The Completeness of S4

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Abstract

In this paper, we redefine notion of topo-bisimulation via bi-open relation and review the properties of topo-bisimulation. We also define new dyadic tree. Finally, by means of this dyadic tree, the proof of “S4 is complete with respect to $\mathbb{Q}$” in [3] is examined.

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1 Introduction

Topo-bisimulations, bisimulations for topological models, were introduced in [1]. It is well-known that bisimulation between topological models is a generalization of the notion of bisimulation between preorder models. In this paper, we redefine notion of topo-bisimulation via bi-open relation. This is a useful tool about what is topo-bisimulation.

The correspondence between elementary topology and the modal logic $\textbf{S4}$ was first established by McKinsey. In [9], he introduced the topological interpretation of $\textbf{S4}$, where the necessitation operator box, $\Box$, is interpreted as the topological interior, and showed that $\textbf{S4}$ is complete for the class of all topological spaces. Later more mathematically interesting results such as “$\textbf{S4}$ is complete for any dense-in-itself separable metric space” where obtained by McKinsey and Tarski [10] ([7]). In particular, $\textbf{S4}$ is complete modal logic of the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the Cantor space $\mathbb{C}$. An alternative proof
of completeness of $S4$ with respect to $Q$ were given in [3]. In this paper, we define new dyadic tree. By means of this dyadic tree, the alternative proof of completeness of $S4$ with respect to $Q$ in [3] is considered.

Recall that topologic space is a pair $(E, \mathcal{O}(E))$, where $E$ is a nonempty set and $\mathcal{O}(E)$ is a collection of subsets of $E$ satisfying the following three conditions:

(1) $\emptyset, E \in \mathcal{O}(E)$;
(2) if $U \in \mathcal{O}(E)$ and $V \in \mathcal{O}(E)$ then $U \cap V \in \mathcal{O}(E)$;
(3) if $U_i \in \mathcal{O}(E)$ for all $i \in I$, with $I$ some index set, then $\bigcup_{i \in I} U_i \in \mathcal{O}(E)$.

**Definition 1.1** The basic modal language $\mathcal{ML}$ consists of a countable infinite set of propositional letters $P = \{p_1, p_2, \ldots\}$, the constat falsity $\bot$, the Boolean connectives $\lor, \land$ and the modal operator $\Box$.

Well-formed formulas $\phi$ of $\mathcal{ML}$ are given by the following grammar:

$\phi ::= \bot | p_i | \phi \land \phi | \neg \phi | \Box \phi$

We use $\Box \phi$ as an abbreviation for $\neg \Box \neg \phi$.

**Definition 1.2** The modal logic $S4$ is the smallest set of modal formulas which contains all instances of propositional tautologies, the following axioms: $p \rightarrow \Box p$

$\Box \Box p \rightarrow \Box p$

$\Box (p \lor q) \leftrightarrow \Box p \lor \Box q$

$\Box \bot \leftrightarrow \bot$

and is closed under modus ponens, substitution and $\vdash \varphi \rightarrow \phi \rightarrow \vdash \Box \varphi \rightarrow \Box \phi$. For details, see [4].

**Definition 1.3** (Topological Models) Given a collection of proposition letters $P$, a topological model is a triple $\mathcal{M} = (E, \mathcal{O}(E), \mu)$ where $(E, \mathcal{O}(E))$ is a topological space and $\mu : P \rightarrow \mathcal{P}(E)$ a valuation. Then for $x \in E$, the model operators $\Box$ and $\Diamond$ are interpreted as follows.

$(\mathcal{M}, x) \models \Diamond \varphi$ if $$(\forall U \in \mathcal{O}(E))(x \in U \implies (\exists y \in U)(y \models \varphi))$$

$(\mathcal{M}, x) \models \Box \varphi$ if $$(\exists U \in \mathcal{O}(E))(x \in U \land (\forall y \in U)(y \models \varphi)).$$
2 Topo-Bisimulations

2.1 Bi-open Relations

Let \((E, O(E))\) and \((F, O(F))\) topological spaces and \(R\) be relation between \(E\) and \(F\). We will say that \(R\) is bi-open whenever the following two conditions are satisfied.

