Positive-Normal Operators in Semi-Hilbertian Spaces

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Abstract

Given a bounded positive linear operator A on a Hilbert space $\mathcal{H}$ we consider the semi-Hilbertian space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$, where $\langle \xi | \eta \rangle_A := \langle A\xi | \eta \rangle$. In this paper we introduce a class of operators on a semi Hilbertian space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle_A$. We call the elements of this class A -positive-normal or A -posinormal. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be A -posinormal if there exists a A -positive operator $P \in \mathcal{B}(\mathcal{H})$ (i.e., $AP \geq 0$) such that $TAT^* = T^*APT$. We study some basic properties of these operators. Also we study the relationship between a special case of this class with the other kinds of classes of operators in semi-Hilbertian spaces.

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1 Introduction

We consider a Hilbert space $\mathcal{H}$ with an additional semi inner product defined by a positive semidefinite operator A ; namely $\langle \xi | \eta \rangle_A = \langle A\xi | \eta \rangle$ for every $\xi, \eta \in \mathcal{H}$. It must be observed from [2] and [3] that this additional structure induces an adjoint operation. However, this operation is defined for not every bounded linear operator on $\mathcal{H}$, unless A is invertible. For those operators T which admit an adjoint with respect to $\langle \cdot | \cdot \rangle_A$, we choose one, denoted by $T^{(*)}_A$, which has similar, but not identical, properties as the classical T^* . Since not every operator admits an A -adjoint and, in case it admits one, it may have many others, then the extensions of normal operator, quasi-normal operators, isometries, unitary, partial isometries, quasi-isometry and m -isometry are not trivial. These classes of operators have been generalized to semi-Hilbertian spaces by many authors. Such operators appear in different contexts in [2], [3], [4], [14], [28], [29], [32] and other papers. The aim of this work is to continue this process of generalization to posinormal operators. The inspiration for our investigation comes from [2], [3], [4], [14], [28], [29], [32]. In this paper section 2 contains basic results on A -operators. There is also a description of the range inclusion theorem of R. G. Douglas [11], which is a key for some results of this paper. At the end of this section we give some characterizations of A -quasinormal operators inspired from [26] and [27]. In section 3 we study the concept of an A -posinormal operators and we investigate various structural properties of this class of operators. In the final section of the paper we consider the tensor product of some classes of A -operators.

2 Definitions and basic facts about semi-Hilbertian space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$.

Along this work $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})^+$ is the cone of positive (semidefinite) operators of $\mathcal{B}(\mathcal{H})$, i.e., $\mathcal{B}(\mathcal{H})^+ := \{T \in \mathcal{B}(\mathcal{H}) \mid \langle T\xi | \xi \rangle \geq 0 \ \forall \xi \in \mathcal{H}\}$ and $\mathcal{B}(\mathcal{H})_{cr}$ is the subset of $\mathcal{B}(\mathcal{H})$ of all operators with closed range. For every $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for, respectively, the null space, the range and the closure of the range of T . Its adjoint operator by T^* . In addition, if $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ then $T_1 \geq T_2$ means that $T_1 - T_2 \in \mathcal{B}(\mathcal{H})^+$. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$. On the other hand, $T^\dagger$ stands for the Moore-Penrose inverse of $T \in \mathcal{B}(\mathcal{H})$.

Given $A \in \mathcal{B}(\mathcal{H})^+$, the functional

$$\langle \cdot | \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \quad \langle \xi | \eta \rangle_A = \langle A\xi | \eta \rangle$$

is a semi-inner product on $\mathcal{H}$. By $\|\cdot\|_A$ we denote the seminorm induced

by $\langle \cdot | \cdot \rangle_A$, i.e., $\|\xi\|_A = \langle \xi | \xi \rangle_A^{\frac{1}{2}}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in \mathcal{N}(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is an injective operator, and the seminormed space $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed. Moreover, $\langle \cdot | \cdot \rangle_A$ induces a semi-norm on a certain subspace of $\mathcal{B}(\mathcal{H})$, namely, on the subspace

$$\{ T \in \mathcal{B}(\mathcal{H}) / \exists c > 0 : \|T\xi\|_A \leq c\|\xi\|_A \ \forall \xi \in \mathcal{H} \}.$$

In such case it holds

$$\begin{aligned} \|T\|_A &= \sup_{\substack{\xi \in \mathcal{R}(A) \\ \xi \neq 0}} \frac{\|T\xi\|_A}{\|\xi\|_A} = \sup_{\|\xi\|_A \leq 1} \|T\xi\|_A = \sup\{\|T\xi\|_A : \|\xi\|_A = 1\} \\ &= \inf\{c > 0 : \|T\xi\|_A \leq c\|\xi\|_A, \ \xi \in \mathcal{H}\} < \infty. \end{aligned}$$

Moreover

$$\|T\|_A = \sup\{\langle T\xi | \eta \rangle_A; \ \xi, \eta \in \mathcal{H}, \ : \ \|\xi\| \leq 1, \ \|\eta\| \leq 1\}.$$

For $\xi, \eta \in \mathcal{H}$, we say that ξ and η are A -orthogonal if $\langle \xi | \eta \rangle_A = 0$. Define

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) := \{ T \in \mathcal{B}(\mathcal{H}) : \|T\xi\|_A \leq c\|\xi\|_A \text{ for every } \xi \in \mathcal{H} \}$$

It is easy to see that $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$. For more details about the class $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ see [2, 3, 4].

Note that given $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, in general $T^* \notin \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ (see [4]).

Definition 2.1 ([2]) For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A -adjoint of T if for every $\xi, \eta \in \mathcal{H}$

$$\langle T\xi | \eta \rangle_A = \langle \xi | S\eta \rangle_A,$$

i.e., $AS = T^*A$; we say that T is A -selfadjoint if $AT = T^*A$.

or which is equivalent, if S is a solution of the equation $AX = T^*A$.

Remark 2.1 The existence of an A -adjoint operator is not guaranteed. Observe that T admits an A -adjoint if and only if the equation $AX = T^*A$ has solution. This kind of equation can be studied applying the next theorem due to Douglas (for its proof see [11], [14]).

Theorem 2.1 Let $A, B \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalents.

- (1) $\mathcal{R}(B) \subset \mathcal{R}(A)$.
- (2) There exists a positive number λ such that $BB^* \leq \lambda AA^*$.
- (3) There exists $C \in \mathcal{B}(\mathcal{H})$ such that $AC = B$.

If one of these conditions holds then there exists a unique operator $D \in \mathcal{B}(\mathcal{H})$ such that $AD = B$ and $\mathcal{R}(D) \subseteq \overline{\mathcal{R}(A^*)}$ and $\mathcal{N}(D) = \mathcal{N}(B)$. Moreover

$$\|D\| = \inf\{\lambda > 0 / BB^* \leq \lambda AA^*\}.$$

This solution will be called a reduced solution of the equation $BX = C$.

If we denote by $\mathcal{B}_A(\mathcal{H})$ the subalgebra of $\mathcal{B}(\mathcal{H})$ of all bounded operators which admit an A -adjoint operator then

$$\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : T^* \mathcal{R}(A) \subset \mathcal{R}(A) \}.$$

Furthermore, applying Douglas theorem we can see that

$$\begin{aligned} \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) &= \{ T \in \mathcal{B}(\mathcal{H}) : T^* \mathcal{R}(A^{\frac{1}{2}}) \subset \mathcal{R}(A^{\frac{1}{2}}) \} \\ &= \{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(A^{\frac{1}{2}} T^* A^{\frac{1}{2}}) \subseteq \mathcal{R}(A) \}. \end{aligned}$$

The relationship between the above sets is proved in [4].

Proposition 2.1 *Let $A \in \mathcal{B}(\mathcal{H})^+$ then $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$.*

Remark 2.2 $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^s}(\mathcal{H})$ for all $s \in (0, 1)$. More generally, if $0 < s < s' < 1$ then $\mathcal{B}_{A^{s'}}(\mathcal{H}) \subseteq \mathcal{B}_{A^s}(\mathcal{H})$. Moreover, $\mathcal{B}_{A^{s'}}(\mathcal{H}) = \mathcal{B}_{A^s}(\mathcal{H})$ if and only if $\mathcal{R}(A)$ is closed. See [4].

If an operator equation $BX = C$ has solution then it is easy to see that the distinguished solution of Douglas theorem is given by $B^\dagger C$. Therefore, given $T \in \mathcal{B}_A(\mathcal{H})$, if we denote by $T^{(*)A}$ the unique A -adjoint operator of T whose range is included in $\overline{\mathcal{R}(A)}$ then

$$T^{(*)A} = A^\dagger T^* A.$$

In view of Theorem 2.1,

$$AT^{(*)A} = T^* A, \mathcal{R}(T^{(*)A}) \subset \overline{\mathcal{R}(A)} \text{ and } \mathcal{N}(T^{(*)A}) = \mathcal{N}(T^* A).$$

Note that if S is an A -adjoint of T then $S = T^{(*)A} + Z$, with $Z \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(Z) \subset \mathcal{N}(A)$.

