On the Integration of L^2 – Solutions of Non-Oscillatory Solutions to $x''+a(t)x'+k^2x=0$

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Abstract. In this note, sufficient conditions are given to guarantee when all solutions to the linear homogeneous differential equation $x''+a(t)x'+k^2x=0$ are both bounded and elements in $L^2[0,\infty)$. Furthermore, when the solutions are non-oscillatory, they are shown to approach 0 as $t\rightarrow\infty$ and upper bounds are given for $\int_0^\infty x(s)^2 ds$ and $\int_0^\infty x'(s)^2 ds$. These results may be extended to the equation $x''+a(t)x'+b(t)x=0$. Finally, a short discussion about the L^2 -solution to $x''+q(t)x=0$ then follows.

Keywords: bounded, non-oscillatory, L^2 – convergence, monotonic

In this note, we will discuss some properties of the linear homogeneous differential equation,

$$(1) \quad x'' + a(t)x' + k^2x = 0.$$

Specifically, we shall show under conditions all the solutions are bounded, in $L^2[0, \infty)$, and approach 0 as $t \rightarrow \infty$. Furthermore, we shall calculate bounds for the values of the improper integrals of the solutions and their derivatives, i.e., $\int_0^\infty x(s)^2 ds$ and $\int_0^\infty x'(s)^2 ds$. For a discussion of some previous results concerning the non-oscillatory behavior of solutions see the recent papers by Li and Yeh [7] and Kroopnick [6]. Theorems and proofs now follow.

Theorem I. Given equation (1). Suppose $a(\cdot)$ is a positive element in $C[0, \infty)$ such that $A_0 > a(t) > a_0 > 0$ for some positive constants A_0 and a_0 , then all solutions to (1) are bounded. Moreover, if any solution $x(\cdot)$ is non-oscillatory then both $x(t) \rightarrow 0$ and $x'(t) \rightarrow 0$ as $t \rightarrow \infty$. Finally, the solution and its derivative are both elements of $L^2[0, \infty)$.

Proof. We first prove the boundedness of all solutions. Multiply equation (1) by $2x'$ and integrate from 0 to t obtaining

$$(2) \quad x'(t)^2 + 2 \int_0^t a(s)x'(s)^2 ds + k^2x(t)^2 = x'(0)^2 + k^2x(0)^2.$$

From equation (2) we may conclude that since all terms on the LHS of (2) are positive and since the RHS of (2) is bounded, both $|x(\cdot)|$ and $|x'(\cdot)|$ must remain bounded as

$t \rightarrow \infty$. Otherwise, the LHS of (2) would become infinite which is impossible.

Furthermore, since $a(t) > a_0 > 0$, it easily follows that $|x'(\cdot)|$ is in $L^2[0, \infty)$ since the integral must be bounded as $t \rightarrow \infty$.

Next, we show that $x(\cdot)$, too, is in $L^2[0, \infty)$ if the solution is non-oscillatory. Multiply (1) by $x(t)$ and integrate from 0 to t and then integrate the first term by parts to obtain

$$(3) \quad x(t)x'(t) - \int_0^t x(s)^2 ds + \int_0^t a(s)x'(s)x(s) ds + k^2 \int_0^t x(s)^2 ds = k^2 x(0)x'(0).$$

Now, if we can show that $x'(\cdot)$ eventually does not change sign, then $x(\cdot)$ must eventually be monotonic. We will then show that $x(\cdot)$ is also an element of $L^2[0, \infty)$. These two facts imply along with what has been proven before will show that both $x(\cdot)$ and $x'(\cdot)$ must approach 0 as $t \rightarrow \infty$. Otherwise, the L^2 -convergence of the solution and its derivative could not occur. Suppose $x'(\cdot)$ does change sign infinitely often, then it is oscillatory. Consequently, $x'(t) = 0$ infinitely often. However, this means that if $x(t) > 0$ then $x''(t) < 0$. So, $x(t)$ has an infinite number of consecutive critical points which are all relative maxima which is impossible. Likewise, should $x(t) < 0$, then we have an infinite number of consecutive relative minima which is also impossible. Consequently, $x'(t)$ must be non-oscillatory. This implies that $x(t)x'(t)$ does not change sign. Therefore, we may invoke the mean value theorem for integrals and integrate the third term of (3) to yield

$$(4) \quad x(t)x'(t) - \int_0^t x'(s)^2 ds + \frac{1}{2}a(u)x(t)^2 + k^2 \int_0^t x(s)^2 ds = k^2 x(0)x'(0) + \frac{1}{2}a(u)x(0)^2$$

where $0 < u < t$. Since the RHS of (4) is both positive and bounded and all terms on the LHS are either positive or bounded we may conclude that $|x(\cdot)|$ is in $L^2[0, \infty)$ and therefore both $x(t) \rightarrow 0$ and $x'(t) \rightarrow 0$ as $t \rightarrow \infty$.

We now state and prove Theorem II which deals with the integration of $x(\cdot)$ and $x'(\cdot)$ over $[0, \infty)$.

