Positive-Normal Operators in Semi-Hilbertian Spaces

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Abstract

Given a bounded positive linear operator $A$ on a Hilbert space $\mathcal{H}$ we consider the semi-Hilbertian space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$, where $\langle \xi | \eta \rangle_A := \langle A\xi | \eta \rangle$. In this paper we introduce a class of operators on a semi Hilbertian space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle_A$. We call the elements of this class $A$-positive-normal or $A$-posinormal. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $A$-posinormal if there exists a $A$-positive operator $P \in \mathcal{B}(\mathcal{H})$ (i.e., $AP \geq 0$) such that $TAT^* = T^*APT$. We study some basic properties of these operators. Also we study the relationship between a special case of this class with the other kinds of classes of operators in semi-Hilbertian spaces.

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1 Introduction

We consider a Hilbert space $H$ with an additional semi inner product defined by a positive semidefinite operator $A$; namely $\langle \xi | \eta \rangle_A = \langle A\xi | \eta \rangle$ for every $\xi, \eta \in H$. It must be observed from [2] and [3] that this additional structure induces an adjoint operation. However, this operation is defined for not every bounded linear operator on $H$, unless $A$ is invertible. For those operators $T$ which admit an adjoint with respect to $\langle | \rangle_A$, we choose one, denoted by $T^{(\ast)}_A$, which has similar, but not identical, properties as the classical $T^\ast$. Since not every operator admits an $A$-adjoint and, in case it admits one, it may have many others, then the extensions of normal operator, quasi-normal operators, isometries, unitary, partial isometries, quasi-isometry and $m$-isometry are not trivial. These classes of operators have been generalized to semi-Hilbertian spaces by many authors. Such operators appear in different contexts in [2], [3], [4], [14], [28], [29], [32] and other papers. The aim of this work is to continue this process of generalization to posinormal operators. The inspiration for our investigation comes from [2],[3],[4],[14],[28],[29],[32]. In this paper section 2 contains basic results on $A$-operators. There is also a description of the range inclusion theorem of R. G. Douglas [11], which is a key for some results of this paper. At the end of this section we give some characterizations of $A$-quasinormal operators inspired from [26] and [27]. In section 3 we study the concept of an $A$-posinormal operators and we investigate various structural properties of this class of operators. In the final section of the paper we consider the tensor product of some classes of $A$-operators.

2 Definitions and basic facts about semi-Hilbertian space $(H, \langle | \rangle_A)$.

Along this work $H$ denotes a complex Hilbert space with inner product $\langle | \rangle$. $\mathcal{B}(H)$ is the algebra of all bounded linear operators on $H$, $\mathcal{B}(H)^+$ is the cone of positive (semidefinite) operators of $\mathcal{B}(H)$, i.e., $\mathcal{B}(H)^+ := \{ T \in \mathcal{B}(H) | \langle T\xi | \xi \rangle \geq 0 \ \forall \xi \in H \}$ and $\mathcal{B}(H)_{cr}$ is the subset of $\mathcal{B}(H)$ of all operators with closed range. For every $T \in \mathcal{B}(H)$, $\mathcal{N}(T), \mathcal{R}(T)$ and $\overline{\mathcal{R}(T)}$ stand for, respectively, the null space, the range and the closure of the range of $T$, its adjoint operator by $T^\ast$. In addition, if $T_1, T_2 \in \mathcal{B}(H)$ then $T_1 \geq T_2$ means that $T_1 - T_2 \in \mathcal{B}(H)^+$. Given a closed subspace $S$ of $H$, $P_S$ denotes the orthogonal projection onto $S$. On the other hand, $T^\dagger$ stands for the Moore-Penrose inverse of $T \in \mathcal{B}(H)$.

Given $A \in \mathcal{B}(H)^+$, the functional

$$\langle | \rangle_A : H \times H \longrightarrow \mathbb{C}, \langle \xi | \eta \rangle_A = \langle A\xi | \eta \rangle$$

is a semi-inner product on $H$. By $\| |,| \rangle_A$ we denote the seminorm induced
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by \langle \cdot \mid \cdot \rangle_A, i.e., \|\xi\|_A = (\langle \xi \mid \xi \rangle_A^\frac{1}{2}). Observe that \|\xi\|_A = 0 if and only if \xi \in \mathcal{N}(A). Then \|\cdot\|_A is a norm if and only if \( A \) is an injective operator, and the seminormed space \((H, \|\cdot\|_A)\) is complete if and only if \( \mathcal{R}(A) \) is closed. Moreover, \( \langle \cdot \mid \cdot \rangle_A \) induces a semi-norm on a certain subspace of \( \mathcal{B}(H) \), namely, on the subspace

\[ \{ T \in \mathcal{B}(H) / \exists c > 0 : \|T\xi\|_A \leq c\|\xi\|_A \ \forall \xi \in H \} . \]

In such case it holds

\[
\|T\|_A = \sup_{\xi \in \mathcal{R}(A)} \frac{\|T\xi\|_A}{\|\xi\|_A} = \sup_{\|\xi\|_A \leq 1} \|T\xi\|_A = \sup\{\|T\xi\|_A : \|\xi\|_A = 1 \}
\]

\[
= \inf\{c > 0 : \|T\xi\|_A \leq c\|\xi\|_A, \ \xi \in H \} < \infty .
\]

Moreover

\[
\|T\|_A = \sup\{\langle T\xi \mid \eta \rangle_A ; \ \xi, \eta \in H, : \|\xi\| \leq 1, \|\eta\| \leq 1 \} .
\]

For \( \xi, \eta \in H \), we say that \( \xi \) and \( \eta \) are \( A \)-orthogonal if \( \langle \xi \mid \eta \rangle_A = 0 \). Define

\[ \mathcal{B}_{A^\perp}(H) := \{ T \in \mathcal{B}(H) : \|T\xi\|_A \leq c\|\xi\|_A \ \text{for every} \ \xi \in H \} \]

It is easy to see that \( \mathcal{B}_{A^\perp}(H) \) is a subspace of \( \mathcal{B}(H) \). For more details about the class \( \mathcal{B}_{A^\perp}(H) \) see [2, 3, 4].

Note that given \( T \in \mathcal{B}_{A^\perp}(H) \), in general \( T^* \notin \mathcal{B}_{A^\perp}(H) \) (see [4]).

**Definition 2.1** ([2]) For \( T \in \mathcal{B}(H) \), an operator \( S \in \mathcal{B}(H) \) is called an \( A \)-adjoint of \( T \) if for every \( \xi, \eta \in H \)

\[ \langle T\xi \mid \eta \rangle_A = \langle \xi \mid S\eta \rangle_A , \]

i.e., \( AS = T^*A \); we say that \( T \) is \( A \)-selfadjoint if \( AT = T^*A \).

or which is equivalent, if \( S \) is a solution of the equation \( AX = T^*A \).

**Remark 2.1** The existence of an \( A \)-adjoint operator is not guaranteed. Observe that \( T \) admits an \( A \)-adjoint if and only if the equation \( AX = T^*A \) has solution. This kind of equation can be studied applying the next theorem due to Douglas (for its proof see [11, 14]).

**Theorem 2.1** Let \( A, B \in \mathcal{B}(H) \). The following conditions are equivalents.

1. \( \mathcal{R}(B) \subset \mathcal{R}(A) \).
2. There exists a positive number \( \lambda \) such that \( BB^* \leq \lambda AA^* \).
3. There exists \( C \in \mathcal{B}(H) \) such that \( AC = B \).
If one of these conditions holds then there exists a unique operator $D \in B(H)$ such that $AD = B$ and $\mathcal{R}(D) \subseteq \overline{\mathcal{R}(A^*)}$ and $\mathcal{N}(D) = \mathcal{N}(B)$. Moreover

$$\|D\| = \inf\{\lambda > 0 : BB^* \leq \lambda AA^*\}.$$ 

This solution will be called a reduced solution of the equation $BX = C$.

If we denote by $B_A(H)$ the subalgebra of $B(H)$ of all bounded operators which admit an $A$-adjoint operator then

$$B_A(H) = \{ T \in B(H) : T^* R(A) \subset R(A) \}.$$ 

Furthermore, applying Douglas theorem we can see that

$$B_A^s(H) = \{ T \in B(H) : T^* R(A^s) \subset R(A^s) \} = \{ T \in B(H) : R(A^s T^* A^s) \subseteq R(A) \}.$$ 

The relationship between the above sets is proved in [4].

**Proposition 2.1** Let $A \in B(H)^+$ then $B_A(H) \subseteq B_A^s(H)$.

**Remark 2.2** $B_A(H) \subseteq B_A^s(H)$ for all $s \in (0,1)$. More generally, if $0 < s < s' < 1$ then $B_A^s(H) \subseteq B_A^{s'}(H)$. Moreover, $B_A^s(H) = B_A^{s'}(H)$ if and only if $R(A)$ is closed. See [4].