Observe that if T is A -selfadjoint it does not mean, in general, that $T = T^{(*)A}$. In fact $T = T^{(*)A}$ if and only if T is A -selfadjoint and $\mathcal{R}(T) \subset \overline{\mathcal{R}(A)}$. It is also clear that T has a unique A -adjoint (namely $T^{(*)A}$) if and only if A is injective. If this is the case, then we get the equality $(T^{(*)A})^{(*)A} = T$.

In the following proposition we collect some properties of $T^{(*)A}$. For its proof see [2, 3].

Proposition 2.2 *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.*

- (1) $T^{(\ast)_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{(\ast)_A})^{(\ast)_A} = P_{R(A)} T P_{R(A)}$ and $(T^{(\ast)_A})^{(\ast)_A} (\ast)_A = T^{(\ast)_A}$.
- (2) If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{(\ast)_A} = S^{(\ast)_A} T^{(\ast)_A}$.
- (3) $T^{(\ast)_A} T$ and $TT^{(\ast)_A}$ are A -selfadjoint.
- (4) $\|T\|_A = \|T^{(\ast)_A}\|_A = \|T^{(\ast)_A} T\|_A^{\frac{1}{2}} = \|TT^{(\ast)_A}\|_A^{\frac{1}{2}}$.
- (5) $\|S\|_A = \|T^{(\ast)_A}\|_A$ for every $S \in \mathcal{B}(\mathcal{H})$ which is an A -adjoint of T .
- (6) If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$.

Nevertheless, $T^{(\ast)_A}$ is not in general the unique A -adjoint of T that realizes the minimal norm.

The following classes of operators are studied in [2, 28, 32].

Any operator $T \in \mathcal{B}(\mathcal{H})$ is

- (1) A -contraction if $\|T\xi\|_A \leq \|\xi\|_A$ for every $\xi \in \mathcal{H}$, or equivalently if $T^*AT \leq A$.
- (2) A -isometry if $T^*AT = A \iff \|T\xi\|_A = \|\xi\|_A \ \forall \xi \in \mathcal{H}$.
- (3) A -normal if $T^*AT = TAT^* \iff \|T\xi\|_A = \|T^*\xi\|_A \ \forall \xi \in \mathcal{H}$.
- (4) A -partial isometry if $\|T\xi\|_A = \|\xi\|_A \ \forall \xi \in N(AT)^{\perp_A}$.
- (5) A -unitary if $T^*AT = TAT^* = A \iff \|T^*\xi\|_A = \|T\xi\|_A = \|\xi\|_A \ \forall \xi \in \mathcal{H}$.
- (6) A -hyponormal if $TAT^* \leq T^*AT \iff \|T^*\xi\|_A \leq \|T\xi\|_A \ \forall \xi \in \mathcal{H}$.
- (7) A -quasi-isometry if and only if, $T^*AT = T^{*2}AT^2 \iff \|T\|_A = \|T^2\|_A$.
- (8) A - m -isometry if $\sum_{k=0}^m (-1)^k \binom{m}{k} T^{*m-k} AT^{m-k} = 0$
 $\iff \sum_{k=0}^m (-1)^k \binom{m}{k} \|T^{m-k}\xi\|_A^2 = 0 \ \forall \xi \in \mathcal{H}$.

The following proposition gives a necessary and sufficient condition for with $T \in \mathcal{B}_A(\mathcal{H})$ belongs to $\mathcal{B}_{A^2}(\mathcal{H})$. For its proof see [24].

Proposition 2.3 (1) *If $T \in \mathcal{B}_A(\mathcal{H})$. Then $T \in \mathcal{B}_{A^2}(\mathcal{H})$ if and only if*

$$\left(T^{(\ast)_A^{\frac{1}{2}}}\right)^* \in \mathcal{B}_A(\mathcal{H}).$$

- (2) *If $T \in \mathcal{B}_A(\mathcal{H})$ is such that $T^* \in \mathcal{B}_A(\mathcal{H})$, then $T^{(\ast)_A^{\frac{1}{2}}} \in \mathcal{B}_{A^2}(\mathcal{H})$ and*

$$\left(T^{(\ast)_A^{\frac{1}{2}}}\right)^{(\ast)_A} = \left(T^*\right)^{(\ast)_A^{\frac{1}{2}}}, \text{ or equivalently } \left(T\right)^{(\ast)_A^{\frac{1}{2}}} = \left[\left(T^*\right)^{(\ast)_A^{\frac{1}{2}}}\right]^{(\ast)_A}.$$

Definition 2.2 ([28]) $T \in \mathcal{B}_A(\mathcal{H})$ is an A -normal operator if

$$TT^{(*)}_A = T^{(*)}_AT.$$

Definition 2.3 ([29]) An operator $T \in \mathcal{B}_A(\mathcal{H})$ is called A -quasinormal if T commutes with $T^{(*)}_AT$ i.e.,

$$TT^{(*)}_AT = T^{(*)}_AT^2.$$

Remark 2.3 Every A -normal operator is an A -quasinormal operator.

In the following theorem we give conditions under which an A -quasinormal operator T is an A -normal operator.

Theorem 2.2 Let $T \in \mathcal{B}_A(\mathcal{H})$. The following properties hold

1. If T is invertible A -quasinormal then T is A -normal.
2. Assume that $\mathcal{N}(A)$ is a invariant subspace for T , then
 - (a) If T and $T - I$ are A -quasinormal then T is A -normal, where I indicates the identity operator.
 - (b) $T - \lambda I$ is A -quasinormal for all $\lambda \in \mathbb{C}$ if and only if T is A -normal.

Proof. (1) $TT^{(*)}_AT = T^{(*)}_AT^2 \implies TT^{(*)}_A = T^{(*)}_AT$ as T is invertible.

(2) First we see that the condition on $T - I$ implies

$$\begin{aligned} & TT^{(*)}_AT - TT^{(*)}_A - TP_{\overline{\mathcal{R}(A)}}T + 2T - T^{(*)}_AT + T^{(*)}_A - P_{\overline{\mathcal{R}(A)}} \\ &= T^{(*)}_AT^2 - 2T^{(*)}_AT + T^{(*)}_A - P_{\overline{\mathcal{R}(A)}}T^2 + 2PT - P_{\overline{\mathcal{R}(A)}}. \end{aligned}$$

Therefore, since T is A -quasinormal, we have $TT^{(*)}_A = T^{(*)}_AT$.

(3) Assume that $T - \lambda I$ is A -quasisinormal for all $\lambda \in \mathbb{C}$, we have

$$(T - \lambda I)(T - \lambda I)^{(*)}_A(T - \lambda I) = (T - \lambda I)^{(*)}_A(T - \lambda I)^2, \quad \forall \lambda \in \mathbb{C}$$

which implies that

$$TT^{(*)}_AT - T^{(*)}_AT^2 - \lambda(TT^{(*)}_A - T^{(*)}_AT) = 0, \quad \forall \lambda \in \mathbb{C}.$$

We deduce that T is A -normal.

The following proposition gives a characterization of an A -quasinormal operator.

Proposition 2.4 Let $T \in \mathcal{B}_A(\mathcal{H})$, $X = T + T^{(\ast)_A}T$ and $Y = T - T^{(\ast)_A}T$. Then

- (1) T is A -quasinormal if and only if X commutes with Y .
- (2) If T is A -quasinormal then $TT^{(\ast)_A}T$ commutes with X and Y .

Proof. (1) A simple computation shows that

$$XY = YX \iff TT^{(\ast)_A}T = T^{(\ast)_A}T^2.$$

- (2) By the hypothesis

$$TT^{(\ast)_A}T(T \pm T^{(\ast)_A}T) = (T \pm T^{(\ast)_A}T)TT^{(\ast)_A}T.$$

3 Classes of operators on semi-Hilbertian spaces (A -posinormal operators)

In this section we define the class of A -posinormal operators and give some equivalent relation about it.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be posinormal (the word 'posinormal' stands for positive-normal) if there exists a $P \in \mathcal{B}(\mathcal{H})^+$ such that $TT^* = T^*PT$. Or equivalently, $T \in \mathcal{B}(\mathcal{H})$ is posinormal if there exists a co-isometry $V^* \in \mathcal{B}(\mathcal{H})$ a positive operator $P \in \mathcal{B}(\mathcal{H})$ such that $T = T^*PV^*$. The operator T is said to be conditionally totally posinormal, shortened to $T \in CTP(\mathcal{H})$, if $T - \lambda I$ is posinormal for all $\lambda \in \mathbb{C}$, and is totally posinormal, shortened to $T \in TP(\mathcal{H})$, if all operators $T - \lambda I$, $\lambda \in \mathbb{C}$; are posinormal and have a common interrupter positive operator. The class of posinormal operators contains in particular, the classes consisting of hyponormal operators ($TT^* \leq T^*T$), M -hyponormal ($|(T - \lambda I)^*|^2 \leq M|(T - \lambda I)|^2$ for some real $M > 0$). It is known that $T \in CTP(\mathcal{H})$ if and only if it is dominant operators ($|(T - \lambda I)^*|^2 \leq M_\lambda|(T - \lambda I)|^2$ for some real number $M > 0$ and all complex number λ). Posinormal operators were first introduced and studied by H. C. Rhaly [30] and have also been studied by some authors; see, for instance, the papers by M. Itoh [18] and by I. H. Jeon, S. H. Kim, E. Ko [17], S. Mecheri [25] and B. P. Duggal and C. Kubrusly [12], A. Bucur [6].

