On Subgroups of Quasi-Graph Groups

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Abstract

A graph is called a quasi-graph if the case of an edge of the graph equals its inverse is allowed. A graph of groups is called a quasi-graph of groups if the corresponding graph is a quasi-graph. A group is termed quasi-graph group if it is a fundamental group of a non-trivial quasi-graph of groups. In this paper we show that a subgroup of a quasi-graph group is a quasi-graph group.

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1 Introduction

In [4], Mahmood introduced the concepts of quasi-graphs of groups and their fundamental groups. The main result of this paper is to show that if G is a fundamental group of a quasi-graph of groups and if H is a subgroup of G, then we use the results of [5] and [6] to show that H is a fundamental group of a quasi-graph of groups. This paper is divided in to 4 sections. In section 2, we introduce the concepts quasi graphs, groups acting on trees with inversions, and their fundamental domains. In section 3, we use the results of section 2 to obtain the structures of quasi-graphs of groups induced by the fundamental domains for groups acting on trees with inversions. In section 4, we apply the results of section 3 to obtain the structures of quasi-graphs of groups induced by the subgroups of the fundamental groups of quasi-graphs of groups.
2 Quasi-Graphs Induced by Fundamental Domains

A quasi graph $X$ consists of two disjoint sets $V(X)$, (the set of vertices of $X$) and $E(X)$, (the set of edges of $X$), with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions that $\partial_0 \eta = \partial_1$ and $\partial_1 \eta = \partial_0$. For simplicity, if $e \in E(X)$, we write $\partial_0 (e) = o(e)$, $\partial_1 (e) = t(e)$, and $\eta (e) = \bar{e}$. This implies that $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$, and $\bar{e} = e$. The case $e = \bar{e}$ is allowed. There are obvious definitions of subgraphs, circuits, trees, morphisms of graphs and $\text{Aut}(X)$, the set of all automorphisms of the graph $X$ which is a group under the composition of morphisms of graphs. For more details, the interested readers are referred to in [1], [2] and [9]. We say that a group $G$ acts on a graph $X$ if there is a group homomorphism $\phi : G \to \text{Aut}(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. Thus, if $g \in G$, and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\bar{y}) = g(y)$. The case the action with inversion is allowed. That is; $g(y) = (\bar{y})$ is allowed for some $g \in G$, and $y \in E(X)$. In this case we say that $g$ is an inverter element of $G$ and $y$ is called an inverted edge. If the group $G$ acts on the graph $X$ and $x \in X$, (x is a vertex or edge), then

1. The stabilizer of $x$, denoted $G_x$, is defined to be the set $G_x = \{g \in G : g(x) = x\}$. It is clear that $G_x \leq G$, and if $x \in E(X)$, and $u \in \{o(x), t(x)\}$, then $G_x = G_u$ and $G_x \leq G_u$.

2. The orbit of $x$ is the set $G(x) = \{g(x) : g \in G\}$. It is clear that $G$ acts on the graph $X$ without inversions if and only if $G(\bar{e}) \neq e$ for any $e \in E(X)$.

3. The set of orbits is denoted by $G/X = \{G(x) : x \in X\}$. In [6, Prop. 2.1], it is proved that $G/X$ forms a quasi-graph.

**Definition.** Let $G$ be a group acting on a connected quasi-graph $X$ with inversions, and let $T$ and $Y$ be two subtrees of $X$ such that $T \subseteq Y$, and each edge of $Y$ has at least one end in $T$. Assume that $T$ and $Y$ are satisfying the following.

(i) $T$ contains exactly one vertex from each vertex orbit.

(ii) $Y$ contains exactly one edge $y$(say) from edge orbit if $G(y) \neq G(\bar{y})$ and exactly one pair $x$, $\bar{x}$ from each edge orbit if $G(x) = G(\bar{x})$. Then

(1) $T$ is called a tree of representatives for the action of $G$ on $X$.

(2) $Y$ is called a transversal for the action of $G$ on $X$.