If an operator equation $BX = C$ has solution then it is easy to see that he distinguished solution of Douglas theorem is given by $B^1C$. Therefore, given $T \in B_A(H)$, if we denote by $T^{(s),A}$ the unique $A$-adjoint operator of $T$ whose range is included in $\overline{R(A)}$ then

$$T^{(s),A} = A^s T^* A.$$ 

In view of Theorem 2.1,

$$AT^{(s),A} = T^* A, \quad R(T^{(s),A}) \subseteq \overline{R(A)} \quad \text{and} \quad N(T^{(s),A}) = N(T^* A).$$

Note that if $S$ is an $A$-adjoint of $T$ then $S = T^{(s),A} + Z$, with $Z \in B(H)$ such that $R(Z) \subset N(A)$.

Observe that if $T$ is $A$-selfadjoint it is does not mean, in general, that $T = T^{(s),A}$. In fact $T = T^{(s),A}$ if and only if $T$ is $A$-selfadjoint and $R(T) \subset \overline{R(A)}$. It is also clear that $T$ has a unique $A$-adjoint (namely $T^{(e),A}$) if and only if $A$ is injective. If this is the case, then we get the equality $(T^{(s),A})^{(s),A} = T$.

In the following proposition we collect some properties of $T^{(s),A}$. For its proof see [2, 3].
Proposition 2.2 Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.

1. $T^{(*)}_A \in \mathcal{B}_A(\mathcal{H})$, $(T^{(*)}_A)^{(s)}_A = P_{\mathcal{R}(A)}TP_{\mathcal{R}(A)}$ and $(T^{(*)}_A)^{(s)}_A = T^{(*)}_A$.
2. If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{(s)}_A = S^{(*)}AT^{(*)}_A$.
3. $T^{(*)}_AT$ and $TT^{(*)}_A$ are $A$-selfadjoint.
4. $\|T\|_A = \|T^{(*)}_A\|_A = \|T^{(*)}AT\|_A = \|TT^{(*)}_A\|_A$.
5. $\|S\|_A = \|T^{(*)}_A\|_A$ for every $S \in \mathcal{B}(\mathcal{H})$ which is an $A$-adjoint of $T$.
6. If $S \in \mathcal{B}_A(\mathcal{H})$ then $\|TS\|_A = \|ST\|_A$.

Nevertheless, $T^{(*)}_A$ is not in general the unique $A$-adjoint of $T$ that realizes the minimal norm.

The following classes of operators are studied in [2, 28, 32].

Any operator $T \in \mathcal{B}(\mathcal{H})$ is

1. $A$-contraction if $\|T\xi\|_A \leq \|\xi\|_A$ for every $\xi \in \mathcal{H}$, or equivalently if $T^*AT \leq A$.
2. $A$-isometry if $T^*AT = A \iff \|T\xi\|_A = \|\xi\|_A \quad \forall \xi \in \mathcal{H}$.
3. $A$-normal if $T^*AT = TAT^* \iff \|T\xi\|_A = \|T^*\xi\|_A \quad \forall \xi \in \mathcal{H}$.
4. $A$-partial isometry if $\|T\xi\|_A = \|\xi\|_A \quad \forall \xi \in N(A^*)^{\perp\perp}$.
5. $A$-unitary if $T^*AT = TAT^* = A \iff \|T^*\xi\|_A = \|T\xi\|_A = \|\xi\|_A \quad \forall \xi \in \mathcal{H}$.
6. $A$-hyponormal if $TAT^* \leq T^*AT \iff \|T^*\xi\|_A \leq \|T\xi\|_A \quad \forall \xi \in \mathcal{H}$.
7. $A$-quasi-isometry if and only if, $T^*AT = T^*AT^2 \iff \|T\|_A = \|T^2\|_A$.
8. $A$-m-isometry if $\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{m-k}AT^{m-k} = 0$

$\iff \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{m-k}\xi\|_A^2 = 0 \quad \forall \xi \in \mathcal{H}$.

The following proposition gives a necessary and sufficient condition for with $T \in \mathcal{B}_A(\mathcal{H})$ belongs to $\mathcal{B}_{A^2}(\mathcal{H})$. For its proof see [24].

Proposition 2.3 (1) If $T \in \mathcal{B}_A(\mathcal{H})$. Then $T \in \mathcal{B}_{A^2}(\mathcal{H})$ if and only if

$$\left(\frac{T^{(*)}_A}{\sqrt{2}}\right)^* \in \mathcal{B}_A(\mathcal{H}).$$

(2) If $T \in \mathcal{B}_A(\mathcal{H})$ is such that $T^* \in \mathcal{B}_A(\mathcal{H})$, then $T^{(*)}_A \in \mathcal{B}_{A^2}(\mathcal{H})$ and

$$T^{(*)}_A \in \mathcal{B}_{A^2}(\mathcal{H}) \quad \text{and} \quad \left(\frac{T^{(*)}_A}{\sqrt{2}}\right)^* = \left(\frac{T^*}{\sqrt{2}}\right)^*, \quad \text{or equivalently} \quad \left(\frac{T^{(*)}_A}{\sqrt{2}}\right)^* = \left[\left(\frac{T^*}{\sqrt{2}}\right)^*\right]^{(*)}_A.$$
Definition 2.2 ([28]) $T \in \mathcal{B}_A(H)$ is an $A$-normal operator if

$$TT^{(s)A} = T^{(s)A}T.$$ 

Definition 2.3 ([29]) An operator $T \in \mathcal{B}_A(H)$ is called $A$-quasinormal if $T$ commutes with $T^{(s)A}T$ i.e.,

$$TT^{(s)A}T = T^{(s)A}T^2.$$

Remark 2.3 Every $A$-normal operator is an $A$-quasinormal operator.

In the following theorem we give conditions under which an $A$-quasinormal operator $T$ is an $A$-normal operator.

Theorem 2.2 Let $T \in \mathcal{B}_A(H)$. The following properties hold

1. If $T$ is invertible $A$-quasinormal then $T$ is $A$-normal.

2. Assume that $N(A)$ is a invariant subspace for $T$, then

(a) If $T$ and $T - I$ are $A$-quasinormal then $T$ is $A$-normal, where $I$ indicates the identity operator.

(b) $T - \lambda I$ is $A$-quasinormal for all $\lambda \in \mathbb{C}$ if and only if $T$ is $A$-normal.

Proof. (1) $TT^{(s)A}T = T^{(s)A}T^2 \implies TT^{(s)A} = T^{(s)A}T$ as $T$ is invertible.

(2) First we see that the condition on $T - I$ implies

$$TT^{(s)A}T - TT^{(s)A} - TP_{\mathcal{R}(A)}T + 2T - T^{(s)A}T + T^{(s)A} - P_{\mathcal{R}(A)} = T^{(s)A}T^2 - 2T^{(s)A}T + T^{(s)A} - P_{\mathcal{R}(A)}T^2 + 2PT - P_{\mathcal{R}(A)}.$$ 

Therefore, since $T$ is $A$-quasinormal, we have $TT^{(s)A} = T^{(s)A}T$.

(3) Assume that $T - \lambda I$ is $A$-quasisinormal for all $\lambda \in \mathbb{C}$, we have

$$(T - \lambda I)(T - \lambda I)^{(s)A}(T - \lambda I) = (T - \lambda I)^{(s)A}(T - \lambda I)^2, \ \forall \lambda \in \mathbb{C}$$

which implies that

$$TT^{(s)A}T - T^{(s)A}T^2 - \lambda(TT^{(s)A} - T^{(s)A}T) = 0, \ \forall \lambda \in \mathbb{C}.$$ 

We deduce that $T$ is $A$-normal.

The following proposition gives a characterization of an $A$-quasinormal operator.
Proposition 2.4 Let $T \in B_A(\mathcal{H})$, $X = T + T^{(*)}T$ and $Y = T - T^{(*)}T$. Then

(1) $T$ is $A$-quasinormal if and only if $X$ commutes with $Y$.

(2) If $T$ is $A$-quasinormal then $TT^{(*)}T$ commutes with $X$ and $Y$.

Proof. (1) A simple computation shows that

$$XY = YX \iff TT^{(*)}T = T^{(*)}T^2.$$ 

(2) By the hypothesis

$$TT^{(*)}T(T \pm T^{(*)}T) = (T \pm T^{(*)}T)TT^{(*)}T.$$ 

3 Classes of operators on semi-Hilbertian spaces (A-posinormal operators)

In this section we define the class of $A$-posinormal operators and give some equivalent relation about it.

An operator $T \in B(\mathcal{H})$ is said to be posinormal (the word 'posinormal' stands for positive-normal) if there exists a $P \in B(\mathcal{H})^+$ such that $TT^* = T^*PT$. Or equivalently, $T \in B(\mathcal{H})$ is posinormal if there exists a co-isometry $V^* \in B(\mathcal{H})$ a positive operator $P \in B(H)$ such that $T = T^*PV^*$. The operator $T$ is said to be conditionally totally posinormal, shortened to $T \in CTP(\mathcal{H})$, if $T - \lambda I$ is posinormal for all $\lambda \in \mathbb{C}$, and is totally posinormal, shortened to $T \in TP(\mathcal{H})$, if all operators $T - \lambda I$, $\lambda \in \mathbb{C}$, are posinormal and have a common interrupter positive operator. The class of posinormal operators contains in particular, the classes consisting of hyponormal operators ($TT^* \leq T^*T$), $M$-hyponormal ($|(T - \lambda I)^*|^2 \leq M|(T - \lambda I)|^2$ for some real $M > 0$). It is known that $T \in CTP(\mathcal{H})$ if and only if it is dominant operators ($|(T - \lambda I)^*|^2 \leq M\lambda|(T - \lambda I)|^2$) for some real number $M > 0$ and all complex number $\lambda$.