An operator T is said to be p -hyponormal if $(T^*T)^p \geq (TT^*)^p$, and p -posinormal if $c^2(T^*T)^p \geq (TT^*)^p$ for some $c > 0$. It is clear that 1-hyponormal and 1-posinormal are hyponormal and posinormal, respectively. For a positive integer k and a positive number $0 < p \leq 1$, An operator T is said to be

- (1) (p, k) -quasihyponormal if

$$T^{*k} \left((T^*T)^p - (TT^*)^p \right) T^k \geq 0,$$

(2) (p, k) -quasiposinormal if

$$T^{*k} \left(c(T^*T)^p - (TT^*)^p \right) T^k \geq 0 \quad \text{for some } c > 0.$$

It is easy that every $(p, 1)$ -quasihyponormal is p -quasihyponormal and $(p, 1)$ -quasiposinormal is p -quasiposinormal. By the definition, it is clear that

$$p\text{-hyponormal} \subset p\text{-posinormal} \subset (p, k)\text{-quasiposinormal}.$$

p -Hyponormal, p -posinormal, p -quasihyponormal, and (p, k) -quasihyponormal operators have been studied by many authors see for instance [19], [22] and references therein. and it is known that hyponormal operators have many interesting properties similar to those of normal operators(See [6], [7], [8]).

Theorem 3.1 ([18]) *For $T \in \mathcal{B}(\mathcal{H})$, the following statements are equivalent:*

- (1) T is posinormal
- (2) $\mathcal{R}(T) \subset \mathcal{R}(T^*)$
- (3) $TT^* \leq \lambda^2 T^*T$ for some $\lambda > 0$; and
- (4) there exists $S \in \mathcal{B}(\mathcal{H})$ such that $T = T^*S$.

Moreover, if (1),(2),(3) and (4) hold, then there is a unique operator S such that

- (a) $\|S\| = \inf\{\mu | TT^* \leq \mu T^*T\}$;
- (b) $\mathcal{N}(T) = \mathcal{N}(S)$; and
- (c) $\mathcal{R}(S) \subset \overline{\mathcal{R}(T)}$.

Definition 3.1 *We say that $T \in \mathcal{B}(\mathcal{H})$ is an A -positive if $AT \in \mathcal{B}(\mathcal{H})^+$ or equivalently*

$$\langle T\xi | \xi \rangle_A \geq 0 \quad \forall \xi \in \mathcal{H}.$$

Example 3.1 *If $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{(*)A}T$ and $TT^{(*)A}$ are A -positive.*

Remark 3.1 *An operator T is A -positive if and only if $A^{\frac{1}{2}}T$ is $A^{\frac{1}{2}}$ -positive.*

Lemma 3.1 ([15]) *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. The following assertions are equivalent:*

- (1) T is A -positive operator.
- (2) $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ and $\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}} \in \mathcal{B}(\mathcal{H})^+$ where $\overline{A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}}$ denotes the unique bounded linear extension of $A^{\frac{1}{2}}T(A^{\frac{1}{2}})^{\dagger}$ on $\mathcal{B}(\mathcal{H})$.

Lemma 3.2 ([15]) *Let $A, T \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent:*

- (1) *T is an A -positive operator ,*
- (2) *T is an $A^{\frac{1}{2}}$ -positive operator.*

The following Lemma is inspired from the work of M.C.Gonzalez (see [2], [3], [4]).

Lemma 3.3 *Let $A \in \mathcal{B}(\mathcal{H})^+$, then we have*

- (1) $\mathcal{N}(A) = \mathcal{N}(A^{\frac{1}{2}})$
- (2) $\mathcal{R}(A) \subset \mathcal{R}(A^{\frac{1}{2}}) \subset \overline{\mathcal{R}(A)}$.
- (3) $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$.

Proof.

(1) it is clear that $\mathcal{N}(A^{\frac{1}{2}}) \subset \mathcal{N}(A)$. Conversely, if $\xi \in \mathcal{N}(A)$ then $A^{\frac{1}{2}}\xi \in \mathcal{R}(A^{\frac{1}{2}}) \cap \mathcal{N}(A^{\frac{1}{2}}) = \mathcal{R}(A^{\frac{1}{2}}) \cap \mathcal{R}(A^{\frac{1}{2}})^{\perp} = \{0\}$. Hence, $A^{\frac{1}{2}}\xi = 0$ and $\mathcal{N}(A) = \mathcal{N}(A^{\frac{1}{2}})$.

(2) $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^{\frac{1}{2}})}$ obviously. Let $u = \xi + \eta \in \mathcal{H}$ with $\xi \in \mathcal{N}(A^{\frac{1}{2}})$ and $\eta = \lim_{n \rightarrow \infty} A^{\frac{1}{2}}\eta_n \in \overline{\mathcal{R}(A^{\frac{1}{2}})} = \mathcal{N}(A^{\frac{1}{2}})^{\perp} = \mathcal{N}(A)^{\perp}$. then $A^{\frac{1}{2}}u = \lim_{n \rightarrow \infty} A\eta_n \in \overline{\mathcal{R}(A)}$. Thus, $\mathcal{R}(A^{\frac{1}{2}}) \subset \overline{\mathcal{R}(A)}$.

(3) If $\mathcal{R}(A)$ is closed, (2) $\implies \mathcal{R}(A^{\frac{1}{2}}) = \overline{\mathcal{R}(A)}$. Conversely, if $\mathcal{R}(A^{\frac{1}{2}}) = \overline{\mathcal{R}(A)}$, then for each $\xi \in \mathcal{N}(A)^{\perp}$ there exists $\eta \in \mathcal{N}(A)^{\perp}$ such that $A^{\frac{1}{2}}\xi = A\eta$. Then $A^{\frac{1}{2}}(\xi - A^{\frac{1}{2}}\eta) = 0$ and $\xi = A^{\frac{1}{2}}\eta \in \mathcal{R}(A^{\frac{1}{2}})$. Therefore $\overline{\mathcal{R}(A^{\frac{1}{2}})} \subset \mathcal{R}(A^{\frac{1}{2}})$ and so $A^{\frac{1}{2}}$ has closed range. Now, by hypothesis $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$, then $\overline{\mathcal{R}(A)} \subset \mathcal{R}(A) \subset \overline{\mathcal{R}(A)}$, namely, $\mathcal{R}(A)$ is closed.

Definition 3.2 *T is said to be A -posinormal if there exists a A -positive operator P such that*

$$TAT^* = T^*APT.$$

Here, an operator P is called A -interrupter. The class of all A -posinormal operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{P}_A(\mathcal{H})$. T is A -coposinormal if T^ is A -posinormal.*

Remark 3.2 *Every A -normal operator is an A -posinormal with interrupter is $P = I$.*

Remark 3.3 *Every A -hyponormal operator with A invertible is A -posinormal.*

Example 3.2 Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$ and $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)^+$. By a simple computation we have that $TAT^* \neq T^*AT$. Hence T is not A -normal. On the other hand, consider a operator $P = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. Then $AP \geq 0$ and $TAT^* = T^*APT$. Hence T is A -posinormal.

From Example 3.2 we conclude that A -posinormal operators are not necessary A -normal.

Definition 3.3 We say that $T \in \mathcal{B}(\mathcal{H})$ is

- (1) conditionally totally A -posinormal provided that $T - \lambda I$ is A -posinormal for all $\lambda \in \mathbb{C}$.
- (2) totally A -posinormal if all operators $T - \lambda I$ are A -posinormal for all $\lambda \in \mathbb{C}$ and have a common A -interrupter A -positive operator.

Theorem 3.2 Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$ such that T is A -posinormal. The following statements hold.

