On Generalized f -Derivations of BE -Algebras

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Abstract

In this paper, we introduce the notion of a generalized f -derivation in a BE -algebra, and obtain some properties in generalized f -derivations. Also, we characterized isotone generalized f -derivations and $KerD$.

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1 Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK -algebras and BCI -algebras([3, 4]). It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH -algebras. They have shown that the class of BCI -algebras is a proper subclass of the class of BCH -algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE -algebra as a dualization of a generation of a BCK -algebras. In this paper, we introduce the

notion of a generalized f -derivation in a BE - algebra, and obtain some properties in generalized f -derivations. Also, we characterized isotone generalized f -derivations and $Ker D$.

2 Preliminaries

In what follows, let X denote an BE -algebra unless otherwise specified.

By an BE -algebra we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation “ $*$ ” that satisfies the following identities: for any $x, y, z \in X$,

- (BE1) $x * x = 1$ for all $x \in X$,
- (BE2) $x * 1 = 1$ for all $x \in X$,
- (BE3) $1 * x = x$ for all $x \in X$,
- (BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

An BE -algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of an BE -algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. For any x, y in a BE -algebra X , we define $x \vee y = (y * x) * x$.

Let X be a BE -algebra. We define the binary operation “ $\leq$ ” as the following,

$$x \leq y \Leftrightarrow x * y = 1$$

for all $x, y \in X$.

In an BE -algebra, the following identities are true:

- (p1) $x * (y * x) = 1$.
- (p2) $x * ((x * y) * y) = 1$.
- (p3) Let X be a distributive BE -algebra. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$ for all $x, y, z \in X$.

Let $(X, *, 1)$ be an BE -algebra and F a non-empty subset of X . Then F is called a *filter* of X if

- (F1) $1 \in F$,

(F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

Definition 2.1. Let X be a BE -algebra. A function $d : X \rightarrow X$ is called an f -derivation on X if there exists an endomorphism $f : X \rightarrow X$ such that

$$d(x * y) = (f(x) * d(y)) \vee (d(x) * f(y))$$

for every $x, y \in X$.

Proposition 2.2. Let d be a f -derivation of X . Then we have

$$(1) \ d(1) = 1,$$

$$(2) \ d(x) = d(x) \vee f(x) \text{ for all } x \in X.$$

3 Generalized f -derivations of BE -algebras

Definition 3.1. Let X be a BE -algebra and $f : X \rightarrow X$ be an endomorphism. A self-map $D : X \rightarrow X$ is called a *generalized f -derivation* on X if there exists a f -derivation $d : X \rightarrow X$ such that

$$D(x * y) = (f(x) * D(y)) \vee (d(x) * f(y))$$

for all $x, y \in X$.

Remark. If $D = d$, then D is a f -derivation.

Example 3.2. Let $X = \{1, a, b\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is a BE -algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a \\ a & \text{if } x = b \end{cases}$$

and define a map $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a, b \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X . Also, define a map $D : X \rightarrow X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1, b \\ b & \text{if } x = a. \end{cases}$$

Then it is easy to check that D is a generalized f -derivation of X .

Example 3.3. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

and define a map $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a \\ b & \text{if } x = c \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X . Also, define a map $D : X \rightarrow X$ by

$$D(x) = \begin{cases} 1 & \text{if } x = 1, b \\ b & \text{if } x = c. \\ a & \text{if } x = a. \end{cases}$$

It is easy to verify that D is a generalized f -derivation of X .

Proposition 3.4. Let D be a generalized f -derivation of X . Then we have

- (1) $D(1) = 1$,
- (2) $D(x) = D(x) \vee f(x)$ for all $x \in X$.

Proof. (1) Let D be a generalized f -derivation of X . Then we have

$$\begin{aligned} D(1) &= D(1 * 1) = (f(1) * D(1)) \vee (d(1) * f(1)) = (1 * D(1)) \vee (1 * 1) \\ &= D(1) \wedge 1 = (1 * D(1)) * D(1) = D(1) * D(1) = 1. \end{aligned}$$

(2) For all $x \in X$, we have

$$\begin{aligned} D(x) &= D(1 * x) = (f(1) * D(x)) \vee (d(1) * f(x)) \\ &= (1 * D(x)) \vee (1 * f(x)) = D(x) \vee f(x). \end{aligned}$$

□

Proposition 3.5. *Let D be a generalized f -derivation of X . Then the following identities hold:*

(1) $f(x) \leq D(x)$ for all $x \in X$,

(2) If X is a self distributive BE-algebra, then $D(x) * y \leq f(x) * y$ for all $x, y \in X$.