Posinormal operators were first introduced and studied by H. C. Rhaly [30] and have also been studied by some authors; see, for instance, the papers by M. Itoh [18] and by I. H. Jeon, S. H. Kim, E. Ko [17], S. Mecheri [25] and B. P. Duggal and C. Kubrusly [12], A. Bucur [6].

An operator $T$ is said to be $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$, and $p$-posinormal if $c^2(T^*T)^p \geq (TT^*)^p$ for some $c > 0$. It is clear that 1-hyponormal and 1-posinormal are hyponormal and posinormal, respectively. For a positive integer $k$ and a positive number $0 < p \leq 1$, An operator $T$ is said to be

(1) $(p, k)$-quasihyponormal if

$$T^k((T^*T)^p - (TT^*)^p)T^k \geq 0,$$
(2) \((p,k)\)-quasiposinormal if
\[
T^k \left( c(T^*T)^p - (TT^*)^p \right) T^k \geq 0 \quad \text{for some} \; c > 0.
\]

It is easy that every \((p,1)\)-quasihyponormal is \(p\)-quasihyponormal and \((p,1)\)-quasiposinormal is \(p\)-quasiposinormal. By the definition, it is clear that
\[
p - \text{hyponormal} \subset p - \text{posinormal} \subset (p,k) - \text{quasiposinormal}.
\]

\(p\)-Hyponormal, \(p\)-posinormal, \(p\)-quasihyponormal, and \((p,k)\)-quasihyponormal operators have been studied by many authors see for instance [19], [22] and references therein. and it is known that hyponormal operators have many interesting properties similar to those of normal operators( See [6], [7], [8]).

**Theorem 3.1** ([18]) For \(T \in \mathcal{B}(\mathcal{H})\), the following statements are equivalent:

1. \(T\) is posinormal
2. \(\mathcal{R}(T) \subset \mathcal{R}(T^*)\)
3. \(TT^* \leq \lambda^2 T^*T\) for some \(\lambda > 0\); and
4. there exists \(S \in \mathcal{B}(\mathcal{H})\) such that \(T = T^*S\).

Moreover, if (1),(2),(3) and (4) hold, then there is a unique operator \(S\) such that

(a) \(||S|| = \inf\{\mu | TT^* \leq \mu T^*T\}\)
(b) \(\mathcal{N}(T) = \mathcal{N}(S)\); and
(c) \(\mathcal{R}(S) \subset \overline{\mathcal{R}(T)}\).

**Definition 3.1** We say that \(T \in \mathcal{B}(\mathcal{H})\) is an \(A\)-positive if \(AT \in \mathcal{B}(\mathcal{H})^+\) or equivalently
\[
\langle T\xi | \xi \rangle_A \geq 0 \quad \forall \xi \in \mathcal{H}.
\]

**Example 3.1** If \(T \in \mathcal{B}_A(\mathcal{H})\), then \(T^{(\star)A}\) and \(TT^{(\star)A}\) are \(A\)-positive.

**Remark 3.1** An operator \(T\) is \(A\)-positive if and only if \(A^\dagger T\) is \(A^\dagger\)-positive.

**Lemma 3.1** ([15]) Let \(A \in \mathcal{B}(\mathcal{H})^+\) and \(T \in \mathcal{B}(\mathcal{H})\). The following assertions are equivalent:

1. \(T\) is \(A\)-positive operator.
2. \(T \in \mathcal{B}_{A^\dagger}(\mathcal{H})\) and \(A^\dagger T(A^\dagger)^\dagger \in \mathcal{B}(\mathcal{H})^+\) where \(A^\dagger T(A^\dagger)^\dagger\) denotes the unique bounded linear extension of \(A^\dagger T(A^\dagger)^\dagger\) on \(\mathcal{B}(\mathcal{H})\).
Lemma 3.2 ([15]) Let $A, T \in \mathcal{B}(\mathcal{H})^+$. The following assertions are equivalent:

1. $T$ is an $A$-positive operator ,
2. $T$ is an $A^{\frac{1}{2}}$-positive operator.

The following Lemma is inspired from the work of M.C. Gonzalez (see [2], [3], [4]).

Lemma 3.3 Let $A \in \mathcal{B}(\mathcal{H})^+$, then we have

1. $\mathcal{N}(A) = \mathcal{N}(A^{\frac{1}{2}})$
2. $\mathcal{R}(A) \subset \mathcal{R}(A^{\frac{1}{2}}) \subset \overline{\mathcal{R}(A)}$.
3. $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$.

Proof.

1. It is clear that $\mathcal{N}(A^{\frac{1}{2}}) \subset \mathcal{N}(A)$. Conversely, if $\xi \in \mathcal{N}(A)$ then $A^{\frac{1}{2}}\xi \in R(A^{\frac{1}{2}}) \cap \mathcal{N}(A^{\frac{1}{2}}) = \mathcal{R}(A^{\frac{1}{2}}) \cap \mathcal{R}(A^{\frac{1}{2}}) \perp = \{0\}$. Hence, $A^{\frac{1}{2}}\xi = 0$ and $\mathcal{N}(A) = \mathcal{N}(A^{\frac{1}{2}})$.

2. $\mathcal{R}(A) \subset \mathcal{R}(A^{\frac{1}{2}})$ obviously. Let $u = \xi + \eta \in \mathcal{H}$ with $\xi \in \mathcal{N}(A^{\frac{1}{2}})$ and $\eta = \lim_{n \to \infty} A^{\frac{1}{2}}\eta_n \in \overline{\mathcal{R}(A^{\frac{1}{2}})} = \mathcal{N}(A^{\frac{1}{2}}) \perp = \mathcal{N}(A) \perp$. then $A^{\frac{1}{2}}u = \lim_{n \to \infty} A\eta_n \in \overline{\mathcal{R}(A)}$. Thus, $\mathcal{R}(A^{\frac{1}{2}}) \subset \overline{\mathcal{R}(A)}$.

3. If $\mathcal{R}(A)$ is closed, $(2) \implies \mathcal{R}(A^{\frac{1}{2}}) = \overline{\mathcal{R}(A)}$. Conversely, if $\mathcal{R}(A^{\frac{1}{2}}) = \overline{\mathcal{R}(A)}$, then for each $\xi \in \mathcal{N}(A) \perp$ there exists $\eta \in \mathcal{N}(A) \perp$ such that $A^{\frac{1}{2}}\xi = A\eta$. Then $A^{\frac{1}{2}}(\xi - A^{\frac{1}{2}}\eta) = 0$ and $\xi = A^{\frac{1}{2}}\eta \in \mathcal{R}(A^{\frac{1}{2}})$. Therefore $\mathcal{R}(A^{\frac{1}{2}}) \subset \mathcal{R}(A^{\frac{1}{2}})$ and so $A^{\frac{1}{2}}$ has closed range. Now, by hypothesis $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$, then $\overline{\mathcal{R}(A)} \subset \mathcal{R}(A) \subset \overline{\mathcal{R}(A)}$, namely, $\mathcal{R}(A)$ is closed.

Definition 3.2 $T$ is said to be $A$-posinormal if there exists a $A$-positive operator $P$ such that

$$TAT^* = T^*APT.$$ 

Here, an operator $P$ is called $A$-interrupter. The class of all $A$-posinormal operators in $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{P}_A(\mathcal{H})$. $T$ is $A$-coposinormal if $T^*$ is $A$-posinormal.

Remark 3.2 Every $A$-normal operator is an $A$-posinormal with interrupter is $P = I$.

Remark 3.3 Every $A$-hyponormal operator with $A$ invertible is $A$-posinormal.
Example 3.2 Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$ and $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)^+$. By a simple computation we have that $TAT^* \neq T^*AT$. Hence $T$ is not A-normal. On the other hand, consider a operator $P = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. Then $AP \geq 0$ and $TAT^* = T^*APT$. Hence $T$ is A-posinormal.

From Example 3.2 we conclude that A-posinormal operators are not necessary A-normal.

Definition 3.3 We say that $T \in \mathcal{B}(\mathcal{H})$ is

1. conditionally totally A-posinormal provided that $T - \lambda I$ is A-posinormal for all $\lambda \in \mathbb{C}$.
2. totally A-posinormal if all operators $T - \lambda I$ are A-posinormal for all $\lambda \in \mathbb{C}$ and have a common A-interrupter A-positive operator.

Theorem 3.2 Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$ such that $T$ is A-posinormal. The following statements hold.

1. $TAT^* \leq cT^*AT$ for some $c > 0$.
2. $\mathcal{R}(TA^\frac{1}{2}) \subseteq \mathcal{R}(T^*A^\frac{1}{2})$.
3. There exists $S \in \mathcal{B}(\mathcal{H})$ such that $TA^\frac{1}{2} = T^*A^\frac{1}{2}S$.
4. There exists a positive operator $P$ such that $TAT^* = T^*A^\frac{1}{2}PA^\frac{1}{2}T$.

