Taxicab Equations for Power Two, Three, Four & Five

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Abstract

This paper is primarily concerned with the below mentioned equation.

Equation (A):

\[2(w)^k = a^k + b^k + c^k = d^k + e^k + f^k = g^k + h^k + j^k = l^k + m^k + n^k\]

For \(k=2,3,4\) the above equation has three terms in each of the four chains. For \(k=5\), the right hand side of the equation has six terms in each of the four chains.

Keywords: Equal Sums, Taxicab Equations, Diophantine equations, Number Theory, Pure Math

Parameterization of a chain of equations, namely,

\[(a^k + b^k + c^k) = (d^k + e^k + f^k),\]

which is of two chain lengths has been done by several authors. But parameterization of four chain lengths are not common. This paper has made inroads on this topic in that sense that and as an added bonus equated and tied these chain of equation to twice a kth power. Till now the sum of taxicab powers in math literature have
summed up to an arbitrary integer. So the achievement of this paper is that by fixing the sum to twice a kth power, this equation (A) can be clubbed into the subject heading of equal sums of like powers. Now instead of having the sum equal to one kth power as in many cases in math literature we now have the sums equal to twice a kth power. In this paper we will designate this taxicab chain as k-2-3-3-3-3 where k=2, 3, 4 (degree). Except for 5th powers the equation will be k-2-6-6-6 for reasons we will describe later in the paper.

In connection to our equation (A), there is an interesting anecdote about the conversation Ramanujan & Hardy had about a hundred years ago when Hardy came in a cab to visit Ramanujan. Hardy remarked that his cab’s license plate was an uninteresting number 1729. But Ramanujan who had the ability of having integers as his friends said on the contrary 1729 is the smallest number which could be summed as two cubes in two different ways. Meaning sum of two cubes in two ways that is, \((1729) = 1^3 + 12^3 = 9^3 + 10^3\). Since than mathematicians have called this type of equation as taxicab equation without the restriction on the powers being quadratic, cubic, quartic or quintic.

_Nowadays a grade six student if asked to express 50 as a sum of two squares, he or she might say \(50 = 1^2 + 7^2\)_

_But actually 50 can be summed up as two squares in two ways as\

\[
50 = 1^2 + 7^2 = 5^2 + 5^2 = 2(5)^2. \text{ The higher we go on the number line the more chains we can add.} \\
For example,
\[
2(w)^2 = 2(65)^2 = (13,91)^2 = (89,23)^2 = (85,35)^2 = (79,47)^2
\]

This above chain, having four pair of square numbers, after paramatazation, we get a new numerical solution \((k - 2 - 2 - 2 - 2 - 2)\) where \(k = 2\).

2nd degree taxi cab 2-2-2-2-2-2
Taxicab equations

The parametric form for the above is given by:

\[ 2 \cdot m^2 (x^2 + y^2)^2 \]

\[ = (-ay^2 + ax^2 - 2bxy)^2 + (-by^2 + bx^2 + 2axy)^2 \]

\[ = (-cy^2 + cx^2 - 2dxy)^2 + (-dy^2 + dx^2 + 2cxy)^2 \]

\[ = (-ey^2 + ex^2 - 2fxy)^2 + (-fy^2 + fx^2 + 2efxy)^2 \]

\[ = (-gy^2 + gx^2 - 2hxy)^2 + (-hy^2 + hx^2 + 2gxy)^2 \]

Where \(2(m)^2 = a^2 + b^2 = c^2 + d^2 = e^2 + f^2 = g^2 + h^2\)

This has solution:

\((a,b,c,d,e,f,g,h,m) = (13,91,89,23,85,35,79,47,65)\)

\[ 2 \cdot (65)^2 = (13,91)^2 = (89,23)^2 = (85,35)^2 = (79,47)^2 \]

Consider the below mentioned equation, (1st chain)

\[ 2 \cdot m^2 (x^2 + y^2)^2 \]

\[ = (-ay^2 + ax^2 - 2bxy)^2 + (-by^2 + bx^2 + 2axy)^2 \]

Where \(2(m)^2 = a^2 + b^2 = \frac{x}{y}\)

\[ \text{Let } k = \frac{x}{y} \]

\[ 2 \cdot m^2 (k^2 + 1)^2 = (a^2 + b^2) \cdot (k^2 + 1)^2 \]

\[ = (-a + ak^2 - 2bk)^2 + (-b + bk^2 + 2ak)^2 \]

