that $x^m = x^{m-1}x \leq q$ and $x^{m-1} \nleq q$. Let $n$ be the smallest positive integer holding the condition, that is $n$ holds: $x^n = x^{n-1}x \leq q$ and $x^{n-1} \nleq q$. Since $q$ is uniformly $p$–primary, then $x^{\ord(q)} \leq q$.

Thus statement (2) is satisfied.

($\Leftarrow$) : Assume that statements (1) and (2) are satisfied. Let $x, y \in L_*$ such that $xy \leq q$ and $x \nleq q$. By (1), $y \leq \sqrt{q} = p$ and by (2), there is a positive integer $n$ such that $y^n \leq q$. Thus $q$ is a uniformly $p$–primary element of $L$.

Let $\ord(q) = m$. It is clear that $m$ is the smallest positive integer holding statement (2). Conversely, let $m$ be the smallest positive integer holding statement (2). Let $x, y \in L_*$ such that $xy \leq q$ and $x \nleq q$. By (1), $y \leq \sqrt{q} = p$ and so $y^m \leq q$. Then $\ord(q) \leq m$. By hypothesis, $\ord(q) = m$.

Proposition 2.13 If $q_1 \leq q_2$ are uniformly $p$–primary elements of $L$, then $\ord(q_2) \leq \ord(q_1)$. 
Proof 2.14 Let \( \text{ord}(q_1) = k \) and \( \text{ord}(q_2) = m \). Let \( x, y \in L \) such that \( xy \leq q_2 \), \( x \not\leq q_2 \). Then \( y^m \leq q_2 \) and \( y^{m-1} \not\leq q_2 \). Thus \( y \leq p = \sqrt{q_1} \). By Proposition 2.11, \( y^k \leq q_1 \leq q_2 \). Therefore, \( k > m - 1 \) and so \( k \geq m \).

Theorem 2.15 Let \( \{q_i\}_{i \in I} \) be a family of uniformly \( p \)--primary elements of \( L \) such that \( \max_{i \in I} \{\text{ord}(q_i)\} = k \), where \( k \) is a positive integer. Then \( \bigwedge_{i \in I} q_i \) is a uniformly \( p \)--primary element of \( L \) of order \( k \).

Proof 2.16 Let \( q = \bigwedge_{i \in I} q_i \). Then \( \sqrt{q} = \sqrt{\bigwedge_{i \in I} q_i} = \bigwedge_{i \in I} \sqrt{q_i} = p \). Let \( x, y \in L \) such that \( xy \leq q \), \( x \not\leq q \). Then \( xy \leq q_j \), \( x \not\leq \sqrt{q_j} \) for some \( q_j \in \{q_i\}_{i \in I} \). Since \( q_j \) is a uniformly \( p \)--primary element of \( L \), then \( y^k \leq q_j \). Then \( y \leq \sqrt{q_j} = p = \sqrt{q} \). Hence there is a positive integer \( m \) such that \( y^m \leq q \). Therefore, \( q \) is a uniformly \( p \)--primary element of \( L \). Let \( q_i \in \{q_i\}_{i \in I} \) be a uniformly \( p \)--primary element of \( L \) of order \( k \). By Proposition 2.11, \( k \) is the smallest positive integer such that \( p = \sqrt{\{x \in L \mid x^k \leq q_i\}} \). Hence, there is a compact element \( x \leq p \) such that \( x^{k-1} \not\leq q_i \). Thus \( x^{k-1} \not\leq q \). Consequently, we have \( k = \text{ord}(q) \).

In following example, we show that if \( q_1 \) and \( q_2 \) are uniformly primary elements of \( L \), then \( q_1 \wedge q_2 \) need not be a uniformly primary element of \( L \).

Example 2.17 [2, Example 2.6] Let \( R = \mathbb{Z}[X] + 2Y\mathbb{Z}[X,Y] \) where \( \mathbb{Z} \) is the ring of integer and \( X, Y \) are indeterminates. Let \( L(R) \) be the set of ideals of \( R \). Note that \( L(R) \) is a \( C \)--lattices. Then \( q_1 = XR \) and \( q_2 = 2Y\mathbb{Z}[X,Y] \) are uniformly primary elements of \( L(R) \). Let \( q = q_1 \cap q_2 \). Then \( (2X^2\mathbb{Z}[X,Y])(YR) \subseteq q \), \( 2X^2\mathbb{Z}[X,Y] \subseteq q \) and \( (YR)^n \subseteq q \) for every positive integer \( n \). Thus \( q \) is not uniformly primary element of \( L(R) \).