Proof. (1) Since $T$ is A-posinormal there is A-positive operator $P \in \mathcal{B}(\mathcal{H})$ such that $TAT^* = T^*APT$.

$$TAT^* = T^*APT \implies \langle AT^*\xi \mid T^*\xi \rangle = \langle APT\xi \mid T\xi \rangle$$
$$\implies \langle T^*\xi \mid T^*\xi \rangle_A = \langle PT\xi \mid T\xi \rangle_A$$
$$\implies \|T^*\xi\|_A^2 \leq \|PT\xi\|_A\|T\xi\|_A$$

Since $P$ is A-positive $\|PT\xi\|_A \leq c\|T\xi\|_A$ for some $c > 0$ (see Lemma 3.1)

Hence

$$\|T^*\xi\|_A^2 \leq c\|T\xi\|_A^2.$$ This fact implies that $TAT^* \leq cT^*AT$ and we get that

$$(TA^\frac{1}{2})(TA^\frac{1}{2})^* \leq c(A^\frac{1}{2}T)^*(A^\frac{1}{2}T).$$

By Douglas theorem it follows that (2) and (3) hold.
(4) Assume that $T A^{\frac{1}{2}} = T^* A^{\frac{1}{2}} S$ for some $S \in \mathcal{B}(\mathcal{H})$, then

$$T A^* = T^* A^{\frac{1}{2}} S S^* A^{\frac{1}{2}} T.$$ 

Put $P = SS^*$, $P$ is positive and satisfy $T A^* = T^* A^{\frac{1}{2}} P A^{\frac{1}{2}} T$.

**Proposition 3.1** Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$, the following statements are equivalent:

(1) $T$ is $A$-posinormal.

(2) $T A^* \leq c T^* A T$ for some $c > 0$.

(3) $\mathcal{R}(T A^{\frac{1}{2}}) \subseteq \mathcal{R}(T^* A^{\frac{1}{2}})$.

(4) There exists $S \in \mathcal{B}(\mathcal{H})$ such that $T A^{\frac{1}{2}} = T^* A^{\frac{1}{2}} S$.

(5) There exists a positive operator $P$ such that $T A^* = T^* A^{\frac{1}{2}} P A^{\frac{1}{2}} T$.

**Proof.** By Theorem 3.2 we have that $(1) \implies (2) \implies (3) \implies (4) \implies (5)$.

To prove that $(5) \implies (1)$. If $T A^* = T^* A^{\frac{1}{2}} P A^{\frac{1}{2}} T$ we have

$$T A^* = T^* A A^{-\frac{1}{2}} P A^{\frac{1}{2}} T,$$

As $Q = A^{-\frac{1}{2}} P A^{\frac{1}{2}}$ is $A$-positive, thus $T$ is $A$-posinormal.

**Corollary 3.1** Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Then $T$ is $A$-posinormal if and only if $T$ is posinormal.

**Proof.**

$T$ is posinormal $\iff \mathcal{R}(T) \subseteq \mathcal{R}(T^*) \iff \mathcal{R}(T A^{\frac{1}{2}}) \subseteq \mathcal{R}(T^* A^{\frac{1}{2}})$ (as $A$ invertible) $\iff T$ is $A$ - posinormal (Proposition 3.1).

**Lemma 3.4** Let $T, S \in \mathcal{B}(\mathcal{H})$ then the following statements hold

(1) If $T \geq S$ then $B^* T B \geq B^* S B$, for all $B \in \mathcal{B}(\mathcal{H})$.

(2) If range of $B$ is dense in $\mathcal{H}$, then

$$T \geq S \iff B^* T B \geq B^* S B.$$

**Proof.** (1) Let $T \geq S$. then we get the following relation

$$\langle B^* T B \xi | \xi \rangle = \langle T B \xi | B \xi \rangle \geq \langle S B \xi | B \xi \rangle \geq \langle B^* S B \xi | \xi \rangle, \forall \xi \in \mathcal{H}.$$

(2) Let $B^* T B \geq B^* S T$. Then we have

$$\langle B^* T B \xi | \xi \rangle \geq \langle B^* S B \xi | \xi \rangle \implies \langle T B \xi | B \xi \rangle \geq \langle S B \xi | \xi \rangle, \forall \xi \in \mathcal{H}.$$

Hence $T \geq S$ on $\mathcal{R}(B)$ because $B$ has a dense range in $\mathcal{H}$, we have $T \geq S$ on $\mathcal{H}$.
Corollary 3.2 If $T$ is $A$-posinormal with $A$-interrupter $P$ and $A \geq AP$, then $T$ is $A$-hyponormal.

Proof. Since $T$ is $A$-posinormal we have that $TAT^* = T^*APT$, for some $P$ with $AP \geq 0$. From Lemma 3.4 and the hypothesis $A \geq AP$ we deduce that $T^*AT \geq T^*APT = TAT^*$ and hence, $T$ is $A$-hyponormal.

In the following theorem we collect some properties of the class $\mathcal{P}_A(\mathcal{H})$.

Theorem 3.3 (1) If $T$ is of class $\mathcal{P}_A(\mathcal{H})$ then $\lambda T$ is of class $\mathcal{P}_A(\mathcal{H})$.

(2) If $T, S \in \mathcal{B}(\mathcal{H})$ such that $T$ is self-adjoint, $TS = ST$ and $S$ is of class $\mathcal{P}_A(\mathcal{H})$ then $TS$ is of class $\mathcal{P}_A(\mathcal{H})$.

(3) If $A$ is invertible and $T, S$ are of class $\mathcal{P}_A(\mathcal{H})$ such that $T$ commutes with $S$ and $S^*$ both, then $TS$ is of class $\mathcal{P}_A(\mathcal{H})$.

(4) If $A$ is invertible and $T$ is of class $\mathcal{P}_A(\mathcal{H})$ then so any $S \in \mathcal{B}(\mathcal{H})$ that is $A$-unitary equivalent to $T$ i.e., $S = V^*TV$ where $V$ is $A$-unitary operator.

Proof. (1) clear.

(2) Since $S$ if $A$-posinormal, $SAS^* = S^*APS$. Therefore

$$(TS)A(TS)^* = TSAS^*T = TS^*APST = (TS)^*AP(TS).$$

It follows that $TS$ is $A$-posinormal.

(3) Since $T$ is $A$-posinormal these exist a constant $c > 0$ such that $TAT^* \leq cT^*AT$. By Lemma 3.4 we have

$$TAT^* \leq cT^*AT \implies STAT^*S^* \leq cST^*ATS^*$$
$$\implies T(SAS^*)T^* \leq cT^*(SAS^*)T$$
$$\implies T(SAS^*)T^* \leq cc'T^*S^*AST$$
as $S$ is $A$-posinormal

Hence,

$$TSA(TS)^* \leq cc'(TS)^*A(TS).$$

(4) We have

$$SAS^* = V^*TVAV^*T^*V$$
$$= V^*TAT^*V \text{ (as } V \text{ is } A\text{-unitary})$$
$$\leq cV^*(T^*AT)V \text{ (as } T \text{ is } A\text{-posinormal})$$
$$\leq V^*T^*VAV^*TV$$
$$\leq cS^*AS.$$
Remark 3.4 Theorem 3.3 (1) ensures that a complex multiplication of a $A$-posinormal operator is again $A$-posinormal (i.e., the class of $A$-posinormal operators is closed under scalar multiplication).

Since $\gamma T$ is $A$-posinormal for all $\gamma \geq 0$ whenever $T$ is $A$-posinormal. It follows that the collection of all $A$-posinormal operators is a cone in $B(\mathcal{H})$.

The following example shows that if $T$ is $A$-posinormal it is not necessary that $T^*$ is $A$-posinormal.

Example 3.3 Let $\mathcal{H} = l_2(\mathbb{C})$, the unilateral shift operator on $\mathcal{H}$ is defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$. It is know that $T^*(x_1, x_2, ...) = (x_2, x_3, ...)$, and easily to check that $\mathcal{R}(T) \subset \mathcal{R}(T^*)$ hence $T$ is $I$-posinormal operator. Clearly that $\mathcal{R}(T) \neq \mathcal{R}(T^*)$, therefore $T^*$ is not $I$-posinormal.