For each open subset \(U\) in \(E\), the subset \(R(U)\) is open in \(F\).

For each open subset \(V\) in \(F\), the subset \(R^{-1}(V)\) is open in \(E\).

Let \(f\) be a function from \(E\) to \(F\), and let \(\Gamma = \{(x, f(x)) : x \in E\}\) be the graph of \(f\), so that \(\Gamma\) is a relation between \(E\) and \(F\). Then \(f\) is an open continuous map from \(E\) to \(F\) if and only if \(\Gamma\) is bi-open, as is very easily seen.

Thus, bi-open relations are natural generalizations of open continuous maps.

Clearly, a relation \(R\) is bi-open iff its converse \(R^{-1}\) is.

The reason we introduce this notion of bi-openness is that it is a useful tool for what is now known as topo-bisimulation.

2.2 Topo-bisimulation

We are given topological spaces \((E, O(E))\) and \((F, O(F))\) and a collection of proposition letters, together with maps \(\mu : P \rightarrow P(E)\) and \(\nu : P \rightarrow P(F)\).

In this setting, a topo-bisimulation is usually defined as follows. It is a nonempty relation \(R\) between \(E\) and \(F\) such that the following three conditions hold. If \(xRy\) then

1. \(x \in \mu(p)\) iff \(y \in \nu(p)\);
2. if \(x \in U \in O(E)\) then there is a \(V\) such that \(y \in V \in O(F)\) and for each \(t \in V\) there is a \(z \in U\) such that \(zRt\);
3. if \(y \in V \in O(F)\) then there is a \(U\) such that \(x \in U \in O(E)\) and for each \(z \in U\) there is a \(t \in V\) such that \(zRt\).

Now, condition (2) clearly means that \(R(U) \subseteq O(F)\) for each \(U \subseteq O(E)\). Similarly, condition (3) means that \(R^{-1}(V) \subseteq O(E)\) for each \(V \subseteq O(F)\).

So, conditions (2) and (3) simply boil down to this: \(R\) is a bi-open relation between \(E\) and \(F\).

As to condition (1), it translates into the following

\[ (*) \] \(R(\mu(p)) \subseteq \nu(p)\) and \(R^{-1}(\nu(p)) \subseteq \mu(p)\) for each \(p \in P\).
Thus, topo-bisimulation between \((E, \mu)\) and \((F, \nu)\) is a simply a nonempty bi-open relation \(R\) between the spaces \(E\) and \(F\) satisfying condition (*)

### 2.3 From bi-open to topo-bisimulation

Now, notice condition (*) implies the following condition

\[
(**) \quad R^{-1}R(\mu(p)) \subset \mu(p) \quad \text{for each } p \in P.
\]

Conversely, suppose condition (**) is satisfied and let \(\rho(p) = R(\mu(p))\) for each \(p \in P\). We have \(R^{-1}(\rho(p)) = R^{-1}R(\mu(p)) \subset \mu(p)\), so that \(R\) is a topo-bisimulation between \((E, \mu)\) and \((F, \rho)\).

Summing up, let topological spaces \(E\) and \(F\) be given together with a map \(\mu : P \rightarrow \mathcal{P}(E)\) and a nonempty bi-open relation \(R \subset E \times F\). In order that there exist a map \(\nu : P \rightarrow \mathcal{P}(F)\) such that \(R\) be a topo-bisimulation between \((E, \mu)\) and \((F, \rho)\) it is necessary and sufficient condition (**) be satisfied. One such \(\nu : P \rightarrow \mathcal{P}(F)\) is then always given by \(\nu(p) = R(\mu(p))\) for each \(p \in P\).

Here are two very special cases of this construction. The first appears quite frequently in the literature. The second construction does appear in the literature, though in a much weaker form. (See, [3]).

Consider an open continuous function \(f : E \rightarrow F\) and its graph \(\Gamma \subset E \times F\). We have already noticed that \(\Gamma\) is a bi-open relation, and so is also \(\Gamma^{-1}\) of course.

**First construction.** Given any map \(\nu : P \rightarrow \mathcal{P}(F)\), define \(\mu(p) = f^{-1}(\nu(p))\). Then \(\Gamma\) is a topo-bisimulation between \((E, \mu)\) and \((F, \nu)\).