Example 2.18 Let \( S = \{0, 1, x, y\} \) be a commutative semigroup with multiplication \( x = x^2 \), \( y = y^2 \) and \( xy = 0 = yx \). It is clear that \( L(S) \), the set of ideals of \( S \), is a multiplicative lattice. \((x)\) and \((y)\) are easily seen to be uniformly primary ideal of \( L(S) \). But \((x) \cap (y) = \{0\}\) is not uniformly primary ideal of \( L(S) \). Because \((x) \subseteq (x) \cap (y), (x) \not\subseteq (x) \cap (y) \) and for every positive integer \( n \), \((y^n) \not\subseteq (x) \cap (y) \).

Proposition 2.19 Let \( q \) be an element of \( L \) containing \( a \in L \). Then \( q \) is a uniformly \( p \)--primary element of \( L \) of order \( m \) if and only if \( q \) is a uniformly \( p \)--primary element of \( L/a \) of order \( m \).

Proof 2.20 Let \( q \) be a uniformly \( p \)--primary element of \( L \). Let \( x \circ y = xy \lor a \leq q \) and \( x \not\leq q \) for \( x, y \in L \). Then \( xy \leq q \) and \( x \not\leq q \). By hypothesis, \( y^{\text{ord}_L(q)} \leq q \). Then \( y^{\text{ord}_L(q)} \lor a \leq q \). Thus \( q \) is a uniformly primary element of \( L/a \) of order at most \( \text{ord}_L(q) \). Let \( \text{ord}_L(q) = m \). By Proposition 2.11,
there is a compact element \( x \leq p \) such that \( x^m - 1 \not\leq q \). Then \( x \lor a \leq p \) such that \( x^m \lor a \not\leq q \). Hence \( \text{ord}_{L/a}(q) \) is \( m \). Conversely, let \( q \) is a uniformly \( p \)-primary element of \( L/a \) of order \( m \). Assume that \( x,y \in L^* \) such that \( xy \leq q \) and \( x \not\leq q \). Then \( xy \lor a \leq q \) and \( x \lor a \not\leq q \). Since \( q \) is a uniformly \( p \)-primary element of \( L/a \), then \( y \lor a \leq q \). Thus \( y \not\leq q \). Hence \( \text{ord}_{L/a}(q) = m \). Conversely, let \( q \) is a uniformly \( p \)-primary element of \( L/a \) of order \( m \). Assume that \( x,y \in L^* \) such that \( xy \leq q \) and \( x \not\leq q \). Then \( xy \lor a \leq q \) and \( x \lor a \not\leq q \). Since \( q \) is a uniformly \( p \)-primary element of \( L/a \), then \( y \lor a \leq q \). Thus \( y \not\leq q \). Hence \( \text{ord}_{L/a}(q) = m \). By Proposition 2.11, there is \( x \lor a \leq p \) such that \( x^m \lor a \not\leq q \). Then \( x \leq p \) and \( x^m \not\leq q \). Indeed, if \( x^m \leq q \), then \( x^m \lor a \leq q \) since \( a \leq q \). Thus \( q \) is a uniformly primary element of \( L \).

**Proposition 2.21** Let \( q \) be a \( p \)-primary element of \( L \). If \( p \) is finitely join of compact element of \( L \), then \( q \) is a Noether strongly primary element of \( L \). Hence \( q \) is a uniformly primary element of \( L \).

**Proof 2.22** Let \( p = \sqrt[n]{\sum_{i=1}^{n} a_i} \) for some compact elements \( a_i \). Since \( p = \sqrt[q]{q} \), then \( a_i^{s_i} \leq q \) for some positive integer \( s_i \) for each \( i = 1,2,...,n \). Let \( N = \sum_{i=1}^{n} (s_i - 1) + 1 \). It is clear that \( p^n \) is join of all elements of the form \( a_1^{t_1}a_2^{t_2}...a_n^{t_n} \) where each \( t_i \) is a non negative integer and \( N = \sum_{i=1}^{n} t_i \). Thus these elements are contained by \( q \). Hence \( p^n \leq q \), that is, \( q \) is a Noether strongly primary element of \( L \). Also \( q \) is a uniformly primary element of \( L \) by Proposition 2.7.

**References**


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