Proposition 3.2 Let $T \in B(\mathcal{H})$. If $T$ and $T^*$ are $A$-posinormal operators then

$$\mathcal{R}(TA^\frac{1}{2}) = \mathcal{R}(T^*A^\frac{1}{2}).$$

Proof. Since $T$ and $T^*$ are $A$-posinormal operator $\mathcal{R}(TA^\frac{1}{2}) \subseteq \mathcal{R}(T^*A^\frac{1}{2})$ and $\mathcal{R}(T^*A^\frac{1}{2}) \subseteq \mathcal{R}(TA^\frac{1}{2})$. Hence

$$\mathcal{R}(TA^\frac{1}{2}) = \mathcal{R}(T^*A^\frac{1}{2}).$$

Corollary 3.3 Let $T \in B(\mathcal{H})$ such that $T$ and $T^*$ are $I$-posinormal. Then the following properties hold

1. $\mathcal{R}(T^nT^*) = \mathcal{R}(T^{n+1})$ and $\mathcal{R}(T^*T^n) = \mathcal{R}(T^{n+1}).$

2. $\mathcal{N}(TT^*) = \mathcal{N}(TT^n)$ and $\mathcal{N}(T^*T^n) = \mathcal{N}(T^{n+1}).$

Proof. (1) Let $\xi \in \mathcal{R}(T^nT^*)$ there exists $\eta \in \mathcal{H}$ such that $\xi = T^nT^*\eta$, but there exists $\eta_1 \in \mathcal{H}$ such that $T^*\eta = T_1\eta$, therefore $\xi = T^nT_1\eta = T^{n+1}\eta_1$ hence $\xi \in \mathcal{R}(T^{n+1})$. Now let $\xi \in \mathcal{R}(T^nT^*)$ there exists $\eta \in \mathcal{H}$ such that $\xi = T^nT^*\eta$, but there exists $\eta_1 \in \mathcal{H}$ such that $T\eta = T^*\eta_1$, therefore $\xi = T^nT^*\eta_1$ hence $\xi \in \mathcal{R}(T^nT^*)$ and $\mathcal{R}(T^nT^*) = \mathcal{R}(T^{n+1}).$

by similar way we have $\mathcal{R}(T^nT) = \mathcal{R}(T^{n+1}).$

(2) Since $\mathcal{R}(T^nT^*) = \mathcal{R}(T^{n+1})$ then $\mathcal{R}(T^nT^*)^\perp = \mathcal{R}(T^{n+1})^\perp$ hence $\mathcal{N}((T^nT^*)^*) = \mathcal{N}(T^{n+1})$ so $\mathcal{N}(TT^*) = \mathcal{N}(T^{n+1})$ by the same way we get $\mathcal{N}(T^*T^n) = \mathcal{N}(T^{n+1}).$

Proposition 3.3 If $T$ is of class $\mathcal{P}_A(\mathcal{H})$, then we have

1. $A^\frac{1}{2}TA^\frac{1}{2}$ is posinormal.

2. If $TA = AT$, then $TA$ and $TA^\frac{1}{2}$ are posinormal.

3. If $TA = AT$ and the $A$-interrupter $P$ of $T$ is positive then $T$ is of class $\mathcal{P}_{A^\frac{1}{2}}(\mathcal{H})$. 

**Proof.** (1) Since $T$ is $A$-posinormal, by Theorem 3.2 we have $TAT^* = T^*A^\frac{1}{2}PA^\frac{1}{2}T$ with $P \geq 0$. Hence

$$A^\frac{1}{2}T^*A^\frac{1}{2}T^*A^\frac{1}{2} = A^\frac{1}{2}T^*A^\frac{1}{2}PA^\frac{1}{2}TA^\frac{1}{2}.$$  

Thus,

$$(A^\frac{1}{2}T^\frac{1}{2}A^\frac{1}{2})(A^\frac{1}{2}T^\frac{1}{2}A^\frac{1}{2})^* = (A^\frac{1}{2}T^\frac{1}{2}A^\frac{1}{2})^*PA^\frac{1}{2}TA^\frac{1}{2}.$$  

It follows that $A^\frac{1}{2}TA^\frac{1}{2}$ is posinormal.

(2) If $TA = AT$ we have $TA^\frac{1}{2} = A^\frac{1}{2}T$. Thus $TA$ is posinormal and moreover

$$(TA^\frac{1}{2})^* = P(TA^\frac{1}{2}), \quad P \geq 0.$$  

It follows that $TA^\frac{1}{2}$ is posinormal.

(3) Since $T$ is $A$-posinormal and $AT = TA$,

$$A(TT^* - T^*PT) = 0.$$  

By Lemma 3.3

$$A^\frac{1}{2}(TT^* - T^*PT) = 0,$$  

and

$$TA^\frac{1}{2}T^* = T^*A^\frac{1}{2}PT.$$  

From Lemma 3.2, $P$ is $A^\frac{1}{2}$-positive, hence $T$ is of class $\mathcal{P}_{A^\frac{1}{2}}(\mathcal{H})$.

**Proposition 3.4** Let $A \in \mathcal{B}(\mathcal{H})_+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$ such that $A^\frac{1}{2}TA^\frac{1}{2}$ is posinormal then $T$ is $A$-posinormal.

**Proof.** By the assumption there exist a positive operator $P$ such that

$$A^\frac{1}{2}TAT^*A^\frac{1}{2} = A^\frac{1}{2}T^*A^\frac{1}{2}PA^\frac{1}{2}TA^\frac{1}{2}$$  

and hence

$$TAT^* = T^*A^\frac{1}{2}PA^\frac{1}{2}T = T^*AA\frac{1}{2}PA^\frac{1}{2}T.$$  

**Corollary 3.4** Given $A, P \in \mathcal{B}(\mathcal{H})_+$ and let $T \in \mathcal{B}(\mathcal{H})$ such that $TA = AT$. Then the following statements are equivalents

(1) $T$ is $A$-posinormal with $A$-interrupter $P$.

(2) $T$ is $A^\frac{1}{2}$-posinormal with $A^\frac{1}{2}$-interrupter $P$.

**Proof.** (1) $\implies$ (2) Follows from Proposition 3.3.

(2) $\implies$ (1) $TA^\frac{1}{2}T^* = T^*A^\frac{1}{2}PT \implies TAT^* = T^*APT$ since $AT = TA$. As $P \geq 0$ and $A^\frac{1}{2}$-positive it is $A$-positive by Lemma 3.2. Thus $T$ is $A$-posinormal.
Proposition 3.5 Assume that $A \in \mathcal{B}(\mathcal{H})^+$ has a dense range and let $T \in \mathcal{B}(\mathcal{H})$ such that $A^{\frac{1}{2}}T A^{\frac{1}{2}}$ is posinormal then there exist a constant $c > 0$ for which $T A T^* \leq c T^* A T$.

Proof. Since $A^{\frac{1}{2}}T A^{\frac{1}{2}}$ is posinormal, by Theorem 3.1 there is $c > 0$ such that

$$\left( A^{\frac{1}{2}}T A^{\frac{1}{2}} \right) \left( A^{\frac{1}{2}}T A^{\frac{1}{2}} \right)^* \leq c \left( A^{\frac{1}{2}}T A^{\frac{1}{2}} \right)^* \left( A^{\frac{1}{2}}T A^{\frac{1}{2}} \right)$$

and hence $A^{\frac{1}{2}}T A T^* A^{\frac{1}{2}} \leq c A^{\frac{1}{2}}T^* A T A^{\frac{1}{2}}$. As range of $A$ is dense we have by Lemma 3.4

$$A^{\frac{1}{2}}T A T^* A^{\frac{1}{2}} \leq c A^{\frac{1}{2}}T^* A T A^{\frac{1}{2}} \implies T A T^* \leq c T^* A T$$

Theorem 3.4 If $T$ is $A$-posinormal with $A$-interrupter $P$ such $T$ has dense range and $A$ is one to one, then $P$ is unique.

Proof. Assume $P_1$ and $P_2$ both serve as $A$-interrupters for $T$. Then $T^* A P_1 T = T A T^* = T^* A P_2 T$. Since $T$ has dense range, $T^*$ is one to one and, consequently, $A(P_1 - P_2)T = 0$. We again apply the fact that $T$ has dense range to conclude that $A(P_1 - P_2) = 0$. Since $A$ is one to one, $P_1 - P_2 = 0$.

Proposition 3.6 [(20)] If $T$ is posinormal, then $\mathcal{N}(T) \subseteq \mathcal{N}(T^*)$; in particular $\mathcal{N}(T)$ is a reducing subspace for the posinormal operator $T$.

Lemma 3.5 Let $A \in \mathcal{B}(\mathcal{H})^+$ is one to one and $T$ is an $A$-posinormal operator then

1. $\mathcal{N}(T) \subset \mathcal{N}(T^*)$
2. $\mathcal{N}(T^2) = \mathcal{N}(T)$.
3. $\mathcal{N}(TA^{\frac{1}{2}}) \subset \mathcal{N}(T^* A^{\frac{1}{2}})$.

Proof. (1) Let $\xi \in \mathcal{N}(T)$, since $T$ is $A$-posinormal we have that $T A T^* \xi = 0$, which implies $\| (T A^{\frac{1}{2}})^* \xi \| = 0$, and hence $A^{\frac{1}{2}} T^* \xi = 0$. Thus $T^* \xi \in \mathcal{N}(A^{\frac{1}{2}}) = \mathcal{N}(A)$.

(2) It suffices to show that $\mathcal{N}(T^2) \subset \mathcal{N}(T)$. If $\xi \in \mathcal{N}(T^2)$ then, by (1) $T \xi \in \mathcal{N}(T^*)$ so that $T^* T \xi = 0$ which implies $\| T \xi \|^2 = \langle T^* T \xi \mid \xi \rangle = 0$, and hence $\xi \in \mathcal{N}(T)$.

(3) Let $\xi \in \mathcal{H}$ such that $T A^{\frac{1}{2}} \xi = 0$. Since $T$ is posinormal, $T A T^* A^{\frac{1}{2}} \xi = 0$.