Indeed, condition (**) for \(\Gamma^{-1}\) and \(\nu\) reads \(\Gamma\Gamma^{-1}(\nu(p)) \subset \nu(p)\). Now, given any subset \(Y \subset F\), one clearly has \(f f^{-1}(Y) \subset Y\), that is \(\Gamma\Gamma^{-1}(Y) \subset Y\). Thus condition (**) is clearly satisfied.

**Second construction.** Given a map \(\mu : P \rightarrow \mathcal{P}(E)\), we know that there exists a map \(\nu : P \rightarrow \mathcal{P}(F)\) such that \(\Gamma\) is a topo-bisimulation if (and only if) \(f^{-1}f(\mu(p)) \subset \mu(p)\) for each \(p \in P\), and one such \(\nu\) is always given by \(\nu(p) = f(\mu(p))\).

**Lemma 2.1** Given two topo-bisimulations, \(R\) between \((E, \mu)\) and \((F, \nu)\), and \(S\) between \((F, \nu)\) and \((G, \rho)\). \(T = SR\) is a topo-bisimulation between \((E, \mu)\) and \((G, \rho)\).

**Proof.** \(T\) is easily seen to be bi-open, using the definition of bi-openness, and condition (*) for \((T, \mu, \rho)\) follows immediately from conditions (*) for \((R, \mu, \nu)\) and \((S, \nu, \rho)\). (See, [9]).
2.4 Inverted Topological Models

Now, to each topological model \((E, \mu)\) we will associate the inverted model \((E, \mu^c)\) where \(\mu^c(p) = E \setminus \mu(p)\).

The reason we introduce this notion of inversion is the following quite useful result.

**Lemma 2.2** Let \(R\) be a topo-bisimulation between two topological models \((E, \mu)\) and \((F, \nu)\). Then, \(R\) is still a topo-bisimulation between the two inverted models \((E, \mu^c)\) and \((F, \nu^c)\).

**Proof.** Since \(R\) is still bi-open anyway, there only remains to show the following purely set-theoretical fact: Let \(xRy\) be true. Then, by definition, we have \(x \in \mu(p)\) iff \(y \in \nu(p)\) so that we also have \(x \not\in \mu^c(p)\) iff \(y \not\in \nu^c(p)\). □

This leads to the interesting observation that condition (*) is equivalent to the following condition:

\[ (***) R(\mu^c(p)) \subset \nu^c(p) \text{ and } R^{-1}(\nu^c(p)) \subset \mu^c(p) \text{ for each } p \in P. \]

**Lemma 2.3** Let \(E\) and \(F\) are two topological spaces and \(R\) be bi-open relation between them. Then, given \(X \subseteq E\) and \(Y \subseteq F\), we always have \(R(X^\circ) \subseteq R(X)^\circ\) and \(R^{-1}(Y^\circ) \subseteq R^{-1}(Y)^\circ\).

**Proof.** Since \(R\) is a bi-open relation, the subset \(R(X^\circ)\) is an open subset contained in \(R(X)\), similarly, \(R^{-1}(Y^\circ)\) is an open subset of \(R^{-1}(Y)\). (See, [9]). □

**Theorem 2.4** Let \(R\) be a topo-bisimulation between two topological models \((E, \mu)\) and \((F, \nu)\). For any formula \(\varphi\), if \(xRy\) then \(x \in \mu(\varphi)\) iff \(y \in \nu(\varphi)\).

**Proof.** Use induction on the complexity of the formula \(\varphi\). Atomic and Boolean cases for \(\varphi\) are straightforward. Consider the box case: \(\varphi = \square \psi\). Then the following computation is clear.

\[ R(\mu(\square \psi)) = R(\mu(\psi)^\circ) \text{ (by definition) } \]
\[ \quad \subset R(\mu(\psi))^\circ \text{ (Lemma 2.3) } \]

So by induction hypothesis

\[ R(\mu(\psi))^\circ \subset \nu(\psi)^\circ = \nu(\square \psi). \]

Conversely,

\[ R^{-1}(\nu(\square \psi)) = R^{-1}(\nu(\psi)^\circ) \text{ (by definition) } \]
Thus, by induction hypothesis, again,

\[ R^{-1}(\nu(\psi)) \subseteq \mu(\psi) \]

\( \square \)

2.5 **Alexandroff Topologic spaces**

A topological space is called Alexandroff space if a topological space in which any intersection of open subsets is again an open subset. This definition is easily seen to be equivalent to each of the following:

(i) An Alexandroff space is a topological space in which every point is contained in a smallest open subset.