- (1) $TAT^* \leq cT^*AT$ for some $c > 0$.
- (2) $\mathcal{R}(TA^{\frac{1}{2}}) \subseteq \mathcal{R}(T^*A^{\frac{1}{2}})$.
- (3) There exists $S \in \mathcal{B}(\mathcal{H})$ such that $TA^{\frac{1}{2}} = T^*A^{\frac{1}{2}}S$.
- (4) There exists a positive operator P such that

$$TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T.$$

Proof. (1) Since T is A -posinormal there is A -positive operator $P \in \mathcal{B}(\mathcal{H})$ such that $TAT^* = T^*APT$.

$$\begin{aligned} TAT^* = T^*APT &\implies \langle AT^*\xi \mid T^*\xi \rangle = \langle APT\xi \mid T\xi \rangle \\ &\implies \langle T^*\xi \mid T^*\xi \rangle_A = \langle PT\xi \mid T\xi \rangle_A \\ &\implies \|T^*\xi\|_A^2 \leq \|PT\xi\|_A \|T\xi\|_A \end{aligned}$$

Since P is A -positive $\|PT\xi\|_A \leq c\|T\xi\|_A$ for some $c > 0$ (see Lemma 3.1)

Hence

$$\|T^*\xi\|_A^2 \leq c\|T\xi\|_A^2.$$

This fact implies that $TAT^* \leq cT^*AT$ and we get that

$$(TA^{\frac{1}{2}})(TA^{\frac{1}{2}})^* \leq c(A^{\frac{1}{2}}T)^*(A^{\frac{1}{2}}T).$$

By Douglas theorem it follows that (2) and (3) hold.

(4) Assume that $TA^{\frac{1}{2}} = T^*A^{\frac{1}{2}}S$ for some $S \in \mathcal{B}(\mathcal{H})$, then

$$TAT^* = T^*A^{\frac{1}{2}}SS^*A^{\frac{1}{2}}T.$$

Put $P = SS^*$, P is positive and satisfy $TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T$.

Proposition 3.1 *Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$, the following statements are equivalent :*

- (1) T is A -posinormal.
- (2) $TAT^* \leq cT^*AT$ for some $c > 0$.
- (3) $\mathcal{R}(TA^{\frac{1}{2}}) \subseteq \mathcal{R}(T^*A^{\frac{1}{2}})$.
- (4) There exists $S \in \mathcal{B}(\mathcal{H})$ such that $TA^{\frac{1}{2}} = T^*A^{\frac{1}{2}}S$.
- (5) There exists a positive operator P such that $TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T$.

Proof. By Theorem 3.2 we have that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

To prove that (5) $\implies$ (1). If $TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T$ we have

$$TAT^* = T^*AA^{-\frac{1}{2}}PA^{\frac{1}{2}}T,$$

As $Q = A^{-\frac{1}{2}}PA^{\frac{1}{2}}$ is A -positive, thus T is A -posinormal.

Corollary 3.1 *Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Then T is A -posinormal if and only if T is posinormal.*

Proof.

$$\begin{aligned} T \text{ is posinormal} &\iff \mathcal{R}(T) \subset \mathcal{R}(T^*) \iff \mathcal{R}(TA^{\frac{1}{2}}) \subset \mathcal{R}(T^*A^{\frac{1}{2}}) \text{ (as } A \text{ invertible)} \\ &\iff T \text{ is } A\text{-posinormal (Proposition 3.1).} \end{aligned}$$

Lemma 3.4 *Let $T, S \in \mathcal{B}(\mathcal{H})$ then the following statements hold*

- (1) If $T \geq S$ then $B^*TB \geq B^*SB$, for all $B \in \mathcal{B}(\mathcal{H})$.
- (2) If range of B is dense in $\mathcal{H}$, then

$$T \geq S \iff B^*TB \geq B^*SB.$$

Proof. (1) Let $T \geq S$. then we get the following relation

$$\langle B^*TB\xi \mid \xi \rangle = \langle TB\xi \mid B\xi \rangle \geq \langle SB\xi \mid B\xi \rangle \geq \langle B^*SB\xi \mid \xi \rangle, \forall \xi \in \mathcal{H}.$$

(2) Let $B^*TB \geq B^*ST$. Then we have

$$\langle B^*TB\xi \mid \xi \rangle \geq \langle B^*SB\xi \mid \xi \rangle \implies \langle TB\xi \mid B\xi \rangle \geq \langle SB\xi \mid \xi \rangle \quad \forall \xi \in \mathcal{H}.$$

Hence $T \geq S$ on $\mathcal{R}(B)$ because B has a dense range in $\mathcal{H}$, we have $T \geq S$ on $\mathcal{H}$.

Corollary 3.2 *If T is A -posinormal with A -interrupter P and $A \geq AP$, then T is A -hyponormal.*

Proof. Since T is A -posinormal we have that $TAT^* = T^*APT$, for some P with $AP \geq 0$. From Lemma 3.4 and the hypothesis $A \geq AP$ we deduce that $T^*AT \geq T^*APT = TAT^*$ and hence, T is A -hyponormal.

In the following theorem we collect some properties of the class $\mathcal{P}_A(\mathcal{H})$.

Theorem 3.3 (1) *If T is of class $\mathcal{P}_A(\mathcal{H})$ then λT is of class $\mathcal{P}_A(\mathcal{H})$.*

(2) *If $T, S \in \mathcal{B}(\mathcal{H})$ such that T is self-adjoint, $TS = ST$ and S is of class $\mathcal{P}_A(\mathcal{H})$ then TS is of class $\mathcal{P}_A(\mathcal{H})$.*

(3) *If A is invertible and T, S are of class $\mathcal{P}_A(\mathcal{H})$ such that T commutes with S and S^* both, then TS is of class $\mathcal{P}_A(\mathcal{H})$.*

(4) *If A is invertible and T is of class $\mathcal{P}_A(\mathcal{H})$ then so any $S \in \mathcal{B}(\mathcal{H})$ that is A -unitary equivalent to T i.e., $S = V^*TV$ where V is A -unitary operator.*

Proof. (1) clear.

(2) Since S is A -posinormal, $SAS^* = S^*APS$. Therefore

$$(TS)A(TS)^* = TSAS^*T = TS^*APST = (TS)^*AP(TS).$$

It follows that TS is A -posinormal.

(3) Since T is A -posinormal there exist a constant $c > 0$ such that $TAT^* \leq cT^*AT$. By Lemma 3.4 we have

$$\begin{aligned} TAT^* \leq cT^*AT &\implies STAT^*S^* \leq cST^*ATS^* \\ &\implies T(SAS^*)T^* \leq cT^*(SAS^*)T \\ &\implies T(SAS^*)T^* \leq cc'T^*S^*AST \text{ as } S \text{ is } A\text{-posinormal} \end{aligned}$$

Hence,

$$TSA(TS)^* \leq cc'(TS)^*A(TS).$$

(4) We have

$$\begin{aligned} SAS^* &= V^*TVAV^*T^*V \\ &= V^*TAT^*V \text{ (as } V \text{ is } A\text{-unitary)} \\ &\leq cV^*(T^*AT)V \text{ (as } T \text{ is } A\text{-posinormal)} \\ &\leq V^*T^*VAV^*TV \\ &\leq cS^*AS. \end{aligned}$$

Remark 3.4 Theorem 3.3 (1) ensures that a complex multiplication of a A -posinormal operator is again A -posinormal (i.e., the class of A -posinormal operators is closed under scalar multiplication).

Since γT is A -posinormal for all $\gamma \geq 0$ whenever T is A -posinormal. It follows that the collection of all A -posinormal operators is a cone in $\mathcal{B}(\mathcal{H})$.

The following example shows that if T is A -posinormal it is not necessary that T^* is A -posinormal.

Example 3.3 Let $\mathcal{H} = l_2(\mathbb{C})$, the unilateral shift operator on $\mathcal{H}$ is defined by $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. It is known that $T^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$, and easily to check that $\mathcal{R}(T) \subset \mathcal{R}(T^*)$ hence T is I -posinormal operator. Clearly that $\mathcal{R}(T) \neq \mathcal{R}(T^*)$, therefore T^* is not I -posinormal.