Thus

$$(T A^{\frac{1}{2}})(T A^{\frac{1}{2}})^* A^{\frac{1}{2}} \xi = 0 \implies \| (T A^{\frac{1}{2}})^* A^{\frac{1}{2}} \xi \|^2 = 0
\implies (T A^{\frac{1}{2}})^* A^{\frac{1}{2}} \xi = 0
\implies A^{\frac{1}{2}} T^* A^{\frac{1}{2}} \xi = 0.$$
Remark 3.5 If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. If $T$ is in $\mathcal{B}(\mathcal{H})$, then $T$ can be written as a $2 \times 2$ matrix with operators entries, $T = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ where $W \in \mathcal{B}(\mathcal{M})$, $X \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$, $Y \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{M})$, and $Z \in \mathcal{B}(\mathcal{M}^\perp)$ (cf. Conway [10]).

A subspace $\mathcal{M}$ is a reducing subspace for $T$ (or $\mathcal{M}$ reduces $T$) if it is both $T$ and $T^*$-invariant (equivalently) if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are $T$-invariant.

Proposition 3.7 Let $A \in \mathcal{B}(\mathcal{H})^+$ be one to one and $T$ is an $A$-posinormal operator then $\mathcal{N}(T)$ reduces $T$.

Proof. It is known that $\mathcal{N}(T)$ is $T$-invariant. Consider the decomposition $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$. Since $T = \begin{pmatrix} O & X \\ O & Z \end{pmatrix}$ and $T^* = \begin{pmatrix} O & O^* \\ X^* & Z^* \end{pmatrix}$, for the proof, it suffices to show that $X = O$. We have for all $\xi \in \mathcal{N}(T)$, $T^*(\xi \oplus 0) = 0 \oplus X^*(\xi)$. Then $\mathcal{N}(T)^\perp \subset \mathcal{N}(T^*)$ by Lemma 3.5, so that $X^* = O$ and hence $X = O$, which implies that $T = \begin{pmatrix} O & O \\ O & Z \end{pmatrix}$ that is $\mathcal{N}(T)$ reduces $T$.

Remark 3.6 If $A \geq 0$ and invertible then $A^{-1} \geq 0$.

Corollary 3.5 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $AT = TA$ then $T \in \mathcal{P}_A(\mathcal{H}) \iff T \in \mathcal{P}_{A^{-1}}(\mathcal{H})$.

Proof. By using Proposition 3.1 (2) and Lemma 3.4 we have

\[
TAT^* \leq cT^*AT \iff A^{-1}\left(TAT^*\right)A^{-1} \leq cA^{-1}\left(T^*AT\right)A^{-1} \\
\iff TA^{-1}T^* \leq cT^*A^{-1}T.
\]

Proposition 3.8 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible. If $T$ is invertible operator then $T$ and $T^{-1}$ are $A$-posinormal.

Proof. We have

\[
TAT^* = T^*A\left(A^{-1}(T^*)^{-1}TAT^\ast T^{-1}\right)T.
\]

A direct computation shows that $A^{-1}(T^*)^{-1}TAT^\ast T^{-1}$ is $A$-positive. Hence $T$ is $A$-posinormal.

\[
T^{-1}A(T^{-1})^* = (T^{-1})^*A\left(A^{-1}T^{-1}T^*A(T^{-1})^*T\right)T^{-1}.
\]

By direct computation we show that $A^{-1}T^{-1}A(T^{-1})^*T$ is $A$-positive, hence $T^{-1}$ is $A$-posinormal.
Remark 3.7 Since the set of invertible operators from $\mathcal{B}(\mathcal{H})$ is open in $\mathcal{B}(\mathcal{H})$, this shows that the set of $A$-posinormal operators whenever $A$ is invertible is topologically large.

Corollary 3.6 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$ then $T - \lambda I$ is $A$-posinormal for $\lambda \notin \sigma(T)$ (spectrum of $T$).

Corollary 3.7 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Then $(T - \lambda I)$ is $A$-posinormal for all $\lambda \in \mathcal{C}$ if and only if $(T - \lambda I)$ is $A$-posinormal for all $\lambda \in \mathbb{C}$.

Proof. Assume that $(T - \lambda I)$ is $A$-posinormal for all $\lambda \in \mathcal{C}$ and let $\lambda \notin \sigma(T)$ then $(T - \lambda I)$ is invertible and hence $(T - \lambda I)$ is $A$-posinormal by Corollary 3.5., the other direction is clear.

The proof of the following corollary is straightforward and will be omitted.

Corollary 3.8 Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. Assume that $T$ is invertible. If $P$ serves as the $A$-interrupter for the $A$-posinormal operator $T^*$, then $P$ is invertible and $AP^{-1}A^{-1}$ serves as the $A^{-1}$-interrupter of the $A^{-1}$-posinormal operator $T^{-1}$.

Proposition 3.9 Let $A$ and $B$ are positive operators such that $A + B$ is invertible, then

$$\mathcal{P}_A(\mathcal{H}) \cap \mathcal{P}_B(\mathcal{H}) \subset \mathcal{P}_{A+B}(\mathcal{H})$$

Proof. Assume that $T$ is of class $\mathcal{P}_A(\mathcal{H}) \cap \mathcal{P}_B(\mathcal{H})$, then there exists $A$-positive operator $P_1$ and $B$-positive operator $P_2$ such that

$$TAT^* = T^*AP_1T \quad \text{and} \quad TBT^* = T^*BP_2T.$$ 

It follows that

$$T(A + B)T^* = TAT^* + TBT^* = T^*AP_1T + T^*BP_2T = T^*(AP_1 + BP_2)T.$$ 

From the hypothesis we have

$$T(A + B)T^* = T^*(A + B)(A + B)^{-1}(AP_1 + BP_2)T.$$ 

Since $(A + B)^{-1}(AP_1 + BP_2)$ is $(A + B)$-positive, $T$ is of class $\mathcal{P}_{A+B}(\mathcal{H})$.

Proposition 3.10 Let $T \in \mathcal{P}_A(\mathcal{H})$ with $A$-interruptor $P$. If $P$ is invertible then $PTP^*$ is of class $\mathcal{P}_{(P^*)^{-1}AP^{-1}}(\mathcal{H})$ with same interruptor.
Proof. Since $T$ is of class $\mathcal{P}_A(\mathcal{H}) : TAT^* = T^*APT$.

$$PTP^*(P^*)^{-1}AP^{-1}PTP^* = PT^*P^*(P^*)^{-1}AP^{-1}PTP^*.$$ 

Since $P$ is invertible-A interruptor of $T$ we have

$$\langle (P^*)^{-1}AP^{-1}P\xi | \xi \rangle = \langle A\xi | P^{-1}\xi \rangle = \langle AP(P^{-1}\xi) | P^{-1}\xi \rangle \geq 0.$$ 

Hence, $P$ is $(P^*)^{-1}AP^{-1}$-positive.

**Theorem 3.5** Assume that $T$ is $A$-posinormal with $A$-interruptor $P$ and $Q$ is positive operator satisfying $A \geq QAQ \geq AP$, then the operator $S = QTQ$ is $A$-hyponormal.

**Proof.** Put $[S^*, S]_A = S^*AS - SAS^*$ we have

$$[S^*, S]_A = QT^*QAQTQ - QTQAQT^*Q = QT^*QAQTQ + QT^*APTQ + QTAT^*Q - QTQAQT^*Q = QT^*(QAQ - AP)TQ + QT(A - QAQ)T^*Q$$

Therefore

$$\langle [S^*, S]_A u | u \rangle = \langle (QAQ - AP)TQu | TQu \rangle + \langle (A - QAQ)T^*Qu | T^*Qu \rangle \geq 0.$$ 

**Proposition 3.11** Let $A \in \mathcal{B}(\mathcal{H})^+$ be invertible and $T \in \mathcal{B}(\mathcal{H})$. If $T$ is $A$-posinormal and normal then $T^n$ is $A$-posinormal for $n = 1, 2, ....$

**Proof.** We use an induction. Clearly, it is true for $n = 1$. Suppose $T^kAT^k \leq c_kT^kAT^k$ for $1 \leq k \leq n$ and $c_k \geq 0$. Then

$$T^{n+1}AT^{n+1} = T(T^nAT^n)T^* \leq c_nT(T^nAT^n)T^* \leq c_nT^*(TAT^*)T^n \leq c_nc_1T^{*n+1}AT^{n+1}.$$ 

Hence $T^{n+1}$ is $A$-posinormal.

4 Tensor products of $A$-posinormal operators

Let $\mathcal{H} \otimes \mathcal{H}$ denote the completion, endowed with a reasonable uniform crossnorm, of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$ of $\mathcal{H}$ with $\mathcal{H}$. Given non-zero
Let $T, S \in \mathcal{B}(\mathcal{H})$, let $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denote the tensor product on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, when $T \otimes S$ is defined as follows

$$
\langle T \otimes S(\xi_1 \otimes \eta_1) | (\xi_2 \otimes \eta_2) \rangle = \langle T\xi_1 | \xi_2 \rangle \langle S \eta_1 | \eta_2 \rangle.
$$

The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in \mathcal{B}(\mathcal{H})$, but by no means all of them. Thus, whereas $T \otimes S$ is normal if and only if $T$ and $S$ are normal [16], there exist paranormal operators $T$ and $S$ such that $T \otimes S$ is not paranormal [1]. In [13], Duggal showed that if for non-zero $T, S \in \mathcal{B}(\mathcal{H}), T \otimes S$ is $p$-hyponormal if and only if $T$ and $S$ are $p$-hyponormal. Thus result was extended to $p$-quasihyponormal operators in [19].