The right hand side after simplification becomes

\[ = (a^2(k^2 - 1)^2 - 4ab(k^3 - 1) + 4b^2k^2) + (b^2(k^2 - 1)^2 + 4ab(k^3 - 1) + 4a^2k^2) \]
\[(a^2 + b^2)(k^2 - 1)^2 + 4k^2(a^2 + b^2)\]
\[= (a^2 + b^2) \times (k^2 + 1)^2 \text{ (which equals the left hand side).}\]

Since \(a^2 + b^2 = (c^2 + d^2)\)

By substituting \(a = c \& b = d\) we get the next chain

Similarly by substituting \(c = e \& d = f\) we get third chain

Putting \(d = g\) and \(f = h\) we get the fourth chain

Hence we get after putting values of \((a,b,c,d,e,f,m)\) in equation ‘A’:

\[2 \times (65)^2(x^2 + y^2)^2 =\]
\[(-13y^2 + 13x^2 - 182xy)^2 + (-91y^2 + 91x^2 + 26xy)^2\]
\[= (-89y^2 + 89x^2 - 46xy)^2 + (-23y^2 + 23x^2 + 178xy)^2\]
\[= (-85y^2 + 85x^2 - 70xy)^2 + (-35y^2 + 35x^2 + 170xy)^2\]
\[= (-79y^2 + 79x^2 - 94xy)^2 + (-47y^2 + 47x^2 + 158xy)^2\]

For \(x=3, y=1\) we get, after removing common factors,

\[2(325)^2 = (221, 403)^2 = (287, 359)^2 = (235, 395)^2 = (175, 425)^2\]

The above chain can be made longer for bigger integers than \(w = 325\) since they can be represented by more than four pairs of square numbers.

Since numerical solutions are not available where twice a kth power is equal to two kth powers for \(k=3,4\) we have not attempted to parameterize it. Also it is to be noted that two kth powers have more than two chain lengths for \(k=3 \& 4\) but are not equal to twice a kth power as required in this paper. Hence we have considered equations with three elements in each chain. As given in the start of this paper our taxi cab equation (A) is for \(k=2,3,4 \& 5\). We first parameterise for the 2nd degree & fourth degree for \(k=2 \& 4\). In this context we are aware that half a century ago M. Parasarathy and Albert Gloden gave a parametric solutions for 4-3-3 equation and their results included extending it for a long chain. The difference between their...
parameterization and this is that their paramatization did not sum up to twice a kth power & also our present method is less complicated.

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The paramatized eqn. For K-2-3-3-3-3 is given below, for k=2 & k=4

We have from equation (A) above

Equation (2): \[ 2(w)^k = (a, b, c)^k = (d, e, f)^k = (g, h, j)^k = (l, m, n)^k \]

\[ \text{For } k = 2 \text{ & 4} \]

\[ \text{We have below the well known Identity: For } k = 2 \text{ & 4} \]

\[ 2(pq + q^2 + p^2)^k = (p^2 - q^2)^k + (-p^2 - 2pq)^k + (q^2 + 2pq)^k \]

Substituting \( p = r \) & \( q = s \), we get:

\[ 2(rs + r^2 + s^2)^k = (r^2 - s^2)^k + (r^2 + 2rs)^k + (s^2 + 2rs)^k \]

\[ \text{Hence we get;} \]

\[ 2 = \frac{1}{(pq + q^2 + p^2)^k} \left( (p^2 - q^2)^k + (p^2 + 2pq)^k + (q^2 + 2pq)^k \right) \]

\[ = \frac{1}{(rs + r^2 + s^2)^k} \left( (r^2 - s^2)^k + (-r^2 - 2rs)^k + (s^2 + 2rs)^k \right) \]

\[ \text{Multiplying through out by } (pq + q^2 + p^2)^k (rs + r^2 + s^2)^k \]

\[ \text{We get the first two chains below: Parametric form:} \]

\[ 2(w)^k = (a, b, c)^k = (d, e, f)^k \]

\[ w = (pq + q^2 + p^2)(s^2 + r^2 + rs) \]
a = \((s^2 + r^2 + rs)(p^2 - q^2)\)

b = \((s^2 + r^2 + rs)(q^2 + 2pq)\)

c = \((s^2 + r^2 + rs)(-p^2 - 2pq)\)

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d = \((pq + q^2 + p^2)(r^2 - s^2)\)

e = \((pq + q^2 + p^2)(s^2 + 2rs)\)

f = \((pq + q^2 + p^2)(-r^2 - 2rs)\)

The next chains are,

\(2(w)^k = (g, h, j)^k = (L, m, n)^k\)