(ii) An Alexandroff space is a topological space in which every point has a smallest neighbourhood (which, of course, is open).

Let \((E, R)\) correspond to an Alexandroff topologic space \(E\). Then \(E\) is topological if \(R\) is a preorder on \(E\), as is very easily seen. That is, Alexandroff topologic spaces are in a canonical one to one correspondence with preorders, which is a very well-known fact.

Let \((E, \mu)\) and \((F, \nu)\) be two topological models where \(E\) and \(F\) are Alexandroff topologic spaces. We have two preorder relations \(S\) on \(E\), and \(T\) on \(F\), such that, for points \(x \in E\) and \(y \in F\), the subsets \(S(x)\) and \(T(y)\) are the smallest neighbourhoods of \(x\) and \(y\) respectively.

What is, then, a bi-open relation \(R\) between the spaces \(E\) and \(F\)? It is a relation \(R \subseteq E \times F\) such that the following condition hold. Given \(xRy\), we have

(i) \(S(x) \subseteq R^{-1}(T(y))\) and

(ii) \(T(y) \subseteq R(S(x))\)

We translate,

(i) If \(xSz\), then there exist \(t \in F\) such that \(yTt\) and \(zRt\).

(ii) If \(yTt\) then there exist \(z \in E\) such that \(xSz\) and \(zRt\).

**Conclusion.** It is seen, thus, that topological models are a generalization of the notion of preorder models for modal logic, and bisimulation between topological models is a generalization of the notion of bisimulation between preorder models.
3 The Completeness of $S_4$

3.1 Forests and Trees

A Forest is usually defined to be a partially ordered set $(F, \leq)$ such that, for each $x \in F$, the subset $\{y : y \leq x\}$ is a chain. A tree is a forest having a unique minimal element $r$, its root.

The dyadic tree $T_2$ is usually defined to be the set of finite strings over $\{0, 1\}$, including the empty string (the root of $T_2$), with the partial order $(s_1, s_2, \ldots, s_m) \leq (t_1, t_2, \ldots, t_n)$ iff $m \leq n$ and $s_k = t_k$ for $1 \leq k \leq m$. (See, [11]).

For reasons that will soon appear, we are going to give a new definition for $T_2$. The partially ordered set that we are going to define, new tree, is certainly isomorphic to the classical tree $T_2$, but with a slight twist, a twist that we hope will prove to be quite efficient in applications. Our tree $T_2$ will grow over $\{-1, 1\}$ instead of $\{0, 1\}$. Moreover, the nodes of our tree will always be denoted by infinite strings since they are the infinite paths (or branches) of the tree.

3.2 New Dyadic Tree

The nodes of new dyadic tree $T_2$ are infinite strings $s = (s_1, s_2, \ldots, s_k, \ldots)$ starting with a finite number of $\mp 1$ and ending with only 0’s. To be more precise, a node of the tree is a string $s = (s_1, s_2, \ldots, s_k, \ldots)$ for which there exists an index $n \geq 0$ such that

$s_k = \pm 1$ for each $k \leq n$ and $s_k = 0$ for each $k > n$.

The index $n$ will be called the height of the node $s$ and denoted by $hg(s)$. A partial order $R$ is defined on the set of nodes as follows:

$sRt$ iff $hg(s) \leq hg(t)$ and $s_k = t_k$ for each $k \leq hg(s)$.

That $R$ is a partial order is immediately seen. The string $0 = (0, 0, \ldots, 0, \ldots)$ with only 0’s is, of course, the root of the tree.

The tree is clearly isomorphic to the dyadic tree $T_2$.