Theorem II. Under the conditions of Theorem I, the following inequalities hold:

$$(5) \quad \int_0^\infty x'(s)^2 ds \leq \frac{x'(0)^2 + k^2 x(0)^2}{2a_0}.$$

and

$$(6) \quad \int_0^t x(s)^2 ds \leq x(0)x'(0) + \frac{1/2a(0)x(0)^2}{k^2} + \frac{x(0)^2 + k^2 x'(0)^2}{2a_0 k^2}.$$

Proof. By relation (2), both $x(t) \rightarrow 0$ and $x'(t) \rightarrow 0$ as $t \rightarrow \infty$. Then after dividing by $2a_0$, inequality (5) easily follows. For inequality (6), first rewrite (4) as

$$(7) \quad x(t)x'(t) + 1/2a(u)x(t)^2 + k^2 \int_0^t x(s)^2 ds = k^2 x(0)x'(0) + 1/2a(0)x(0)^2 + \int_0^t x'(s)^2 ds.$$

Next, let $t \rightarrow \infty$ and use inequality (5) in place of the last term $\int_0^t x'(s)^2 ds$ in (4). By removing the positive term $1/2a(s)x(t)^2$ we now have the following inequality

$$(8) \quad k^2 \int_0^\infty x(s)^2 ds \leq k^2 x(0)x'(0) + 1/2a(0)x(0)^2 + \frac{x'(0)^2 + k^2 x'(0)^2}{2a_0}.$$

Finally, dividing (8) by k^2 gives us inequality (6). The proof is complete.

Remark 1. The above analysis may easily be extended to the general linear homogeneous equation

$$(9) \quad x'' + a(t)x' + b(t)x = 0$$

with the same conditions on $b(\cdot)$ as $a(\cdot)$, i.e., that it is continuous on $C[0, \infty)$ such that $B_0 > b(t) \geq b_0 > 0$ for positive constants B_0 and b_0 . Instead of equation (2), we would have

$$(10) \quad x'(t)^2 + 2 \int_0^t a(s)x'(s)^2 ds + b(u)x(t)^2 = x'(0)^2 + b(u)x'(0)^2$$

where $0 < u < t$. Notice that in Theorem I, $a(\cdot)$ need not be differentiable at any point.

In the case of equation (9), neither $a(\cdot)$ or $b(\cdot)$ need to be differentiable at any point. The inequalities for (9) are nearly identical to (5) and (6). Just replace k^2 with b_0 . For numerous applications to equation (9) see [8, chapters 6 and 8].

Example. Consider the differential equation

$$(11) \quad x'' + 10x' + 9x = 0$$

which has non-oscillatory solutions $x(t) = A \exp(-t)$ and $x(t) = B \exp(-9t)$. From Theorem I we can show that the equation

$$(12) \quad x'' + (2 + \cos(t))x' + 9x = 0$$

has non-oscillatory solutions by comparing equation (11) to equation (10) using the Sturm-Picone Comparison Theorem [1]. First, put each equation in self-adjoint form to obtain

$$(13) \quad (e^{10t}x')' + 9e^{10t}x = 0$$

and

$$(14) \quad (e^{(2t+\sin(t))}x')' + 9e^{(2t+\sin(t))}x = 0$$

Since $e^{(2t+\sin(t))} < e^{10t}$, the solutions to equation (14) must be non-oscillatory because the solutions to (13) are non-oscillatory by invoking the Sturm-Picone Comparison Theorem. That is, if the solutions to equation (14) were oscillatory, then the same would be true for the solutions to equation (13) which is certainly not the case by the comparison theorem.

Remark 2. See [5] for non-oscillatory criteria for case when the damping term is absent in (9), i.e., the case when $a(t)=0$. The study of this equation is extremely important in the study of the spectral theory of linear operators [9]. The analysis of this equation is more intricate than what we have discussed here but it is certainly worthy of much serious study. See ([1, ch. 9], [3], [4], and [9]) for an introduction to the classical theory as well as some significant results concerning this important differential equation.

References

- [1]. Coddington, E. A. and Levinson, N., **Theory of Ordinary Differential Equations**, McGraw-Hill, New York, 1955.
- [2]. Diaz, J. B., McLaughlin, Joyce R., *Sturm comparison theorems for ordinary and partial differential equations*. **Bull. Amer. Math. Soc.** (75), 1969, 335–339.
- [3]. Eastham, M.S.P., *On a limit point method of Hartman*, Bull. Lond. Math. Soc. (4), 1972, 340-344.
- [4]. Hartman, P. On the number of L^2 -solutions of $x''+q(t)x=0$, **Amer. J. Math.** (73), 1951, 645-645.
- [5]. Hartman, P, *On nonoscillatory linear differential equations of second order*, **Proc. Amer. Math. Soc.** (64), 1977, 251-259.
- [6]. Kroopnick, A., *Asymptotic behavior and L^2 - properties of non-oscillatory solutions to $x''+a(t)x'+b(t)x=0$* , **International Mathematical Forum** (7), 2012, 2827-2831.
- [7]. Li, Horn-Jann and Yeh, Cheg-Chih, *On the nonoscillatory behavior of solutions of a second order linear differential equation*, **Mathematische Nachrichten** (182), 2006, 295-315.
- [8]. Tenenbaum, Morris and Pollard, Harry, Morris, **Ordinary Differential Equations**, Dover Publications, New York, 1985.
- [9]. Titchmarsh, E.C., **Eigenfunction Expansions, Part II** (second edition), Oxford University Press, London, 1962.

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