Proposition 3.2 Let $T \in \mathcal{B}(\mathcal{H})$. If T and T^* are A -posinormal operators then

$$\mathcal{R}(TA^{\frac{1}{2}}) = \mathcal{R}(T^*A^{\frac{1}{2}}).$$

Proof. Since T and T^* are A -posinormal operator $\mathcal{R}(TA^{\frac{1}{2}}) \subseteq \mathcal{R}(T^*A^{\frac{1}{2}})$ and $\mathcal{R}(T^*A^{\frac{1}{2}}) \subseteq \mathcal{R}(TA^{\frac{1}{2}})$. Hence

$$\mathcal{R}(TA^{\frac{1}{2}}) = \mathcal{R}(T^*A^{\frac{1}{2}}).$$

Corollary 3.3 Let $T \in \mathcal{B}(\mathcal{H})$ such that T and T^* are I -posinormal. Then the following properties hold

$$(1) \quad \mathcal{R}(T^n T^*) = \mathcal{R}(T^{n+1}) \text{ and } \mathcal{R}(T^{*n} T) = \mathcal{R}(T^{*n+1}).$$

$$(2) \quad \mathcal{N}(TT^{*n}) = \mathcal{N}(T^{*n+1}) \text{ and } \mathcal{N}(T^* T^n) = \mathcal{N}(T^{n+1}).$$

Proof. (1) Let $\xi \in \mathcal{R}(T^n T^*)$ there exists $\eta \in \mathcal{H}$ such that $\xi = T^n T^* \eta$, but there exists $\eta_1 \in \mathcal{H}$ such that $T^* \eta = T \eta_1$, therefore $\xi = T^n T \eta_1 = T^{n+1} \eta_1$ hence $\xi \in \mathcal{R}(T^{n+1})$. Now let $\xi \in \mathcal{R}(T^{n+1})$ there exists $\eta \in \mathcal{H}$ such that $\xi = T^{n+1} \eta = T^n T \eta$, but there exists $\eta_1 \in \mathcal{H}$ such that $T \eta = T^* \eta_1$, therefore $\xi = T^n T^* \eta_1$ hence $\xi \in \mathcal{R}(T^n T^*)$ and $\mathcal{R}(T^n T^*) = \mathcal{R}(T^{n+1})$.

by similar way we have $\mathcal{R}(T^{*n} T) = \mathcal{R}(T^{*n+1})$.

(2) Since $\mathcal{R}(T^n T^*) = \mathcal{R}(T^{n+1})$ then $\mathcal{R}(T^n T^*)^\perp = \mathcal{R}(T^{n+1})^\perp$ hence $\mathcal{N}((T^n T^*)^*) = \mathcal{N}((T^{n+1})^*)$ so $\mathcal{N}(TT^{*n}) = \mathcal{N}(T^{*n+1})$ by the same way we get $\mathcal{N}(T^* T^n) = \mathcal{N}(T^{n+1})$.

Proposition 3.3 If T is of class $\mathcal{P}_A(\mathcal{H})$, then we have

$$(1) \quad A^{\frac{1}{2}} T A^{\frac{1}{2}} \text{ is posinormal.}$$

$$(2) \quad \text{If } TA = AT, \text{ then } TA \text{ and } TA^{\frac{1}{2}} \text{ are posinormal.}$$

$$(3) \quad \text{If } TA = AT \text{ and the } A\text{-interrupter } P \text{ of } T \text{ is positive then } T \text{ is of class } \mathcal{P}_{A^{\frac{1}{2}}}(\mathcal{H}).$$

Proof. (1) Since T is A -posinormal, by Theorem 3.2 we have $TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T$ with $P \geq 0$. Hence

$$A^{\frac{1}{2}}TA^{\frac{1}{2}}A^{\frac{1}{2}}T^*A^{\frac{1}{2}} = A^{\frac{1}{2}}T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}TA^{\frac{1}{2}}.$$

thus,

$$(A^{\frac{1}{2}}TA^{\frac{1}{2}})(A^{\frac{1}{2}}TA^{\frac{1}{2}})^* = (A^{\frac{1}{2}}TA^{\frac{1}{2}})^*PA^{\frac{1}{2}}TA^{\frac{1}{2}}.$$

It follows that $A^{\frac{1}{2}}TA^{\frac{1}{2}}$ is posinormal.

(2) If $TA = AT$ we have $TA^{\frac{1}{2}} = A^{\frac{1}{2}}T$. Thus TA is posinormal and moreover $\left(TA^{\frac{1}{2}}\right)\left(TA^{\frac{1}{2}}\right)^* = \left(TA^{\frac{1}{2}}\right)^*P\left(TA^{\frac{1}{2}}\right)$, $P \geq 0$. It follows that $TA^{\frac{1}{2}}$ is posinormal.

(3) Since T is A -posinormal and $AT = TA$,

$$A(TT^* - T^*PT) = 0.$$

By Lemma 3.3

$$A^{\frac{1}{2}}(TT^* - T^*PT) = 0,$$

and

$$TA^{\frac{1}{2}}T^* = T^*A^{\frac{1}{2}}PT.$$

From Lemma 3.2, P is $A^{\frac{1}{2}}$ -positive, hence T is of class $\mathcal{P}_{A^{\frac{1}{2}}}(\mathcal{H})$.

Proposition 3.4 *Let $A \in \mathcal{B}(\mathcal{H})_+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$ such that $A^{\frac{1}{2}}TA^{\frac{1}{2}}$ is posinormal then T is A -posinormal.*

Proof. By the assumption there exist a positive operator P such that

$$A^{\frac{1}{2}}TAT^*A^{\frac{1}{2}} = A^{\frac{1}{2}}T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}TA^{\frac{1}{2}}$$

and hence

$$TAT^* = T^*A^{\frac{1}{2}}PA^{\frac{1}{2}}T = T^*AA^{-\frac{1}{2}}PA^{\frac{1}{2}}T.$$

Corollary 3.4 *Given $A, P \in \mathcal{B}(\mathcal{H})^+$ and let $T \in \mathcal{B}(\mathcal{H})$ such that $TA = AT$. Then the following statements are equivalents*

- (1) T is A -posinormal with A -interrupter P .
- (2) T is $A^{\frac{1}{2}}$ -posinormal with $A^{\frac{1}{2}}$ -interrupter P .

Proof. (1) $\implies$ (2) Follows from Proposition 3.3.

(2) $\implies$ (1). $TA^{\frac{1}{2}}T^* = T^*A^{\frac{1}{2}}PT \implies TAT^* = T^*APT$ since $AT = TA$. As $P \geq 0$ and $A^{\frac{1}{2}}$ -positive it is A -positive by Lemma 3.2. Thus T is A -posinormal.

Proposition 3.5 Assume that $A \in \mathcal{B}(\mathcal{H})^+$ has a dense range and let $T \in \mathcal{B}(\mathcal{H})$ such that $A^{\frac{1}{2}}TA^{\frac{1}{2}}$ is posinormal then there exist a constant $c > 0$ for which $TAT^* \leq cT^*AT$.

Proof. Since $A^{\frac{1}{2}}TA^{\frac{1}{2}}$ is posinormal, by Theorem 3.1 there is $c > 0$ such that

$$\left(A^{\frac{1}{2}}TA^{\frac{1}{2}}\right)\left(A^{\frac{1}{2}}TA^{\frac{1}{2}}\right)^* \leq c\left(A^{\frac{1}{2}}TA^{\frac{1}{2}}\right)^*\left(A^{\frac{1}{2}}TA^{\frac{1}{2}}\right)$$

and hence $A^{\frac{1}{2}}TAT^*A^{\frac{1}{2}} \leq cA^{\frac{1}{2}}T^*ATA^{\frac{1}{2}}$. As range of A is dense we have by Lemma 3.4

$$A^{\frac{1}{2}}TAT^*A^{\frac{1}{2}} \leq cA^{\frac{1}{2}}T^*ATA^{\frac{1}{2}} \implies TAT^* \leq cT^*AT$$

Theorem 3.4 If T is A -posinormal with A -interrupter P such T has dense range and A is one to one, then P is unique.

Proof. Assume P_1 and P_2 both serve as A -interrupters for T . Then $T^*AP_1T = TAT^* = T^*AP_2T$. Since T has dense range, T^* is one to one and, consequently, $A(P_1 - P_2)T = 0$. We again apply the fact that T has dense range to conclude that $A(P_1 - P_2) = 0$. Since A is one to one, $P_1 - P_2 = 0$.

Proposition 3.6 ([20]) If T is posinormal, then $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$; in particular $\mathcal{N}(T)$ is a reducing subspace for the posinormal operator T .

Lemma 3.5 Let $A \in \mathcal{B}(\mathcal{H})^+$ is one to one and T is an A -posinormal operator then

- (1) $\mathcal{N}(T) \subset \mathcal{N}(T^*)$
- (2) $\mathcal{N}(T^2) = \mathcal{N}(T)$.
- (3) $\mathcal{N}(TA^{\frac{1}{2}}) \subset \mathcal{N}(T^*A^{\frac{1}{2}})$.

Proof. (1) Let $\xi \in \mathcal{N}(T)$, since T is A -posinormal we have that $TAT^*\xi = 0$, which implies $\|(TA^{\frac{1}{2}})^*\xi\| = 0$, and hence $A^{\frac{1}{2}}T^*\xi = 0$. Thus $T^*\xi \in \mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A)$.

(2) It suffices to show that $\mathcal{N}(T^2) \subset \mathcal{N}(T)$. If $\xi \in \mathcal{N}(T^2)$ then, by (1) $T\xi \in \mathcal{N}(T^*)$ so that $T^*T\xi = 0$ which implies $\|T\xi\|^2 = \langle T^*T\xi \mid \xi \rangle = 0$, and hence $\xi \in \mathcal{N}(T)$.