The third & fourth chains are given below:

Let, \((r^2 - s^2) = u, (s^2 + 2rs) = v, (-r^2 - 2rs) = z\)

we have \((u + v + z) = 0\),

hence we get after taking different combinations of

\((q^2 - 2pq + p^2)(v + z + u)\)

& \((q^2 - 2pq + p^2)(u + v + z)\)

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We get:

\[ q^2(v + z + u) - 2pq(z + u + v) + p^2(u + v + z) \]

Arranging this in a vertical matrix we get:

(say) \(g = (q^2v - 2pqz + p^2u)\)

\(h = (q^2z - 2pqu + p^2v)\)
similarly we have:

\[ q^2(u+v+z) - 2pq(z+u+v) + p^2(v+z+u) \]

Arranging this in a vertical matrix we get:

(say) \[ l = (q^2u - 2pqz + p^2v) \]
\[ m = (q^2v - 2pqu + p^2z) \]
\[ n = (q^2z - 2pqv + p^2u) \]

Substituting for \( u, v, z \) & factoring we get:

\[ g = (2qr + pr + qs - ps)(pr + qs + ps) \]
\[ h = (2qs + ps + 2pr + qr)(ps - qr) \]
\[ j = -(qs + 2ps + pr - qr)(pr + qr + qs) \]

\[ l = -(2pr + qr + ps - qs)(qr + ps + qs) \]
\[ m = (2qr + pr + 2ps + qs)(pr - qs) \]
\[ n = (ps + 2qs - pr + qr)(ps + pr + qr) \]

Acknowledgment: The fourth chain in the above was received from Dr. Ajai Choudhry with thanks.

Below we show that the third & fourth chain above,

\[ 2(w)^k = (g,h,j)^k \quad -- -- Equation '1' \]

is true for \( p = q \) and \( k = 2 \). We have
\[ g = (q^2v - 2pqz + p^2u) \]
\[ h = (q^2z - 2pqu + p^2v) \]
\[ j = (q^2u - 2pqv + p^2z) \]

We also have:

\[ w = (rs + r^2 + s^2)(pq + q^2 + p^2) \]

For \( p = q \), we get, \[ 2(w)^2 = 2 \cdot 9 \cdot (q)^4 \cdot (rs + r^2 + s^2)^2 \]

Right hand side of equation ‘1’ above is:

we have:

\[ (r^2 - s^2) = u, \quad (s^2 + 2rs) = v, \quad (-r^2 - 2rs) = z. \]

We get:

\[ (g^2 + h^2 + j^2) = \]
\[ (q^2v - 2pqz + p^2u)^2 + (q^2z - 2pqu + p^2v)^2 + (q^2u - 2pqv + p^2z)^2 \]
\[ = q^4((v - 2z + u)^2 + (z - 2u + v)^2 + (u - 2v + z)^2) \]
\[ = q^4(6(u^2 + v^2 + z^2) - 6(uv + uz + vz)) \]
\[ = 6q^4((u^2 + v^2 + z^2) - (uv + uz + vz)) \]
\[ = 6q^4(-3(uv + uz + vz)) \quad \text{[since } u + v + z = 0 \text{ & hence]} \]
\[ ((u^2 + v^2 + z^2) - (uv + uz + vz)) = -3(uv + uz + vz) \]
\[ = -2 \cdot 9(uv + uz + vz) \]

(Substituting values of \( u, v, z \) we get)
\[ = -(2 \cdot 9 \cdot q^4)(-)(s^4 + r^4 + 3r^2s^2 + 2r^3s + 2rs^3) \]
\[ = (2)(9)(q^4)(rs + r^2 + s^2)^2 \]

Hence both sides of equation ‘1’ are equal.
For other values of \((p,q)\) a similar method can be followed for \(k=2,4\)

\[\text{Hence for } k = 2,4 \]

\[2 \cdot ((p^2 + pq + q^2)(r^2 + rs + s^2))^k \]
\[= (s^2 + r^2 + rs)^k((p^2 - q^2)^k + (q^2 + 2pq)^k + (-p^2 - 2pq)^k)\]
\[= (p^2 + q^2 + pq)^k((r^2 - s^2)^k + (s^2 + 2rs)^k + (-r^2 - 2rs)^k)\]
\[= ((2qr + pr + qs - ps)(pr + qs + ps))^k \]
\[+ ((2qs + ps + 2pr + qr)(ps + qr))^k \]
\[+ ((-qs + 2ps + pr - qr)(pr + qr + qs))^k \]

\[= (- (2pr + qr + ps - qs)(qr + ps + qs))^k + ((2qr + pr + 2ps + qs)(pr - qs))^k + ((ps + 2qs - pr + qr)(ps + pr + qr))^k \]