An (infinite) branch of the tree is simply an infinite string $b = (b_1, b_2, \ldots, b_k, \ldots)$ where $b_k = \pm 1$ for each $k \geq 1$. Let us denote the set of branches by $B_2$. The partial order $R$ is easily extended into a partial order $S$ over the whole set of nodes and branches $\overline{T_2} = T_2 \cup B_2$ which is again a tree itself, the complete dyadic tree. Two different branches are never comparable, and the nodes
that precede a given branch \( b = (b_1, b_2, \ldots, b_k, \ldots) \) are those of the forms \( s = (b_1, b_2, \ldots, b_n, \ldots, 0, 0, 0, \ldots) \) for each height \( n \geq 0 \).

We further introduce the following notations. Given any node \( s \) of the tree \( T_2 \), set \( U(s) = S(s) = \{ y : sSy \} \), \( V(s) = R(s) = \{ t : sRt \} \) and \( W(s) = \{ b : b \in B_2 \) ve \( sSb \} \). Clearly, \( V(s) = U(s) \cap T_2 \) and \( W(s) = U(s) \cap B_2 \). The family of subsets \( U(s) \) (resp. \( V(s) \) and \( W(s) \)) form an open base for the (canonical) topology on \( T_2 \) (resp. \( T_2 \) and, \( B_2 \)).

Let \( E(n) \) be the set of all the nodes whose height is \( n \), and let \( F(n) \) be the set of all the nodes whose heights are \( < n \). Notice that \( F(n) \) is a finite and closed subset in \( T_2 \). The \( U(t) \)'s for \( t \in E(n) \), are open, finite in number, two by two disjoint, and cover \( T_2 \setminus F(n) \), so that each of them is not only open but also closed in \( T_2 \setminus F(n) \).

The topologies \( T_2 \) and \( T_2 \) are Alexandroff topologies. The topology on \( B_2 \) is certainly no Alexandroff: given a branch \( b \), for each node \( s \) that precedes \( b \), the subset \( W(s) \) is a neighbourhood of \( b \), and the intersection of all those subsets \( W(s) \) is easily seen to be the singleton \( \{ b \} \) which is not a neighbourhood of \( b \). In fact, it is known that \( B_2 \) is a copy of the Cantor space.

### 3.3 The Essential Map

We define a map \( f : T_2 \rightarrow \mathbb{R} \) as follows, and call it the essential map:

\[
f(s) = \sum_{k>0} \frac{s_k}{3^k}, \text{ for } s = (s_1, s_2, \ldots, s_k, \ldots).
\]

We first notice that \( f \) is well defined and that \( f(x) \) is a real number in the interval \([−1/2, 1/2]\) for each \( x \in T_2 \). Also \( f(−x) = −f(x) \), where \( −x \) is of course a node whenever \( x \) is, and a branch whenever \( x \) is! We also have \( f(0) = 0 \) where \( 0 = (0, 0, \ldots, 0, \ldots) \) is the root of the tree.

Set \( H = f(T_2) \), \( K = f(B_2) \) and \( M = f(T_2) = H \cup K \). Two restrictions of the essential map are important to us, the restriction to the set of nodes, and the restriction to the set of branches. So we get \( g \mid_{T_2} : T_2 \rightarrow H \) and \( h \mid_{B_2} : B_2 \rightarrow K \). We already know that \( f(T_2) \subset [−1/2, 1/2] \). So both \( H \) and \( K \) are subsets of the interval \([−1/2, 1/2]\).

We know state and prove of the properties of the essential map.

**Essential Properties**

1. The set \( H \) is a subset of \( \mathbb{Q} \) contained in the open interval \((-1/2, 1/2)\).

2. The function \( f \) is one-to-one, so that both \( g \) and \( h \) bijective.
(3) The function \( f : T_2 \rightarrow M \) is an open function, therefore \( g : T_2 \rightarrow H \), and \( h : B_2 \rightarrow K \) are also open.

(4) The function \( h : B_2 \rightarrow K \) is also continuous, i.e. \( h \) is a homeomorphism from \( B_2 \) onto \( K \). The space \( K \) is therefore a copy of the Cantor space, since \( B_2 \) is a such a copy, itself.