In the following study we will prove a necessary and sufficient condition for $T \otimes S$ to be $A$-posinormal, where $T$ and $S$ are both non-zero operators.

Recall that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and so, by the uniqueness of positive square roots, $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number $r$. From the density of of the rationals in the reals, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for every positive real number $p$. Observe also that

$$
A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I).
$$

The following elementary results on tensor products of operators will be used often (and without further reference) in the sequel: $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d \neq 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. If $T_k$ and $S_k$ ($k = 1, 2$) are positive operators, then $T_1 \otimes S_1 = T_2 \otimes S_2$ if and only if there exists a scalar $d > 0$ such that $T_1 = dT_2$ and $S_1 = d^{-1}S_2$. The proofs to these results are to be found in the papers by Hou [16] and Stochel [31].

**Lemma 4.1** If $T_1 \geq T_2 \geq 0$ and $S_1 \geq S_2 \geq 0$, then $T_1 \otimes S_1 \geq T_2 \otimes S_2 \geq 0$.

**Proof.** By Assumptions we have $\langle T_1 \xi | \xi \rangle \geq \langle T_2 \xi | \xi \rangle \ \forall \ \xi \in \mathcal{H}$ and $\langle S_1 \eta | \eta \rangle \geq \langle S_2 \eta | \eta \rangle \ \forall \ \eta \in \mathcal{H}$. Thus

$$
\langle T_1 \xi | \xi \rangle \langle S_1 \eta | \eta \rangle \geq \langle T_2 \xi | \xi \rangle \langle S_2 \eta | \eta \rangle,
$$

and hence

$$
\langle T_1 \otimes S_1(\xi \otimes \eta) | \xi \otimes \eta \rangle \geq \langle T_2 \otimes S_2(\xi \otimes \eta) | \xi \otimes \eta \rangle.
$$

**Proposition 4.1** ([31]) Let $T_1, T_2, S_1, S_2 \in \mathcal{B}(\mathcal{H})$ be positive operators. If $T_1 \neq 0$ and $S_1 \neq 0$, then the following conditions are equivalents

1. $T_1 \otimes S_1 \leq T_2 \otimes S_2$
2. There exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$. 

In the following theorem we generalize theorem 2.4 in [31] to the space \((\mathcal{H}, \langle \ | \rangle_A)\).

**Theorem 4.1** Let \(A, B \in \mathcal{B}(\mathcal{H})^+\). If \(T \in \mathcal{B}_A(\mathcal{H})\) and \(S \in \mathcal{B}_B(\mathcal{H})\) are nonzero operators, then the following properties hold.

1. \(T \otimes S\) is \((A \otimes B)\)-quasi-isometry \(\iff\) \(\alpha T\) is \(A\)-quasi-isometry and \(\alpha^{-1} S\) is \(B\)-quasi-isometry for some constant \(\alpha \neq 0\).

2. \(T \otimes S\) is \((A \otimes B)\)-isometric \(\iff\) \(\alpha T\) is \(A\)-isometry and \(\alpha^{-1} S\) is \(B\)-isometry for some constant \(\alpha \neq 0\).

3. \(T \otimes S\) is \((A \otimes B)\)-unitary \(\iff\) \(\alpha T\) is \(A\)-unitary and \(\alpha^{-1} S\) is \(B\)-unitary for some constant \(\alpha \neq 0\).

4. \(T \otimes S\) is \((A \otimes B)\)-selfadjoint \(\iff\) \(\alpha T\) is \(A\)-selfadjoint and \(\alpha^{-1} S\) is \(B\)-selfadjoint for some constant \(\alpha \neq 0\).

5. \(T \otimes S\) is \((A \otimes B)\)-positive \(\iff\) \(\alpha T\) is \(A\)-positive and \(\alpha^{-1} S\) is \(B\)-positive for some constant \(\alpha \neq 0\).

6. \(T \otimes S\) is \((A \otimes B)\)-normal \(\iff\) \(T\) is \(A\)-normal and \(S\) is \(B\)-normal.

7. \(T \otimes S\) is \((A \otimes B)\)-hyponormal \(\iff\) \(T\) is \(A\)-hyponormal and \(S\) is \(B\)-hyponormal.

8. \(T \otimes S\) is \((A \otimes B)\)-quasinormal \(\iff\) \(T\) is \(A\)-quasinormal and \(S\) is \(B\)-quasinormal.

**Proof.** (1)

\[
(T \otimes S) \text{ is } (A \otimes B) - \text{ quasi-isometry} \\
\iff (T \otimes S)^*(A \otimes B)(T \otimes S) = (T \otimes S)^{*2}(A \otimes B)(T \otimes S)^2 \\
\iff T^* AT \otimes S^* BS = T^{*2} AT^{2} \otimes S^{*2} BS^{2} \\
\iff \exists \ d > 0 : T^* AT = dT^{*2} AT^{2} \text{ and } S^* BS = d^{-1} S^{*2} BS^{2} \\
\iff (\sqrt{d}T)^* A(\sqrt{d}T) = (\sqrt{d}T)^{*2} A(\sqrt{d}T)^2 \text{ and} \\
(\sqrt{d^{-1}}S)^* A(\sqrt{d^{-1}}S) = (\sqrt{d^{-1}}S)^{*2} A(\sqrt{d^{-1}}S)^2.
\]

(2) "\(\Rightarrow\)" Assume that \(T \otimes S\) is \((A \otimes B)\)-isometry, then

\[(T \otimes S)^*(A \otimes B)(T \otimes S) = A \otimes B \implies T^* AT \otimes S^* BS = A \otimes B.
\]

Since the operators involved in the above inequalities are positive and nonzero, it follows by Proposition 4.1 that there is a constant \(d > 0\) such that

\[T^* AT = dA \text{ and } S^* BS = d^{-1} B.
\]

This implies that...
\[
\left( \frac{1}{\sqrt{d}} T \right)^* A \left( \frac{1}{\sqrt{d}} T \right) = A \quad \text{and} \quad \left( \sqrt{d} S \right)^* B \left( \sqrt{d} S \right) = B,
\]

we obtain the desired result. The converse implication is obvious. In the same way, we may deduce (3) , (4) and (5).

(6)

\[
\begin{align*}
(T \otimes S) & \text{ is } (A \otimes B) - \text{normal} \\
& \iff (T \otimes S)(T \otimes S)^{(*) \otimes B} = (T \otimes S)^{(*) \otimes B} (T \otimes S) \\
& \iff TT^{(*) \Lambda} \otimes BS^{(*) \Lambda} = T^{(*) \Lambda} T \otimes S^{(*) \Lambda} S.
\end{align*}
\]

(1) First case: if $A$ or $B$ is injective.

Multiplying the both side of this equality by $(A \otimes B)$ we obtained

\[
ATT^{(*) \Lambda} \otimes BSS^{(*) \Lambda} = AT^{(*) \Lambda} T \otimes BS^{(*) \Lambda} S.
\]

Since the operators involved in the above equality are positive, it follows that there exists a scalar $d > 0$ such that

\[
ATT^{(*) \Lambda} = dAT^{(*) \Lambda} T \quad \text{and} \quad BSS^{(*) \Lambda} = d^{-1} BS^{(*) \Lambda} S.
\]

Hence

\[
ATT^{(*) \Lambda} = AdT^{(*) \Lambda} T \quad \text{and} \quad BSS^{(*) \Lambda} = Bd^{-1} S^{(*) \Lambda} S.
\]

Thus,

\[
TT^{(*) \Lambda} = dT^{(*) \Lambda} T \quad \text{and} \quad SS^{(*) \Lambda} = d^{-1} S^{(*) \Lambda} S,
\]

and it follows that

\[
\|TT^{(*) \Lambda}\|_A = d\|T^{(*) \Lambda} T\| \quad \text{and} \quad \|SS^{(*) \Lambda}\| = d^{-1}\|S^{(*) \Lambda} S\|.
\]

Consequently $d = 1$.

(2) General case: $A$ and $B$ are not necessary injective. There exists a scalar $d \neq 0$ such that

\[
TT^{(*) \Lambda} = dT^{(*) \Lambda} T \quad \text{and} \quad SS^{(*) \Lambda} = d^{-1} S^{(*) \Lambda} S.
\]

We deduce that $|d| = |d^{-1}| = 1$ and hence $d = 1$. The desired results are proved.