**Numerical examples:**

For \((p, q, r, s) = (2, 1, 3, 1)\) we get the four chains for \(k = 2 \& 4\)
\[2(91)^k = (39, 65, 104)^k = (49, 56, 105)^k = (19, 80, 99)^k = (11, 85, 96)^k\]

For \((p, q, r, s) = (4, 3, 5, 2)\), after taking out common factors we get
\[2(481)^k = (91, 429, 520)^k = (259, 296, 555)^k = (544, 175, 369)^k = (551, 336, 215)^k\]

Next we parameterize for power three, \(k=3\) the cubic version, namely 3-2-3-3-3-3.

**Consider the below mentioned equation,**

\[2(m)^3 = (m + n)^3 + (m - n)^3 + (z)^3 - \cdots \text{ equation '1'} \]

Let \(m = a^3b^3, \ n = 6p^3c^3\)

**Solving for \(z\), we get** \[z = -6abc^2p^2\]
Substituting values of m, n & z in equation ‘1’ we get

\[ 2(a^3b^3)^3 = (a^3b^3 + 6p^3c^3)^3 + (a^3b^3 - 6p^3c^3)^3 + (-6abc^2p^2)^3 \]

Substituting suitable values of p as (q, r, s, - - -) we get the chain equation.

Equation (A)
\[ 2(a^3b^3)^3 = (a^3b^3 + 6p^3c^3)^3 + (a^3b^3 - 6p^3c^3)^3 + (-6p^2abc^2)^3 \]
\[ = (a^3b^3 + 6q^3c^3)^3 + (a^3b^3 - 6q^3c^3)^3 + (-6q^2abc^2)^3 \]
\[ = (a^3b^3 + 6r^3c^3)^3 + (a^3b^3 - 6r^3c^3)^3 + (-6r^2abc^2)^3 \]
\[ = (a^3b^3 + 6s^3c^3)^3 + (a^3b^3 - 6s^3c^3)^3 + (-6s^2abc^2)^3 \]

where \( (a, b, c) \) are parameters & \((p, q, r, s, - - -) \) takes integer values.

The above chain can be extended as long as needed by substituting different values for \( x \) in the equation \((a^3b^3 + 6x^3c^3)^3 + (a^3b^3 - 6x^3c^3)^3 + (-6x^2abc^2)^3 \) and adding it to the existing chains in equation (3) above. Namely, ‘x’ represents \((p, q, r, s, - - -) \) having integer values.

The parametric form for 3-2-3-3-3-3, is given below:

We get: For \( p = 1, q = 2 & r = 3, s = 5 \), respectively,

\[ 2(a^3b^3)^3 = (a^3b^3 + 6c^3)^3 + (a^3b^3 - 6c^3)^3 + (-6abc^2)^3 \]
\[ = (a^3b^3 + 48c^3)^3 + (a^3b^3 - 48c^3)^3 + (-24abc^2)^3 \]
\[ = (a^3b^3 + 162c^3)^3 + (a^3b^3 - 162c^3)^3 + (-54abc^2)^3 \]
\[ = (a^3b^3 + 750c^3)^3 + (a^3b^3 - 750c^3)^3 + (-150abc^2)^3 \]

Numerical example is: for \( a = 3, b = 1, c = 1 \)

After removing common factors we get:

\[ 2(9)^3 = (11)^3 + (7)^3 + (-6)^3 = (25)^3 + (-7)^3 + (-24)^3 \]
\[ = (63)^3 + (-45)^3 + (-54)^3 \]
\[ = (259)^3 + (-241)^3 + (-150)^3 \]
Next we parameterize for power five, \( k = 5 \) the quintic,

( Note that, since there are no numerical solutions available for \( 5 - 2 - 3 \) \(- 3 \) or \( 5 - 2 - 4 - 4 \) or \( 5 - 2 - 5 - 5 \) we selected \( 5 - 2 - 6 \) \(- 6 \) to paramatise since numerical solutions are available for it.)