(5) The function \( g : T_2 \rightarrow H \) is not continuous otherwise \( T_2 \), an Alexandroff space, would be homeomorphic to \( H \), a Hausdorff space.

The Proofs.

(1) Obviously, for each \( s \in T_2 \), the number \( f(s) \) is rational since \( \sum_{k \geq 1} s_k/3^k \) is then the sum of a finite number of rational numbers. Moreover, observe that \( |f(s)| < \sum_{k \geq 1} 1/3^k = 1/2 \) for each \( s \in T_2 \). Therefore \( H \subset (-1/2, 1/2) \cap \mathbb{Q} \).

(2) The function \( f : T_2 \rightarrow \mathbb{R} \) is one-to-one. Indeed, let \( f(x) = f(y) \) for two points \( x, y \in T_2 \). We have \( \sum_{k \geq 1} x_k/3^k = \sum_{k \geq 1} y_k/3^k \). Define \( I = \{k : x_k > y_k \} \) and \( J = \{k : x_k < y_k \} \) to get \( \sum_{i \in I} (x_i - y_i)/3^i = \sum_{j \in J} (y_j - x_j)/3^j \). Notice that \( I \) and \( J \) are disjoint, and that both sides of the equality are representations of real numbers in base 3. We examine three cases and, in each, prove that both \( I \) and \( J \) are empty.

When \( x \) and \( y \) are both nodes, both sides are representations of rational numbers. Since \( I \) and \( J \) are disjoint, both sides must be equal to 0, so that \( I \) and \( J \) are both empty.

When \( x \) and \( y \) are both branches, then each \( x_i - y_i \) and each \( y_j - x_j \) is equal to 2, so that we have \( \sum_{i \in I} 1/3^i = \sum_{j \in J} 1/3^j \). Both sides of this equality are representations of real numbers in base 3, none of which has a coefficient equal to 2. Since \( I \) and \( J \) are disjoint, and due to the uniqueness of representations, both \( I \) and \( J \) are both empty.

When \( x \) is a node and \( y \) a branch, both sides of the equality have at most a finite number of coefficients equal to 2. The results follows as in the previous cases.

(3) The function \( f : T_2 \rightarrow M \) is open. Indeed, given any node \( s \) of height \( n \), it suffices to prove that \( f(U(s)) \) is an open subset in \( M \). Set \( e = \sum_{k \geq n} 1/3^k = 1/(3^n) \) and \( A = [f(s) - e, f(s) + e] \), an interval in \( \mathbb{R} \). Clearly, we have \( f(U(s)) = A \cap M \), a closed subset in \( M \). Let \( E \) be set of all the nodes whose height is \( n \). The subsets \( f(U(t)) \) for \( t \in E \) are closed, finite in number, two by two disjoint, and the cover \( f(T_2) = M \) except for a finite number of points (a closed subset in \( M \)). So each of the subsets \( f(U(t)) \) is also open in \( M \).
(4) The function $h$ is continuous. For each infinite branch $b$ and each
neighbourhood $V$ of $f(b)$ in $\mathbb{R}$, there is a neighbourhood $W$ of $b$ in $B_2$ such
that $h(W) \subset V$. The $W(s)$’s, for each $s$ that precedes $b$, are neigh-
bourhoods of $b$ in $B_2$. When the height of $s$ tends to infinity, $f(s)$ can clearly
be made as close to $h(b)$ as one wishes. The neighbourhood $V$ contains
an interval $[h(b) - 2t, h(b) + 2t]$ with $t > 0$. We can take $n$ large enough
to get $e(n) < t$, and $s$ so high as to get $|h(b) - f(s)| < t$, so that we have
$h(W(s)) \subset V$. □

3.4 A special labelling

In the new proof van Benthem et al., a labelling of the rationals in $L \subset \mathbb{Q}$
by nodes of $T_2$ plays an important part. So let us have a look at this special
labelling. First of all, what exactly is the subset $L$?