For simplicity we say that $(T; Y)$ is a fundamental domain for the action of $G$ on $X$. For more details, the readers are referred to [3]. The properties of fundamental domains for the actions of groups on connected quasi-graphs imply that if $G$ is a group acting on a connected quasi-graph $X$ with inversions, and $(T; Y)$ is a fundamental domain for the action of $G$ on $X$, then for any $v \in V(Y)$ and any
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Let $G$ be a group acting on a connected quasi-graph $X$ with inversions, and let $(T; Y)$ be a fundamental domain for the action of $G$ on $X$. Let $G/(X; Y; T) = \{G(v) : v \in V(T)\} \cup \{G(e) : e \in E(Y)\}$ and $G/(X; T; T) = \{G(v) : v \in V(T)\} \cup \{G(e) : e \in E(T)\}$. Then $G/(X; Y; T)$ forms a connected quasi-graph and $G/(X; T; T)$ is a maximal subtree of $G/(X; Y; T)$.

**Proof.** Consider the sets $V(G/(X; Y; T)) = \{G(v) : v \in V(T)\}$ and $E(G/(X; Y; T)) = \{G(e) : e \in E(Y)\}$. Since $V(X) \neq \emptyset$, therefore $V(G/(X; Y; T)) \neq \emptyset$. The fact that for $v \in V(X)$ and $e \in E(X)$, the orbits $G(v)$ and $G(e)$ are disjoint implies that $V(G/(X; Y; T)) \cap E(G/(X; Y; T)) = \emptyset$. For $e \in E(Y)$, $G(e) \in E(G/(X; Y; T))$, define $o(G(e)) = G((o(e))^*)$, $t(G(e)) = G((t(e))^*)$, and $\overline{G(e)} = G(\overline{e})$. Then $G/(X; Y; T)$ forms a graph. Similar to the proof of Prop. 2.1 of [6] we can show that $G/(X; Y; T)$ forms a connected quasi-graph.

Since $V(G/(X; T; T)) = V(G/(X; Y; T))$, [ see Prop. 11 of 9], therefore $G/(X; T; T)$ is a maximal subtree of $G/(X; Y; T)$. This completes the proof.

**Note.** $G/(X; Y; T)$ is called the quotient graph induced by the fundamental domain $(T; Y)$ for the action of $G$ on $X$.

3 Quasi-graphs of groups Induced by Fundamental Domains

The concepts of quasi-graphs of groups and their fundamental groups introduced in [4] are modified and defined as follows.

A quasi-graph of groups is defined to be a pair $\Phi = (Z; \Gamma)$ where $Z$ is a connected quasi-graph and $\Gamma$ is a mapping from $Z$ into the class of all groups; where the
image of each element (vertex or edge) \( x \in Z \) under \( \Gamma \) is denoted by \( \Gamma_x \), i.e. 
\( \Gamma(x) = \Gamma_x \) such that for each edge \( e \in E(Z) \) the following hold.

1. \( \Gamma_e = \Gamma_e \);  
2. There exist monomorphisms denoted \( \lambda_e : \Gamma_e \to \Gamma_{e(e)} \) and \( \lambda_x : \Gamma_x \to \Gamma_{o(e)} \);  
3. There exists an element denoted \( \delta_e \in \Gamma_e \) such that \( \delta_e = 1 \) if \( \bar{e} \neq e \);  
4. There exists an automorphism \( \mu_e : \Gamma_e \to \Gamma_e \) satisfying the conditions that \( \mu_e(\delta_e) = \delta_e, \mu_e(a) = a \) if \( \bar{e} \neq e \), and \( \mu_e^2(a) = \delta_e a \delta_e^{-1} \) for all \( a \in \Gamma_e \).

**Note.** For simplicity, we write \( \Phi(Z) \) to mean \( \Phi(Z; \Gamma) \), the quasi graph of groups.

**Notation.** Given the quasi-graph of groups \( \Phi(Z; \Gamma) \). For the edge \( e \in E(Z) \) and the element \( a \in \Gamma_e \). We have the following notation.