Consider equation (A), Given below,

\((u, v, z)\) are parameters to be determined.

\[ 2(t)^5(x^2 + 3y^2)^5 = \]

\[ = (ax^2 + 3by^2 + uxy)^5 + (bx^2 + 3ay^2 - uxy)^5 + (cx^2 + 3dy^2 + vxy)^5 \]

\[ + (dx^2 + 3cy^2 - vxy)^5 + (ex^2 + 3fy^2 + zxy)^5 + (fx^2 + 3ey^2 - zxy)^5 \]

Where, \( 2(t)^5 = a^5 + b^5 + c^5 + d^5 + e^5 + f^5 \)

Where \( a + b = c + d = e + f = p \)

Hence \( a = p - b, c = p - d, e = p - f \)

We have numerical solutions like the one given below.

\[ 2(21)^5 = (13)^5 + (1)^5 + (-3)^5 + (17)^5 + (23)^5 + (-9)^5 \]

Where \((t, a, b, c, d, e, f, p)\) are known quantities,

Also, \( p = a + b = 13 + 1, c + d = -3 + 17 = 14, \)

\( e + f = 23 - 9 = 14 \) \& \( t = 21 \)

As in the above example \((t, a, b, c, d, e, f, p) = (21, 13, 1, -3, 17, 23, -9, 14)\)

Expanding both sides of equation 'A' & equating the coefficients of 'x' we get after some algebra the equations below.

Substituting value of \((a, c, e)\) in equation ‘A’ and solving using Maple Math Software
\[
v = \frac{p^3 - 2p^2f + 3pf^2 - 2f^3 - 2p^2b + 3pb^2 - 2b^3}{-p^3 + 2p^2d - 3pd^2 + 2d^3 + 2p^2b - 3pb^2 + 2b^3}
\]

\[
u = \left(\frac{p^3z - 4p^2zf + 6pf^2 - 4zf^3 + p^3v - 4p^2vd + 6pv^2d - 4vd^3}{-p^3 + 4p^2b - 6pb^2 + 4b^3}\right)
\]

\[
\text{and } \quad \frac{v}{z} = \frac{(b^4 - e^4 + f^4 - a^4)}{(a^4 + c^4 - b^4 - d^4)}
\]

Where \( u = (v + z) \)

The three equations above are solved for the three variables \((u,v,z)\)

Using the equation solve feature in Maple Math Software we get the below mentioned solutions for \((u,v,z)\)

\[
u = 2(d - f) \\
v = 2(p - b - f) \\
z = 2(d + b - p)
\]

Also we have \( p = a + b = c + d = e + f \)

And \( p = 2(d - b + f) \)

We have initial solution:

\[
2(3)^5 = (-1)^5 + (3)^5 + (-1)^5 + (3)^5 + (1)^5 + (1)^5-------(G1) \text{ (say)}
\]

Where \((a, b, c, d, e, f, p)\)

= \((-1, 3, -1, 3, 1, 1, 2)\) we get after substituting in the equations for \((u, v, z)\) above

we get \( t = 3, p = 2, u = 4, v = -4, z = 8 \)

Thus after putting the values of \((a, b, c, d, e, f, u, v, z, t)\)

\[
2(3)^5(x^2 + 3y^2)^5 =
\]
From above for \((a, b, c, d, e, f) = (13, 1, -3, 17, 23, -9)\) & from equations for \((u, v, z)\) above we get equation,

\[2(21)^{5}(x^2 + 3y^2)^5 = (13x^2 + 3y^2 + 52xy)^5 + (x^2 + 39y^2 - 52xy)^5 + (-3x^2 + 51y^2 + 44xy)^5 + (17x^2 - 9y^2 - 44xy)^5 + (23x^2 - 27y^2 + 8xy)^5 + (-9x^2 + 69y^2 - 8xy)^5\]

for \(x = 2, y = 1\)

The result is say \((H1) = 2(35)(7)^5 = 2(21)^5 = (13, 1, -3, 17, 23, -9)^5\)

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For above we get for \((a, b, c, d, e, f) = (159, -61, 127, -29, 81, 17)\)

and \(t = 147, p = 98, u = -92, v = 284, \ z = -376\)

we get for \(x = 2, y = 1\)

\[2(147)^{5}(x^2 + 3y^2)^5 = (159x^2 - 183y^2 - 92xy)^5 + (-61x^2 + 477y^2 + 92xy)^5 + (127x^2 - 87y^2 + 284xy)^5 + (-29x^2 + 381y^2 - 284xy)^5 + (81x^2 + 51y^2 - 376xy)^5 + (17x^2 + 243y^2 + 376xy)^5\]

For above we get for \(x = 2, y = 1, \)

\((a, b, c, d, e, f) = (269, 417, 989, -303, -377, 1063), \ t = 1029\)