$L$ is the set of all rational numbers $q$ of the form $q = \sum_{1 \leq k \leq h} t_k/3^m$ where $t_k = ±1$ for $1 \leq k \leq h$ and $0 \leq m_1 < m_2 < \ldots < m_h$ is a strictly increasing sequence
of integers, for some integer $h \geq 0$. Each $q \in L$ has a unique representation
of the given form: This can easily be seen. Now $q$ is given the label $l(q) = (t_1, t_2, \ldots, t_h, \ldots, 0, 0, 0, \ldots)$, a node of $T_2$. Of course [for $h = 0$], we have $0 \in L$
and $l(0) = 0$, the root of $T_2$. Moreover, if $q \neq 0$ is said to be of stage $s(q) = m_h$.
We also set $s(0) = -1$.

It should be noticed that $L$ is a tree with the partial order $\preceq$ defined as follows:

$q \preceq q$ iff $q = \sum_{1 \leq k \leq j} t_k/3^m$ and $p = \sum_{1 \leq k \leq i} t_k/3^m$ for some $i \leq j$,

that is $q \preceq q$ whenever $p$ is an initial segment of $q$. Clearly $q \preceq q$ implies
$l(p) R l(q)$.

It will be shown that the function $l : L \rightarrow T_2$ is open and continuous, and
that the subspace $L$ of $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}$.

3.5 The Steps

For each $n$, Set $L(n) = \{q : q \in L$ and $q \leq n\}$. For instance, we have
$L(-1) = \{0\}$ ve $L(0) = \{-1, 0, 1\}$.

(1) We have $q \in L(n + 1)$ iff $q = p + e/3^{n+1}$ with $p \in L(n)$ and $e \in \{-1, 0, 1\}$.

(2) Given $p \neq q$ in $L$, if both $p$ and $q$ have a stage $\leq n$, then $|p - q| \geq 1/3^n$.

The proof is by induction over $n$, using (1).

(3) Here is the crucial step in the proof.
Given $q \in L$, let $U(q, n) = [q - 1/3^n, q + 1/3^n] \cap L$. Set $m = s(q)$. The $U(q, n)$'s, for $n > m$, form a base of open neighbourhoods for $q$. For each integer $n > m$, we have $l(U(q, n)) = V(l(q))$.

**Proof.**

(i) Given $p \in U(q, n)$, $p \neq q$, we show that $l(q) \not R l(p)$. Indeed, we have $|p - q| \geq 1/3^n$ in view of (2). Therefore $p = r + \sum_{k>m} e_k/3^k$ with $r \in L(m)$ and $e_k \in \{-1, 0, 1\}$ for each $k \geq m$. We claim that $r = q$. Otherwise, we would have $|q - r| \geq 1/3^n$ and

$$|p - q| \geq |q - r| - \sum_{k>m} 1/3^k \geq 1/3^n - 1/(2.3^m) \geq 1/3^n.$$

Hence the result, i.e. $l(p) \in V(l(q))$.

(ii) Next, let $l(q) = (t_1, t_2, \ldots, t_i, \ldots, 0, 0, 0, \ldots)$. Given a node $t = (t_1, \ldots, t_i, \ldots, t_j, \ldots, 0, 0, 0, \ldots) \in V(l(q))$, $i < j$, set $p = q + \sum_{i<k \leq j} t_k/3^{m+k}$. We clearly have $t = l(p)$ and, which is more $p \in U(q, n)$ since $|p - q| < 1/3^n$.

(4) The function $l : L \to T_2$ is both open and continuous. This follows immediately from the crucial step (3).

(5) Given $p < q$ in $L$, $p - 1/3^n < p < p + 1/3^n < q < q + 1/3^n$ are all in $L$ for large enough $n$, which means that the linear order on $L$ is dense and without endpoints.

**Theorem 3.1** (Cantor) Every countable dense linear ordering without endpoints is isomorphic to $\mathbb{Q}$.

**Proof.** For a proof see, e.g., [8, Page 217, Theorem 2].

**Theorem 3.2** (van Benthem-Gabbay) $S_4$ is complete with respect to the dyadic tree $T_2$.

**Proof.** For detail, see [5].

**Theorem 3.3** $S_4$ is complete with respect to $\mathbb{Q}$.

**Proof.** Since $l$ is a topo-bisimulation between $L$ and $\mathbb{Q}$, and the linear order on $L$ is dense and without endpoints, proof follows from Theorem 3.1 and Theorem 3.2.

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References


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