Proof. (1) By Proposition 3.4(2) and (BE4), we have for all $x \in X$,

$$\begin{aligned} f(x) * D(x) &= f(x) * (D(x) \vee f(x)) = f(x) * ((f(x) * D(x)) * D(x)) \\ &= (f(x) * D(x)) * (f(x) * D(x)) = 1 \end{aligned}$$

which implies $f(x) \leq D(x)$.

(2) By (1) and (p3), we have $D(x) * y \leq f(x) * y$. □

Proposition 3.6. *If D is a generalized f -derivation of X , then we have $D(D(x) * x) = 1$ for all $x \in X$.*

Proof. Let D be a generalized f -derivation of X . Then we have

$$\begin{aligned} D(D(x) * x) &= (D(x) * D(x)) \vee (d(D(x))) * f(x) \\ &= 1 \vee (d(D(x))) * x = 1 \end{aligned}$$

for all $x \in X$. □

Proposition 3.7. *Let X be a BE-algebra. A generalized f -derivation $D : X \rightarrow X$ is an identity map if it satisfies $x * D(y) = D(x) * y$ for all $x, y \in X$*

Proof. Let $x, y \in X$ be such that $x * D(y) = D(x) * y$. Now $D(x) = D(1 * x) = 1 * D(x) = D(1) * x = 1 * x = x$. Thus d is an identity map. □

Proposition 3.8. *Let X be a BE-algebra. Then*

$$f(x) \leq D_n(D_{n-1}(\dots(D_2(D_1(x))\dots)))$$

for $n \in \mathbb{N}$, where $D_1, D_2, \dots, D_n$ are generalized f -derivations of X .

Proof. For $n = 1$, we have

$$\begin{aligned} D_1(x) &= D_1(1 * x) = (f(1) * D_1(x)) \vee (d_1(1) * f(x)) \\ &= (1 * D_1(x)) \vee (1 * f(x)) = D_1(x) \vee f(x) = (f(x) * D_1(x)) * D_1(x), \end{aligned}$$

which implies $f(x) * D_1(x) = 1$. Thus $f(x) \leq D_1(x)$.

Let $n \in \mathbb{N}$ and $f(x) \leq D_n(D_{n-1}(\dots(D_2(D_1(x))\dots)))$. For simplicity, let

$$T_n = D_n(D_{n-1}(\dots(D_2(D_1(x))\dots))).$$

Then

$$\begin{aligned} D_{n+1}(T_n) &= D_{n+1}(1 * T_n) = (f(1) * D_{n+1}(T_n)) \vee (d_{n+1}(1) * T_n) \\ &= (1 * D_{n+1}(T_n)) \vee (1 * T_n) \\ &= D_{n+1}(T_n) \vee T_n = (T_n * D_{n+1}(T_n)) * D_{n+1}(T_n). \end{aligned}$$

Hence $T_n * T_{n+1} = 1$, which implies $T_n \leq T_{n+1}$. By assumption, $f(x) \leq T_n \leq T_{n+1}$. $\square$

Let D be a generalized f -derivation of X . Define a $\text{Ker}D$ by

$$\text{Ker}D = \{x \mid D(x) = 1\}$$

for all $x \in X$.