1. Let \( \lambda_e(\Gamma_e) = \Gamma^e, \lambda_x(\Gamma_x) = \Gamma^x \) and let \( \lambda_e(a) = a^e \). It is clear that \( \Gamma^e \leq \Gamma_{e(e)} \), \( \Gamma^x \leq \Gamma_{o(e)} \), \( a^e \in \Gamma^e \), and \( \lambda_e(\delta_e) = \delta_e^e \in \Gamma^e \).
2. Let \( \phi_e : \Gamma^e \to \Gamma^x \) be the mapping given by \( \phi_e(a^e) = (\mu_e(a))^x \). It is clear that \( \phi_e \) is an isomorphism and if \( \bar{e} \neq e \) then \( \phi_e(a^e) = a^x \).

**Definition.** Given the quasi-graph of groups \( \Phi(Z, \Gamma) \). Let \( \Lambda \) be a maximal subtree of \( Z \). Define \( \pi\Phi(Z; \Gamma; \Lambda) \) to be the group of the presentation:

\[
\langle \text{gen}(\Gamma_e), e | \text{rel}(\Gamma_e), e \Gamma^e.e^{-1} = \Gamma^x, e\bar{e} = \delta_e^e, e=1 \text{if } e \in E(\Lambda) \rangle,
\]

where \( v \in V(Z) \), and \( e \in E(Z) \).

The notation \( e \Gamma^e.e^{-1} = \Gamma^x \) stands for the set of relations of the form \( e.a^e.e^{-1} = \phi_e(a^e) = (\mu_e(a))^x, a \in \Gamma_e \).

\( \pi\Phi(Z; \Gamma; \Lambda) \) is called the fundamental group of \( \Phi(Z, \Gamma) \) relative to \( \Lambda \).

For each \( e \in E(Z) \), let \( t_e \) be the value of \( e \) in \( \pi\Phi(Z; \Gamma; \Lambda) \) where no confusion will be caused by the notations \( t_e \) and \( t(e) \). It is clear that the relations of \( \pi\Phi(Z; \Gamma; \Lambda) \) imply the following.

1. \( \pi\Phi(Z; \Gamma; \Lambda) \) is generated by elements \( t_e \) and \( g \), where \( g \in \Gamma_v \) and \( v \in V(Z) \);
2. \( t_e \notin \Gamma_v, v \in V(Z) \);
3. \( t_e = 1 \) if \( e \in E(\Lambda) \).
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(4) $t_e g^e t_e^{-1} = (\phi_e(g))^\varepsilon$, $g \in \Gamma^e$;
(5) $t_e t_e = \delta^e$.

It is proved in [4, p. 148, 5.7 Corollary] that if $\Lambda_1$ and $\Lambda_2$ are two maximal subtrees of $Z$ then $\pi \Phi(Z; \Gamma; \Lambda_1)$ and $\pi \Phi(Z; \Gamma; \Lambda_2)$ are isomorphic.

The fundamental group of $\Phi(Z, \Gamma)$ denoted $\pi \Phi(Z; \Gamma)$ is defined to be the fundamental group $\pi \Phi(Z; \Gamma)$ relative to a maximal subtree $\Lambda$ of $Z$.

In view of above, $\pi \Phi(Z; \Gamma)$ is independent of any maximal subtree of $Z$.

The main result of this section is the following theorem.

**Theorem 3.1.** Let $G$ be a group acting on a connected quasi-graph $X$ with inversions, and $(Y; T)$ be a fundamental domain for the action of $G$ on $X$.

For each $v \in V(T)$ and $e \in E(Y)$, let $\Gamma_{G(v)} = G_v$ and $\Gamma_{G(e)} = G_e$.