(7) By similar argument.
(8) 

\[(T \otimes S) \text{ is } (A \otimes B) - \text{quasinormal} \]
\[\iff (T \otimes S)(T \otimes S)^{\ast}_{A \otimes B}(T \otimes S) = (T \otimes S)^{\ast}_{A \otimes B}(T \otimes S)^2 \]
\[\iff TT^{\ast}_A \otimes SS^{\ast}_B S = T^{\ast}T^2 \otimes S^{\ast}B \]
\[\iff \exists d \neq 0 : TT^{\ast}_A T = dT^{\ast}_A T^2 \text{ and } SS^{\ast}_B S = d^{-1}S^{\ast}B S^2 . \]

This in turn implies that

\[\left( T^{\ast}_A T \right)^2 = d\left( T^{\ast}_A T \right)^2 T^2 = d\left( T^2 \right)^{\ast}_A T^2 \]
and

\[\left( S^{\ast}_B S \right)^2 = d^{-1}\left( S^{\ast}_B S \right)^2 S^2 = d^{-1}\left( S^2 \right)^{\ast}B S^2 \]

Consequently

\[\| (T^{\ast}_A T)^2 \|_A = \|d\| \| (T^2 \right)^{\ast}_A T^2 \|_A \]
and

\[\| (S^{\ast}_B S)^2 \|_B = \|d^{-1}\| \| (S^2 \right)^{\ast}B S^2 \|_B , \]

which yields \( d = 1 \). Therefore \( T \) is \( A \)-quasinormal and \( S \) is \( B \)-quasinormal.

**Theorem 4.2** Let \( A, B \in \mathcal{B}(\mathcal{H})^+ \) are invertible. Take nonzero \( T \) and \( S \) \( \mathcal{B}(\mathcal{H}) \). The tensor product \( T \otimes S \) is \( (A \otimes B) \)-posinormal if and only if \( T \) is \( A \)-posinormal and \( S \) is \( B \)-posinormal.

**Proof.** Assume that \( T \) is \( A \)-posinormal and \( S \) is \( B \)-posinormal. By Theorem 3.1 there are a positive constants \( \alpha \) and \( \beta \) such that

\[ TAT^{\ast} \leq \alpha T^{\ast}AT \text{ and } SBS^{\ast} \leq \beta S^{\ast}BS . \]

Since the operators involved in the above inequalities are positive, it follows that

\[ TAT^{\ast} \otimes SBS^{\ast} \leq \alpha \beta T^{\ast}AT \otimes S^{\ast}BS \]

(See Lemma 4.1) , and therefore

\[ (T \otimes S)(A \otimes B)(T \otimes S)^{\ast} \leq \alpha \beta (T \otimes S)^{\ast}(A \otimes B)(T \otimes S) \]

so that \( T \otimes S \) is \( (A \otimes B) \)-posinormal as \( (A \otimes B) \) is invertible.

Conversely, if \( T \otimes S \) is \( (A \otimes B) \)-posinormal, there exists a positive constant \( d \) such that

\[ (T \otimes S)(A \otimes B)(T \otimes S)^{\ast} \leq d(T \otimes S)^{\ast}(A \otimes B)(T \otimes S) , \]
which means that
\[ TAT^* \otimes SBS^* \leq (d^{1/2}T^*AT) \otimes (d^{1/2}S^*BS). \]
Since the operators involved in the above inequalities are positive and nonzero, it follows by Proposition 4.1 that there is a constant \( \gamma > 0 \) such that
\[ TAT^* \leq \gamma d^{1/2}T^*AT \] and \( SBS^* \leq \gamma^{-1}d^{1/2}S^*BS \)
so that \( T \) and \( S \) are \( A \)-posinormal and \( B \)-posinormal respectively.
In the general case we have the following generalization

**Theorem 4.3** Let \( A_i \in \mathcal{B}(\mathcal{H})^+ \) and let \( T_i \in \mathcal{B}_{A_i}(\mathcal{H}) \) for \( i = 1, 2, \ldots, n \) and \( T_1 \otimes T_2 \otimes \ldots \otimes T_n \neq 0 \). Then the tensor product \( T_1 \otimes T_2 \otimes \ldots \otimes T_n \) on the Hilbert space \( \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H} \) is \((A_1 \otimes A_2 \otimes \ldots \otimes A_n)\)-posinormal if and only if \( T_i \) is \( A_i \)-posinormal operator for \( i = 1, 2, \ldots, n \).

**Proof.** By induction, it suffices to show that \( T_1 \otimes T_2 \) is \((A_1 \otimes A_2)\)-posinormal if and only if both \( T_1 \) is \( A_1 \)-posinormal and \( T_2 \) is \( A_2 \)-posinormal.

Assume that \( T_1 \otimes T_2 \neq 0 \) is \((A_1 \otimes A_2)\)-posinormal operator, then:

\[(T_1 \otimes T_2)(A_1 \otimes A_2)(T_1 \otimes T_2)^* = (T_1 \otimes T_2)(A_1 \otimes A_2)(P_1 \otimes P_2)(T_1 \otimes T_2).\]

Thus
\[ T_1 A_1 T_1^* \otimes T_2 A_2 T_2^* = T_1^* A_1 P_1 T_1 \otimes T_2^* A_2 P_2 T_2. \]
Since the operators involved in the above equality are positive, it follows that there exists a scalar \( d > 0 \) such that
\[ T_1 A_1 T_1^* = d T_1^* A_1 P_1 T_1 \quad \text{and} \quad T_2 A_2 T_2^* = d^{-1} T_2^* A_2 P_2 T_2. \]

Hence
\[ T_1 A_1 T_1^* = T_1^* A_1 (dP_1) T_1 \quad \text{and} \quad T_2 A_2 T_2^* = T_2^* A_2 (d^{-1} P_2) T_2, \]
Since \( dP_1 \) is \( A_1 \)-positive and \( d^{-1} P_2 \) is \( A_2 \)-positive, it follows that \( T_1 \) is \( A_1 \)-posinormal and \( T_2 \) is \( A_2 \)-posinormal.

Conversely assume that \( T_i \) is \( A_i \)-posinormal for \( i = 1, 2 \), then
\[ T_1 A_1 T_1^* = T_1^* A_1 P_1 T_1 \quad \text{and} \quad T_2 A_2 T_2^* = T_2^* A_2 P_2 T_2 \]
\[(T_1 \otimes T_2)(A_1 \otimes A_2)(T_1 \otimes T_2)^* = T_1 A_1 T_1^* \otimes T_2 A_2 T_2^* \]
\[ = T_1^* A_1 P_1 T_1 \otimes T_2^* A_2 P_2 T_2 \]
\[ = (T_1 \otimes T_2)^*(A_1 \otimes A_2)(P_1 \otimes P_2)(T_1 \otimes T_2). \]
Since \( P_1 \otimes P_2 \) is \((A_1 \otimes A_2)\)-positive it follows that \( T_1 \otimes T_2 \) is \((A_1 \otimes A_2)\)-posinormal.
Definition 4.1 Let $T, S \in \mathcal{B}(\mathcal{H})$. The tensor sum of $T$ and $S$ is the transformation $T \boxplus S : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ defined by

$$T \boxplus S = (T \otimes I) + (I \otimes S)$$

which is an operator in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

Remark 4.1 It is easily seen that $T \in \mathcal{B}(\mathcal{H})$ is $A$-posinormal for some $A \in \mathcal{B}(\mathcal{H})^+$ if and only if $T \otimes I$ is $(A \otimes I)$-posinormal (resp., $I \otimes T$ is $(I \otimes A)$-posinormal).

Basic operations with tensor sum of Hilbert space operators are summarized in the next proposition. For its proof see [21].

Proposition 4.2 Let $T, S, T_k, S_k \in \mathcal{B}(\mathcal{H})$ $k = 1, 2$ and $\alpha, \beta \in \mathbb{C}$. The following properties hold:

(1) $(\alpha + \beta)(T \boxplus S) = \alpha T \boxplus \beta S + \beta T \boxplus \alpha S$
(2) $(T_1 + T_2) \boxplus (S_1 + S_2) = T_1 \boxplus S_1 + T_2 \boxplus S_2$
(3) $(T_1 \boxplus S_1)(T_2 \boxplus S_2) = T_1 \otimes S_2 + T_2 \otimes S_1 + T_1 T_2 \boxplus S_1 S_2$
(4) $(T \boxplus S)^* = T^* \boxplus S^*$
(5) $\|T \boxplus S\| \leq \|T\| + \|S\|$.

In the following proposition we generalized the normality of $T \boxplus S$ proved in [20] to hyponormality.

Theorem 4.4 If $T$ and $S$ are hyponormal then $T \boxplus S$ is hyponormal.

Proof. Since $T$ and $S$ are hyponormal we have that $T^*T \geq TT^*$ and $S^*S \geq SS^*$.

In view of the fact that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$$

we have

$$(T \boxplus S)(T \boxplus S)^* = T \otimes S^* + T^* \otimes S + (TT^* \otimes I + I \otimes SS^*)$$
$$= T \otimes S^* + T^* \otimes S + (T^* \otimes I)(I \otimes S) + (T \otimes S)(I \otimes S^*)$$
$$\leq T \otimes S^* + T^* \otimes S + (T^* \otimes I)(T \otimes I) + (S^* \otimes I)(S \otimes I)$$
$$\leq T \otimes S^* + T^* \otimes S + T^* T \boxplus S^* S$$
$$\leq (T \boxplus S)^*(T \boxplus S).$$

It follows that $T \boxplus S$ is hyponormal.
References


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