(3) Let $\xi \in \mathcal{H}$ such that $TA^{\frac{1}{2}}\xi = 0$. Since T is posinormal, $TAT^*A^{\frac{1}{2}}\xi = 0$.

Thus

$$\begin{aligned} (TA^{\frac{1}{2}})(TA^{\frac{1}{2}})^*A^{\frac{1}{2}}\xi = 0 &\implies \|(TA^{\frac{1}{2}})^*A^{\frac{1}{2}}\xi\|^2 = 0 \\ &\implies (TA^{\frac{1}{2}})^*A^{\frac{1}{2}}\xi = 0 \\ &\implies A^{\frac{1}{2}}T^*A^{\frac{1}{2}}\xi = 0. \end{aligned}$$

So, $T^*A^{\frac{1}{2}}\xi \in \mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A)$ and, consequently $T^*A^{\frac{1}{2}}\xi = 0$.

Remark 3.5 If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. If T is in $\mathcal{B}(\mathcal{H})$, then T can be written as a 2×2 matrix with operators entries, $T = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ where $W \in \mathcal{B}(\mathcal{M})$ $X \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$ $Y \in \mathcal{B}(\mathcal{M}; \mathcal{M}^\perp)$, and $Z \in \mathcal{B}(\mathcal{M}^\perp)$ (cf. Conway [10]).

A subspace $\mathcal{M}$ is a reducing subspace for T (or $\mathcal{M}$ reduces T) if it is both T and T^* -invariant (equivalently) if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are T -invariant.

Proposition 3.7 Let $A \in \mathcal{B}(\mathcal{H})^+$ is one to one and T is an A -posinormal operator then $\mathcal{N}(T)$ reduces T .

Proof. It is know that $\mathcal{N}(T)$ is T -invariant. Consider the decomposition $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$. Since $T = \begin{pmatrix} O & X \\ O & Z \end{pmatrix}$ and $T^* = \begin{pmatrix} O & O \\ X^* & Z^* \end{pmatrix}$. For the proof, it suffices to show that $X = O$. We have for all $\xi \in \mathcal{N}(T)$ $T^*(\xi \oplus 0) = 0 \oplus X^*(\xi)$ ($X^* : \mathcal{N}(T) \rightarrow \mathcal{N}(T)^\perp$). Since T is A -posinormal . then $\mathcal{N}(T) \subset \mathcal{N}(T^*)$ by Lemma 3.5, so that $X^* = O$ and hence $X = O$, which implies that $T = \begin{pmatrix} O & O \\ O & Z \end{pmatrix}$ that is $\mathcal{N}(T)$ reduces T .

Remark 3.6 If $A \geq 0$ and invertible then $A^{-1} \geq 0$.

Corollary 3.5 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $AT = TA$ then

$$T \in \mathcal{P}_A(\mathcal{H}) \iff T \in \mathcal{P}_{A^{-1}}(\mathcal{H}).$$

Proof. By using Proposition 3.1 (2) and Lemma 3.4 we have

$$\begin{aligned} TAT^* \leq cT^*AT &\iff A^{-1}\left(TAT^*\right)A^{-1} \leq cA^{-1}\left(T^*AT\right)A^{-1} \\ &\iff TA^{-1}T^* \leq cT^*A^{-1}T. \end{aligned}$$

Proposition 3.8 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible. If T is invertible operator then T and T^{-1} are A -posinormal.

Proof. We have

$$TAT^* = T^*A(A^{-1}(T^*)^{-1}TAT^*T^{-1}).T$$

A direct computation shows that $A^{-1}(T^*)^{-1}TAT^*T^{-1}$ is A -positive. Hence T is A -posinormal.

$$T^{-1}A(T^{-1})^* = (T^{-1})^*A(A^{-1}T^*T^{-1}A(T^{-1})^*T)T^{-1}.$$

By direct computation we show that $A^{-1}T^*T^{-1}A(T^{-1})^*T$ is A -positive, hence T^{-1} is A -posinormal.

Remark 3.7 Since the set of invertible operators from $\mathcal{B}(\mathcal{H})$ is open in $\mathcal{B}(\mathcal{H})$, this shows that the set of A -posinormal operators whenever A is invertible is topologically large.

Corollary 3.6 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$ then $T - \lambda I$ is A -posinormal for $\lambda \notin \sigma(T)$ (spectrum of T).

Corollary 3.7 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Then $(T - \lambda I)$ is A -posinormal for all $\lambda \in \sigma(T)$ if and only if $(T - \lambda I)$ is A -posinormal for all $\lambda \in \mathbb{C}$.

Proof. Assume that $(T - \lambda I)$ is A -posinormal for all $\lambda \in \sigma(T)$ and let $\lambda \notin \sigma(T)$ then $(T - \lambda I)$ is invertible and hence $(T - \lambda I)$ is A -posinormal by Corollary 3.5., the other direction is clear.

The proof of the following corollary is straightforward and will be omitted.

Corollary 3.8 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Assume that T is invertible. If P serves as the A -interrupter for the A -posinormal operator T^* , then P is invertible and $AP^{-1}A^{-1}$ serves as the A^{-1} -interrupter of the A^{-1} -posinormal operator T^{-1} .

Proposition 3.9 Let A and B are positive operators such that $A + B$ is invertible, then

$$\mathcal{P}_A(\mathcal{H}) \cap \mathcal{P}_B(\mathcal{H}) \subset \mathcal{P}_{A+B}(\mathcal{H})$$

Proof. Assume that T is of class $\mathcal{P}_A(\mathcal{H}) \cap \mathcal{P}_B(\mathcal{H})$, then there exists A -positive operator P_1 and B -positive operator P_2 such that

$$TAT^* = T^*AP_1T \quad \text{and} \quad TBT^* = T^*BP_2T.$$

It follows that

$$T(A + B)T^* = TAT^* + TBT^* = T^*AP_1T + T^*BP_2T = T^*(AP_1 + BP_2)T.$$

From the hypothesis we have

$$T(A + B)T^* = T^*(A + B)(A + B)^{-1}(AP_1 + BP_2)T.$$

Since $(A + B)^{-1}(AP_1 + BP_2)$ is $(A + B)$ -positive, T is of class $\mathcal{P}_{A+B}(\mathcal{H})$

Proposition 3.10 Let $T \in \mathcal{P}_A(\mathcal{H})$ with A -interruptor P . If P is invertible then PTP^* is of class $\mathcal{P}_{(P^*)^{-1}AP^{-1}}(\mathcal{H})$ with same interruptor.

Proof. Since T is of class $\mathcal{P}_A(\mathcal{H}) : TAT^* = T^*APT$.

$$PTP^*(P^*)^{-1}AP^{-1}PTP^* = PT^*P^*(P^*)^{-1}AP^{-1}PPTP^*.$$

Since P is invertible- A interruptor of T we have

$$\begin{aligned} \langle (P^*)^{-1}AP^{-1}P\xi | \xi \rangle &= \langle A\xi | P^{-1}\xi \rangle \\ &= \langle AP(P^{-1}\xi) | P^{-1}\xi \rangle \geq 0. \end{aligned}$$

Hence, P is $(P^*)^{-1}AP^{-1}$ -positive.

Theorem 3.5 Assume that T is A -posinormal with A -interruptor P and Q is positive operator satisfying $A \geq QAQ \geq AP$, then the operator $S = QTQ$ is A -hyponormal.

Proof. Put $[S^*, S]_A = S^*AS - SAS^*$ we have

$$\begin{aligned} [S^*, S]_A &= QT^*QAQTQ - QTQAQT^*Q \\ &= QT^*QAQTQ + QT^*APTQ + QTAT^*Q - QTQAQT^*Q \\ &= QT^*(QAQ - AP)TQ + QT(A - QAQ)T^*Q \end{aligned}$$

Therefore

$$\langle [S^*, S]_A u | u \rangle = \langle (QAQ - AP)TQu | TQu \rangle + \langle (A - QAQ)T^*Qu | T^*Qu \rangle \geq 0.$$

Proposition 3.11 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. If T is A -posinormal and normal then T^n is A -posinormal for $n = 1, 2, \dots$

Proof. We use an induction. Clearly, it is true for $n = 1$. Suppose $T^k AT^{*k} \leq c_k T^{*k} AT^k$ for $1 \leq k \leq n$ and $c_k \geq 0$. Then

$$\begin{aligned} T^{n+1} AT^{*n+1} &= T(T^n AT^{*n})T^* \\ &\leq c_n T(T^{*n} AT^n)T^* \\ &\leq c_n T^{*n} (TAT^*)T^n \\ &\leq c_n c_1 T^{*n+1} AT^{n+1}. \end{aligned}$$

Hence T^{n+1} is A -posinormal.