Then $\Phi = (G/(X; Y; T); \Gamma)$ forms a quasi-graph of groups, and the fundamental group $\pi(\Phi) = \pi((G/(X; Y; T); \Gamma))$ has the presentation

$\langle \text{gen}(G_v), e | \text{rel}(G_v), e.G_{G_v}(\varepsilon), e^{-1} = G_{G_v}, e\varepsilon = \delta_{G_v}, e = 1 \text{ if } e \in E(T) \rangle$ \ldots (*)

where $v \in V(T)$, and $e \in E(Y)$. Furthermore, if $X$ is a tree, then $G$ and $\pi(\Phi) = \pi((G/(X; Y; T); \Gamma))$ are isomorphic.

**Proof.** By Lemma 2.1, $G/(X; Y; T)$ forms a connected quasi-graph. Now we show that $\Gamma$ satisfies the conditions of the definition of quasi-graph of groups.

If $e$ is an edge of $Y$, then $G(e)$ is an edge of $G/(X; Y; T)$.

(1) Now we show that $\Gamma_{G(e)} = \Gamma_{G(e)}$. The fact that $G_e = G_e$ implies that

$\Gamma_{G(e)} = \Gamma_{G(e)}$.

(2) We need to find a monomorphism $\lambda_{G(e)} : \Gamma_{G(e)} \rightarrow \Gamma_{G(e)}$.

We have $t(e) = [e](t(e))^\varepsilon)$. This implies that $G_{t(e)} = G_{t(e)} = [e]G_{t(e)}^e[e^{-1}]$. Then $t(G(e)) = G(t(e))$, $\Gamma_{G(e)} = G_e \leq G_{t(e)} = [e]G_{t(e)}^e[e^{-1}]$. Then $\Gamma_{t(G(e))} = \Gamma_{G(t(e))} = G_{t(e)}^e = [e]^{-1}G_{t(e)}[e]$, and $\Gamma_{G(o(e))} = G_{G(o(e))} = G_{o(e)} = [e]^{-1}G_{t(e)}[e] = [e]G_{o(e)}[e]^{-1}$.

This implies that the mapping $\lambda_{G(e)} : G_e \rightarrow G_{t(e)}$ given by

$\lambda_{G(e)}(g) = \begin{cases} g & \text{if } t(e) \in V(T) \\ [e]^{-1}g[e] & \text{if } o(e) \in V(T) \end{cases}$

yields the required monomorphism $\lambda_{G(e)} : \Gamma_{G(e)} \rightarrow \Gamma_{G(e)}$, and similarly for the mapping $\lambda_{G(e)} = \lambda_{G(e)} : G_e \rightarrow G_{o(e)}$ given by $\lambda_{G(e)}(g) = \begin{cases} [e]g[e]^{-1} & \text{if } t(e) \in V(T) \\ g & \text{if } o(e) \in V(T) \end{cases}$

is the required monomorphism $\lambda_{G(e)} : \Gamma_{G(e)} \rightarrow \Gamma_{G(e)}$. 
(3) Let $\delta_{G(e)} = [e][\bar{e}]$. If $G(e) \neq G(\bar{e})$, then $[e]=\bar{e}^{-1}$, and $\delta_{G(e)} = 1$, and we take $\mu_{G(e)} : G_e \to G_e$ to be the identity automorphism.

If $G(e) = G(\bar{e})$, then $\delta_{G(e)} = [e]^2$.

Let $\mu_{G(e)} : G_e \to G_e$ be the mapping given by $\mu_{G(e)}(g) = [e]g[e]^{-1}$.

It is easy to show that $\mu_{G(e)}$ is an automorphism. Furthermore, $\mu_{G(e)}(\delta_{G(e)}) = \mu_{G(e)}([e]^2) = [e][e][e]^{-1} = [e]^2 = \delta_{G(e)}$, and for $g \in G_e$ we have

$$(\mu_{G(e)}(g))^2 = \mu_{G(e)}([e]g[e]^{-1}) = [e]^2g[e]^{-2} = \delta_{G(e)}g\delta_{G(e)}^{-1}.$$ 

This implies that the mapping $\mu_{G(e)} : \Gamma_{G(e)} \to \Gamma_{G(e)}$ is the required automorphism. This implies that $\Phi = (G/(X;Y;T) ; \Gamma)$ forms a quasi-graph of groups.