Proposition 3.9. *Let D be a generalized f -derivation of X and $y \in \text{Ker}D$. Then we have $x \vee y \in \text{Ker}D$ for all $x \in X$.*

Proof. Let D be a generalized f -derivation of X and $y \in \text{Ker}D$. Then $D(y) = 1$. Hence we have

$$\begin{aligned} D(x * y) &= (f(x) * D(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (1 * f(y)) = 1 \vee f(y) = 1. \end{aligned}$$

$\square$

An BE -algebra X is said to be *commutative* if for all $x, y \in X$,

$$(y * x) * x = (x * y) * y.$$

Proposition 3.10. *Let X be a commutative BE -algebra and let D be a generalized f -derivation. If $x \in \text{Ker}D$ and $x \leq y$, then we have $y \in \text{Ker}D$.*

Proof. Let $x \in \text{Ker} D$ and $x \leq y$. Then $D(x) = 1$ and $x * y = 1$.

$$\begin{aligned} D(y) &= D(1 * y) = D((x * y) * y) \\ &= (f(y * x) * D(x)) \vee (d(y * x) * f(x)) \\ &= (f(y * x) * 1) \vee (d(y * x) * f(x)) \\ &= 1 \vee (d(y * x) * f(x)) = 1, \end{aligned}$$

and so $y \in \text{Ker} D$. This completes the proof. $\square$

Proposition 3.11. *Let D be a generalized f -derivation of X and an endomorphism. Then $\text{Ker} D$ is a filter of X .*

Proof. Clearly, $1 \in \text{Ker} D$. Let $x \in \text{Ker} D$ and $x * y \in \text{Ker} D$. Then we have $D(x) = D(x * y) = 1$, and so

$$1 = D(x * y) = D(x) * D(y) = 1 * D(y) = D(y).$$

This implies $y \in \text{Ker} D$. This completes the proof. $\square$

Proposition 3.12. *Let D be a generalized f -derivation of X and $y \in \text{Ker} D$. Then we have $x * y \in \text{Ker} D$ for all $x \in X$.*

Proof. Let D be a generalized f -derivation of X and $y \in \text{Ker} D$. Then $D(y) = 1$. Hence we have for all $x \in X$,

$$\begin{aligned} D(x * y) &= (f(x) * D(y)) \vee (d(x) * f(y)) \\ &= (x * 1) \vee (d(x) * y) = 1 \vee d(x) * y = 1. \end{aligned}$$

$\square$

Proposition 3.13. *Let D be a generalized f -derivation of X . If D is one-to-one, then $\text{Ker} D = 1$.*

Proof. Suppose that D is one-to-one and $x \in \text{Ker} D$. Then $D(x) = 1 = D(1)$, and thus $x = 1$, i.e., $\text{Ker} D = \{1\}$. $\square$

Definition 3.14. Let X be a BE -algebra. A self-map D is *isotone* if $x \leq y$ implies $D(x) \leq D(y)$.

Theorem 3.15. *Let X be a commutative BE -algebra and let D be a generalized f -derivation of X . Then the following conditions are equivalent:*

- (1) D is an isotone generalized f -derivation of X ,
- (2) $D(x) \vee D(y) \leq D(x \vee y)$ for all $x, y \in X$.

Proof. (1) $\Rightarrow$ (2). Suppose that D is an isotone generalized f -derivation of X . We know that $x \leq x \vee y$ and $y \leq x \vee y$. Since D is isotone, we have $D(x) \leq D(x \vee y)$ and $D(y) \leq D(x \vee y)$. Hence we get $D(x) \vee D(y) \leq D(x \vee y)$.

(2) $\Rightarrow$ (1). Suppose that $D(x) \vee D(y) \leq D(x \vee y)$ and $x \leq y$. Then we get $D(x) \leq D(x) \vee D(y) \leq D(x \vee y) = D(y)$ since $x \vee y = (y * x) * x = (x * y) * y = 1 * y = y$. $\square$

Proposition 3.16. *Let D be a generalized f -derivation of X . If D is an endomorphism of X , then D is isotone.*

Proof. Let $x \leq y$. Then $x * y = 1$. Hence we have

$$D(x) * D(y) = D(x * y) = D(1) = 1,$$

which implies $D(x) \leq D(y)$. This completes the proof. $\square$

Proposition 3.17. *Let D be an isotone generalized f -derivation of X . If $x \leq y$ and $x \in \text{Ker} D$, then $y \in \text{Ker} D$.*

Proof. Let $x \leq y$ and $x \in \text{Ker} D$. Then we have $D(x) = 1$, and so

$$1 = D(x) \leq D(y),$$

which implies $D(y) = 1$. $\square$

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