4 Tensor products of A -posinormal operators

Let $\mathcal{H} \overline{\otimes} \mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. Given non-zero

$T, S \in \mathcal{B}(\mathcal{H})$, let $T \otimes S \in \mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{H})$ denote the tensor product on the Hilbert space $\mathcal{H} \overline{\otimes} \mathcal{H}$, when $T \otimes S$ is defined as follows

$$\langle T \otimes S(\xi_1 \otimes \eta_1) | (\xi_2 \otimes \eta_2) \rangle = \langle T\xi_1 | \xi_2 \rangle \langle S\eta_1 | \eta_2 \rangle.$$

The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{B}(\mathcal{H})$, but by no means all of them. Thus, whereas $T \otimes S$ is normal if and only if T and S are normal [16], there exist paranormal operators T and S such that $T \otimes S$ is not paranormal [1]. In [13], Duggal showed that if for non-zero $T, S \in \mathcal{B}(\mathcal{H})$, $T \otimes S$ is p -hyponormal if and only if T and S are p -hyponormal. This result was extended to p -quasihyponormal operators in [19].

In the following study we will prove a necessary and sufficient condition for $T \otimes S$ to be A -posinormal, where T and S are both non-zero operators.

Recall that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and so, by the uniqueness of positive square roots, $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r . From the density of the rationals in the reals, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for every positive real number p . Observe also that

$$A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I).$$

The following elementary results on tensor products of operators will be used often (and without further reference) in the sequel: $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d \neq 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. If T_k and S_k ($k = 1, 2$) are positive operators, then $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d > 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. The proofs to these results are to be found in the papers by Hou [16] and Stochel [31].

Lemma 4.1 *If $T_1 \geq T_2 \geq 0$ and $S_1 \geq S_2 \geq 0$, then $T_1 \otimes S_1 \geq T_2 \otimes S_2 \geq 0$.*

Proof. By Assumptions we have $\langle T_1\xi | \xi \rangle \geq \langle T_2\xi | \xi \rangle \quad \forall \xi \in \mathcal{H}$ and $\langle S_1\eta | \eta \rangle \geq \langle S_2\eta | \eta \rangle \quad \forall \eta \in \mathcal{H}$. Thus

$$\langle T_1\xi | \xi \rangle \langle S_1\eta | \eta \rangle \geq \langle T_2\xi | \xi \rangle \langle S_2\eta | \eta \rangle,$$

and hence

$$\langle T_1 \otimes S_1(\xi \otimes \eta) | \xi \otimes \eta \rangle \geq \langle T_2 \otimes S_2(\xi \otimes \eta) | \xi \otimes \eta \rangle.$$

Proposition 4.1 ([31]) *Let $T_1, T_2, S_1, S_2 \in \mathcal{B}(\mathcal{H})$ be positive operators. If $T_1 \neq 0$ and $S_1 \neq 0$, then the following conditions are equivalent*

- (1) $T_1 \otimes S_1 \leq T_2 \otimes S_2$
- (2) *there exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.*

In the following theorem we generalize theorem 2.4 in [31] to the space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$.

Theorem 4.1 *Let $A, B \in \mathcal{B}(\mathcal{H})^+$. If $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_B(\mathcal{H})$ are nonzero operators, then the following properties hold.*

(1) $T \otimes S$ is $(A \otimes B)$ -quasi-isometry $\iff \alpha T$ is A -quasi-isometry and $\alpha^{-1}S$ is B -quasi-isometry for some constant $\alpha \neq 0$.

(2) $T \otimes S$ is $(A \otimes B)$ -isometric $\iff \alpha T$ is A -isometry and $\alpha^{-1}S$ is B -isometry for some constant $\alpha \neq 0$.

(3) $T \otimes S$ is $(A \otimes B)$ -unitary $\iff \alpha T$ is A -unitary and $\alpha^{-1}S$ is B -unitary for some constant $\alpha \neq 0$.

(4) $T \otimes S$ is $(A \otimes B)$ -selfadjoint $\iff \alpha T$ is A -selfadjoint and $\alpha^{-1}S$ is B -selfadjoint for some constant $\alpha \neq 0$.

(5) $T \otimes S$ is $(A \otimes B)$ -positive $\iff \alpha T$ is A -positive and $\alpha^{-1}S$ is B -positive for some constant $\alpha \neq 0$.

(6) $T \otimes S$ is $(A \otimes B)$ -normal $\iff T$ is A -normal and S is B -normal.

(7) $T \otimes S$ is $(A \otimes B)$ -hyponormal $\iff T$ is A -hyponormal and S is B -hyponormal.

(8) $T \otimes S$ is $(A \otimes B)$ -quasinormal $\iff T$ is A -quasinormal and S is B -quasinormal.

Proof. (1)

$$\begin{aligned}
 & (T \otimes S) \text{ is } (A \otimes B) - \text{quasi-isometry} \\
 \iff & (T \otimes S)^*(A \otimes B)(T \otimes S) = (T \otimes S)^{*2}(A \otimes B)(T \otimes S)^2 \\
 \iff & T^*AT \otimes S^*BS = T^{*2}AT^2 \otimes S^{*2}BS^2 \\
 \iff & \exists d > 0 : T^*AT = dT^{*2}AT^2 \text{ and } S^*BS = d^{-1}S^{*2}BS^2 \\
 \iff & (\sqrt{dT})^*A(\sqrt{dT}) = (\sqrt{dT})^{*2}A(\sqrt{dT})^2 \text{ and} \\
 & (\sqrt{d^{-1}S})^*A(\sqrt{d^{-1}S}) = (\sqrt{d^{-1}S})^{*2}A(\sqrt{d^{-1}S})^2.
 \end{aligned}$$

(2) " $\implies$ " Assume that $T \otimes S$ is $(A \otimes B)$ -isometry, then

$$(T \otimes S)^*(A \otimes B)(T \otimes S) = A \otimes B \implies T^*AT \otimes S^*BS = A \otimes B.$$

Since the operators involved in the above inequalities are positive and nonzero, it follows by Proposition 4.1 that there is a constant $d > 0$ such that

$$T^*AT = dA \text{ and } S^*BS = d^{-1}B.$$

This implies that

$$(\frac{1}{\sqrt{d}}T)^*A(\frac{1}{\sqrt{d}}T) = A \text{ and } (\sqrt{d}S)^*B(\sqrt{d}S) = B,$$

we obtain the desired result. The converse implication is obvious. In the same way, we may deduce (3) , (4) and (5).

(6)

$$\begin{aligned} & (T \otimes S) \text{ is } (A \otimes B) - \text{normal} \\ \iff & (T \otimes S)(T \otimes S)^{(*)_{A \otimes B}} = (T \otimes S)^{(*)_{A \otimes B}}(T \otimes S) \\ \iff & TT^{(*)_A} \otimes SS^{(*)_B} = T^{(*)_A}T \otimes S^{(*)_B}S. \end{aligned}$$

(1) First case: if A or B is injective.

Multiplying the both side of this equality by $(A \otimes B)$ we obtained

$$ATT^{(*)_A} \otimes BSS^{(*)_B} = AT^{(*)_A}T \otimes BS^{(*)_B}S.$$

Since the operators involved in the above equality are positive, it follows that there exists a scalar $d > 0$ such that

$$ATT^{(*)_A} = dAT^{(*)_A}T \text{ and } BSS^{(*)_B} = d^{-1}BS^{(*)_B}S.$$

Hence

$$ATT^{(*)_A} = AdT^{(*)_A}T \text{ and } BSS^{(*)_B} = Bd^{-1}S^{(*)_B}S.$$

Thus,

$$TT^{(*)_A} = dT^{(*)_A}T \text{ and } SS^{(*)_B} = d^{-1}S^{(*)_B}S,$$

and it follows that

$$\|TT^{(*)_A}\|_A = d\|T^{(*)_A}T\| \text{ and } \|SS^{(*)_B}\| = d^{-1}\|S^{(*)_B}S\|.$$

Consequently $d = 1$.

(2) General case: A and B are not necessary injective. There exists a scalar $d \neq 0$ such that

$$TT^{(*)_A} = dT^{(*)_A}T \text{ and } SS^{(*)_B} = d^{-1}S^{(*)_B}S.$$

We deduce that $|d| = |d^{-1}| = 1$ and hence $d = 1$. The desired results are proved.

(7) By Similar argument.