Let $e$ be an edge of $Y$. Then

$$\Gamma^{G(e)} = \lambda^{G(e)}(\Gamma_{G(e)}) = \lambda^{G(e)}(G_e) = \begin{cases} G_e & \text{if } \tau(e) \in V(T) \\ [e]^{-1}G_e[e] & \text{if } o(e) \in V(T) \end{cases} = G_{v(\tau)}$$

and $\Gamma^{G(e)} = \Gamma^{G(\bar{e})} = G_{v(e)}$. Furthermore, the mapping $\phi_{G(e)} : \Gamma^{G(e)} \to \Gamma^{G(e)}$ given by $\phi_{G(e)}(g) = [e]g[e]^{-1}$ is an isomorphism. Furthermore, $\phi_{G(e)}(\delta_{G(e)}^2) = \phi_{G(e)}([e]^2) = [e][e][e]^{-1} = [e]^2$.

$G/(X;T;T) = \{G(v) : v \in V(T)\} \cup \{G(e) : e \in E(T)\}$. In the presentation (*), we replace $\Gamma_v$ by $G_v$, $\Gamma^{G(e)}$ by $G_{v(e)}$, $\Gamma^{G(\bar{e})}$ by $G_{v(e)}$, and $\delta_{G(e)}^2$ by $= \delta_e$. This leads the presentation of the fundamental group $\pi(\Phi) = \pi((G/(X;Y;T) ; \Gamma))$ of $\Phi = (G/(X;Y;T)$ relative to the maximal subtree $G/(X;T;T) \subseteq G/(X;Y;T)$. If $X$ is a tree, by [8, Corollary 5.2], $G$ has above presentation. This implies that $G$ and $\pi(\Phi) = \pi((G/(X;Y;T) ; \Gamma))$ are isomorphic. This completes the proof.

4 On Subgroups of Quasi-Graph Groups

Recall that a group $G$ is called a quasi-graph group if there exists a quasi-graph of groups $\Phi = (Z;\Gamma)$ where $Z$ contains more than one vertex such that $G = \pi(\Phi)$. The main result of this section is the following theorem.

Theorem 4.1. A subgroup of a quasi-graph group is a quasi-graph group.
Proof. Let $G$ be a fundamental group of a quasi-graph of groups $\Phi(Z, \Gamma)$, where $Z$ contains more than one vertex, $\Lambda$ be a maximal subtree of $Z$, and $H$ be a subgroup of $G$. We need to find a quasi-graph of groups $\Phi_H = (Z_H; \Gamma_H)$, where $H$ is its fundamental group. In [5], a tree $X$ is constructed as follows.

$V(X) = \{[g, v]: v \in V(Z), g \in G\}$, and $E(X) = \{[g, e]: e \in E(Z), g \in G\}$, where $[g, v] = (g \Gamma_v, v)$, and $[g, e] = (g \Gamma^e, e)$. The inverse and the terminals of the edge $[g, e]$ are defined as follows. $[g, e] = [gt_\tau, e^{\tau}]$, $o[g, e] = [gt_\tau, o(e)]$, and $t[g, e] = [gt_\tau, t(e)]$. For $f \in G$, $v \in V(Z)$, and $e \in E(Z)$, define $f[g, v] = [fg, v]$, and $f[g, e] = [fg, e]$. If $\bar{e} = e$, then $[1, e] = [t_\tau, \bar{e}] = [t_\tau, e] = t_\tau[1, e]$. This implies that $G$ acts on $X$ with inversions. The $G$ stabilizers of $[g, v]$ and $[g, e]$ are $G_{[g, v]} = g\Gamma_s g^{-1}$, and $G_{[g, e]} = g\Gamma^e g^{-1}$. The tree of representatives $T$ for the action of $G$ on $X$ is defined as $V(T) = \{[1, v]: v \in V(Z)\}$, and $E(T) = \{[1, e]: e \in E(\Lambda)\}$. Also, $[g, v]^* = [1, v]$. The transversal $Y$ for the action of $G$ on $X$ consists of all edges of the form $[t_\tau, e], e \in E(X)$, with addition to their inverses $[t_\tau, e] = [1, \bar{e}]$, and their terminals $o[t_\tau, e] = [1, o(e)]$, and $t[t_\tau, e] = [t_\tau, t(e)]$. Then $(T; Y)$ is a fundamental domain for the action of $G$ on $X$.

For each vertex $v \in V(T)$, and each edge $e \in E(Y)$, define the following.

(a) $D_v$ is a double coset representative system for $G \bmod (H, G_v)$,

(b) $D^g_v$ and $D^e_v$ are any double coset representative systems for $G_{o(e)} \bmod (g^{-1}Hg \cap G_v)$, and $G_{t(e)}^{\ast} \bmod (g^{-1}Hg \cap [e]^{-1}G_v[e])$ respectively containing 1, but otherwise arbitrary, $g \in G$. For more details of the structures of $D_v, D^g_v$ and $D^e_v$ we refer the readers to [7].

The fundamental domain $(T_H; Y_H)$ for the action of $H$ on $X$ obtained in [7] is defined as follows. $V(T_H) = \{d(v): v \in V(T), d \in D_v\}$, and $E(T_H)$ is the set of edges, $\{ab(e), ab(\bar{e}): e \in E_0(Y) \cup E_1(Y) \cup E_2(Y), a \in D_{o(e)}, b \in D^g_e, ab \in D_{t(e)}^{\ast}\}$, where $E_0(Y) = \{[t_\tau, e]: e \in E(\Lambda)\}; E_1(Y) = \{[t_\tau, e]: e \in A, e \neq e\}$, and $E_2(Y) = \{[t_\tau, e]: e \in A, e = e\}$.

$Y_H$ consists of the edges of the forms $ab(e)$, and $ab(\bar{e})$, where $e \in E_0(Y) \cup E_1(Y) \cup E_2(Y), a \in D_{o(e)}$ and $b \in D^g_e$, and their terminals.

Similar to the sets of edges $E_0(Y), E_1(Y)$, and $E_2(Y)$, the structures of the sets of the edges $E_0(Y_H), E_1(Y_H)$, and $E_2(Y_H)$ can be formed as follows.

$E_0(Y_H) = E(T_H)$ consists of the set of edges of the following forms.
(a) ab(m), where a ∈ D_{o(m)} and b ∈ D_{m}^{a} such that ab ∈ D_{t(m)};
(b) ab(y), where a ∈ D_{o(y)} and b ∈ D_{y}^{a} such that ab[y] ∈ D_{t(y)}^{a};
(c) ab(x), where a ∈ D_{o(x)} and b ∈ D_{x}^{a} such that ab[x] ∈ D_{o(x)} and
H \cap ab[x]G_{b^{-1}a^{-1}} = \emptyset.

E_{1}(Y_{H}) consists of the set of edges of the following forms.
(a) ab(m), where a ∈ D_{o(m)} and b ∈ D_{m}^{a} such that ab \notin D_{t(m)};
(b) ab(y), where a ∈ D_{o(y)} and b ∈ D_{y}^{a} such that ab[y] \notin D_{t(y)}^{a};
(c) ab(x), where a ∈ D_{o(x)} and b ∈ D_{x}^{a} such that ab[x] \notin D_{o(x)} and
H \cap ab[x]G_{b^{-1}a^{-1}} = \emptyset.

E_{2}(Y_{H}) consists of the set of edges of the form: ab(x), where a ∈ D_{o(x)}, and
b ∈ D_{x}^{a} such that ab[x] \notin D_{o(x)}, and H \cap ab[x]G_{b^{-1}a^{-1}} \neq \emptyset.

Theorem 3.1 implies that H is the fundamental group of the quasi-graph of groups
Φ_{H} = (H/(X; Y_{H}; T_{H}); \Gamma ). Consequently H is a quasi-graph group. This completes the proof.

References


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