(8)

$$\begin{aligned}
& (T \otimes S) \text{ is } (A \otimes B) - \text{quasinormal} \\
\iff & (T \otimes S)(T \otimes S)^{\langle * \rangle_{A \otimes B}}(T \otimes S) = (T \otimes S)^{\langle * \rangle_{A \otimes B}}(T \otimes S)^2 \\
\iff & TT^{\langle * \rangle_A}T \otimes SS^{\langle * \rangle_B}S = T^{\langle * \rangle}T^2 \otimes S^{\langle * \rangle}S^2 \\
\iff & \exists d \neq 0 : TT^{\langle * \rangle_A}T = dT^{\langle * \rangle_A}T^2 \text{ and } SS^{\langle * \rangle_B}S = d^{-1}S^{\langle * \rangle_B}S^2.
\end{aligned}$$

This in turn implies that

$$(T^{\langle * \rangle_A}T)^2 = d(T^{\langle * \rangle_A})^2T^2 = d(T^2)^{\langle * \rangle_A}T^2$$

and

$$(S^{\langle * \rangle_B}S)^2 = d^{-1}(S^{\langle * \rangle_B})^2S^2 = d^{-1}(S^2)^{\langle * \rangle_B}S^2$$

Consequently

$$\|(T^{\langle * \rangle_A}T)^2\|_A = |d|\|(T^2)^{\langle * \rangle_A}T^2\|_A$$

and

$$\|(S^{\langle * \rangle_B}S)^2\|_B = |d^{-1}|\|(S^2)^{\langle * \rangle_B}S^2\|_B,$$

which yields $d = 1$. Therefore T is A -quasinormal and S is B -quasinormal.

Theorem 4.2 *Let $A, B \in \mathcal{B}(\mathcal{H})^+$ are invertible. Take nonzero T and $S \in \mathcal{B}(\mathcal{H})$. The tensor product $T \otimes S$ is $(A \otimes B)$ -posinormal if and only if T is A -posinormal and S is B -posinormal.*

Proof. Assume that T is A -posinormal and S is B -posinormal. By Theorem 3.1 there are a positive constants α and β such that

$$TAT^* \leq \alpha T^*AT \text{ and } SBS^* \leq \beta S^*BS.$$

Since the operators involved in the above inequalities are positive, it follows that

$$TAT^* \otimes SBS^* \leq \alpha\beta T^*AT \otimes S^*BS$$

(See Lemma 4.1) , and therefore

$$(T \otimes S)(A \otimes B)(T \otimes S)^* \leq \alpha\beta(T \otimes S)^*(A \otimes B)(T \otimes S)$$

so that $T \otimes S$ is $(A \otimes B)$ -posinormal as $(A \otimes B)$ is invertible.

Conversely, if $T \otimes S$ is $(A \otimes B)$ -posinormal, there exists a positive constant d such that

$$(T \otimes S)(A \otimes B)(T \otimes S)^* \leq d(T \otimes S)^*(A \otimes B)(T \otimes S),$$

which means that

$$TAT^* \otimes SBS^* \leq (d^{\frac{1}{2}}T^*AT) \otimes (d^{\frac{1}{2}}S^*BS).$$

Since the operators involved in the above inequalities are positive and nonzero, it follows by Proposition 4.1 that there is a constant $\gamma > 0$ such that

$$TAT^* \leq \gamma d^{\frac{1}{2}}T^*AT \quad \text{and} \quad SBS^* \leq \gamma^{-1}d^{\frac{1}{2}}S^*BS$$

so that T and S are A -posinormal and B -posinormal respectively.

In the general case we have the following generalization

Theorem 4.3 *Let $A_i \in \mathcal{B}(\mathcal{H})^+$ and let $T_i \in \mathcal{B}_{A_i}(\mathcal{H})$ for $i = 1, 2, \dots, n$ and $T_1 \otimes T_2 \otimes \dots \otimes T_n \neq 0$. Then the tensor product $T_1 \otimes T_2 \otimes \dots \otimes T_n$ on the Hilbert space $\mathcal{H} \overline{\otimes} \mathcal{H} \overline{\otimes} \dots \overline{\otimes} \mathcal{H}$ is $(A_1 \otimes A_2 \otimes \dots \otimes A_n)$ -posinormal if and only if T_i is A_i -posinormal operator for $i = 1, 2, \dots, n$.*

Proof. By induction, it suffices to show that $T_1 \otimes T_2$ is $(A_1 \otimes A_2)$ -posinormal if and only if both T_1 is A_1 -posinormal and T_2 is A_2 -posinormal.

Assume that $T_1 \otimes T_2 \neq 0$ is $(A_1 \otimes A_2)$ -posinormal operator, then:

$$(T_1 \otimes T_2)(A_1 \otimes A_2)(T_1 \otimes T_2)^* = (T_1 \otimes T_2)^{(*)}(A_1 \otimes A_2)(P_1 \otimes P_2)(T_1 \otimes T_2).$$

Thus

$$T_1A_1T_1^* \otimes T_2A_2T_2^* = T_1^*A_1P_1T_1 \otimes T_2^*A_2P_2T_2.$$

Since the operators involved in the above equality are positive, it follows that there exists a scalar $d > 0$ such that

$$T_1A_1T_1^* = dT_1^*A_1P_1T_1 \quad \text{and} \quad T_2A_2T_2^* = d^{-1}T_2^*A_2P_2T_2.$$

Hence

$$T_1A_1T_1^* = T_1^*A_1(dP_1)T_1 \quad \text{and} \quad T_2A_2T_2^* = T_2^*A_2(d^{-1}P_2)T_2,$$

Since dP_1 is A_1 -positive and $d^{-1}P_2$ is A_2 -positive, it follows that T_1 is A_1 -posinormal and T_2 is A_2 -posinormal.

Conversely assume that T_i is A_i -posinormal for $i = 1, 2$, then

$$T_1A_1T_1^* = T_1^*A_1P_1T_1 \quad \text{and} \quad T_2A_2T_2^* = T_2^*A_2P_2T_2$$

$$\begin{aligned} (T_1 \otimes T_2)(A_1 \otimes A_2)(T_1 \otimes T_2)^* &= T_1A_1T_1^* \otimes T_2A_2T_2^* \\ &= T_1^*A_1P_1T_1 \otimes T_2^*A_2P_2T_2 \\ &= (T_1 \otimes T_2)^{(*)}(A_1 \otimes A_2)(P_1 \otimes P_2)(T_1 \otimes T_2). \end{aligned}$$

Since $P_1 \otimes P_2$ is $(A_1 \otimes A_2)$ -positive it follows that $T_1 \otimes T_2$ is $(A_1 \otimes A_2)$ -posinormal.

Definition 4.1 Let $T, S \in \mathcal{B}(\mathcal{H})$. The tensor sum of T and S is the transformation $T \boxplus S : \mathcal{H} \overline{\otimes} \mathcal{H} \longrightarrow \mathcal{H} \overline{\otimes} \mathcal{H}$ defined by

$$T \boxplus S = (T \otimes I) + (I \otimes S)$$

which is an operator in $\mathcal{B}(\mathcal{H} \overline{\otimes} \mathcal{H})$.

Remark 4.1 It is easily seen that $T \in \mathcal{B}(\mathcal{H})$ is A -posinormal for some $A \in \mathcal{B}(\mathcal{H})^+$ if and only if $T \otimes I$ is $(A \otimes I)$ -posinormal (resp., $I \otimes T$ is $(I \otimes A)$ -posinormal).

Basic operations with tensor sum of Hilbert space operators are summarized in the next proposition. For its proof see [21].

Proposition 4.2 Let $T, S, T_k, S_k \in \mathcal{B}(\mathcal{H})$ $k = 1, 2$ and $\alpha, \beta \in \mathbb{C}$. The following properties hold:

- (1) $(\alpha + \beta)(T \boxplus S) = \alpha T \boxplus \beta S + \beta T \boxplus \alpha S$
- (2) $(T_1 + T_2) \boxplus (S_1 + S_2) = T_1 \boxplus S_1 + T_2 \boxplus S_2$
- (3) $(T_1 \boxplus S_1)(T_2 \boxplus S_2) = T_1 \otimes S_2 + T_2 \otimes S_1 + T_1 T_2 \boxplus S_1 S_2$
- (4) $(T \boxplus S)^* = T^* \boxplus S^*$.
- (5) $\|T \boxplus S\| \leq \|T\| + \|S\|$.

In the following proposition we generalized the normality of $T \boxplus S$ proved in [20] to hyponormality.

Theorem 4.4 If T and S are hyponormal then $T \boxplus S$ is hyponormal.

Proof. Since T and S are hyponormal we have that $T^*T \geq TT^*$ and $S^*S \geq SS^*$.

In view of the fact that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$$

we have

$$\begin{aligned} (T \boxplus S)(T \boxplus S)^* &= T \otimes S^* + T^* \otimes S + TT^* \boxplus SS^* \\ &= T \otimes S^* + T^* \otimes S + (TT^* \otimes I + I \otimes SS^*) \\ &= T \otimes S^* + T^* \otimes S + (T \otimes I)(T^* \otimes I) + (I \otimes S)(I \otimes S^*) \\ &\leq T \otimes S^* + T^* \otimes S + (T^* \otimes I)(T \otimes I) + (S^* \otimes I)(S \otimes I) \\ &\leq T \otimes S^* + T^* \otimes S + T^*T \boxplus S^*S \\ &\leq (T \boxplus S)^*(T \boxplus S). \end{aligned}$$

It follows that $T \boxplus S$ is hyponormal.

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