The result is say \((L1) = 2(1029)^{5} = 2 * (147)^{5} * (7)^5\)

and \(t = 1029, p = 688, u = -2732, v = -1588, \ z = -1144\)
\[2(1029)^5(x^2 + 3y^2)^5 = (269x^2 + 1251y^2 - 2732xy)^5\]
\[+ (417x^2 + 807y^2 + 2732xy)^5 + (989x^2 - 909y^2 - 1588xy)^5\]
\[+ (-303x^2 + 2967y^2 + 1588xy)^5\]
\[+ (-377x^2 + 3189y^2 - 1144xy)^5\]
\[+ (1063x^2 - 1131y^2 + 1144xy)^5\]

Taking the Least common multiple for the four chains we get the below equation:
\[2 * (1029)^5 = (343)^5 * (G1) = (49)^5 * (H1) = (7)^5 * (J1) = (L1)\]
Substituting for \((G1), (H1), (J1), (L1)\) from equations above, we get the required taxicab chain, which can be extended as long as needed

The paramatized fifth power equation is given below:
\[2 * (w)^5 = (m_1, m_2, m_3, m_4, m_5, m_6)^5 = (n_1, n_2, n_3, n_4, n_5, n_6)^5\]
\[= (p_1, p_2, p_3, p_4, p_5, p_6)^5 = (q_1, q_2, q_3, q_4, q_5, q_6)^5\]

For \(w = 1029(x^2 + 3y^2)\), we get the four chains below,
\[2 * (1029)^5(x^2 + 3y^2)^5 = 343^5((-x^2 + 9y^2 + 4xy)^5 + (3x^2 - 3y^2 - 4xy)^5\]
\[+ (-x^2 + 9y^2 - 4xy)^5 + (3x^2 - 3y^2 + 4xy)^5\]
\[+ (x^2 + 3y^2 + 8xy)^5 + (x^2 + 3y^2 - 8xy)^5\]

\[= 49^5((13x^2 + 3y^2 + 52xy)^5 + (x^2 + 39y^2 - 52xy)^5\]
\[+ (-3x^2 + 51y^2 + 44xy)^5 + (17x^2 - 9y^2 - 44xy)^5\]
\[+ (23x^2 - 27y^2 + 8xy)^5 + (-9x^2 + 69y^2 - 8xy)^5\]

\[= 7^5(159x^2 - 183y^2 - 92xy)^5 + (-61x^2 + 477y^2 + 92xy)^5\]
\[+ (127x^2 - 87y^2 + 284xy)^5 + (-29x^2 + 381y^2 - 284xy)^5\]
\[+ (81x^2 + 51y^2 - 376xy)^5 + (17x^2 - 243y^2 + 376xy)^5\]
\[= (269x^2 + 1251y^2 - 2732xy)^5 + (417x^2 + 807y^2 + 2732xy)^5 \\
+ (989x^2 - 909y^2 - 1588xy)^5 \\
+ (-303x^2 + 2967y^2 + 1588xy)^5 \\
+ (-377x^2 + 3189y^2 - 1144xy)^5 \\
+ (1063x^2 - 1131y^2 + 1144xy)^5\]

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Numerical example is: For x=1 & y=0, we get, k=5
\[2(w)^k = 2 \times (1029)^5\]
\[= (343)^5 \times (-1,3, -1,3,1,1)^5\]
\[= (49)^5 \times (13,1, -3,15,23, -9)^5\]
\[= (7)^5 \times (17,81, -29,127, -61,159)^5\]
\[= (269,417,989, -303, -377,1063)^5\]

**Summary:**

In conclusion we note that for powers k=2,4 For even powers the maximum chain length can be extended beyond four chain lengths. *For larger values of w > 91.*

\[
\text{where} \quad 2(w)^k = (a^k + b^k + (a + b)^k) \]

**Example for k = 2,4**

\[2 \times (1729)^k = (931,1064,1995)^k = (845,1144,1989)^k \]
\[= (799,1185,1984)^k = (741,1235,1976)^k \]
\[= (656,1305,1961)^k \]
This above numerical solution can be paramatised & we can arrive at nine chains, but it involves using six variables instead of four variables \((p,q,r,s)\) used above & the equations do not have a simple form. Alternatively another approach might help paramatise longer chains of length more than four. For \(k=3\) & \(5\) odd powers our taxicab equation (A) can have as long chains as needed since the terms in the chain are signed, meaning the terms take on plus & minus values. But \(k=5\), remains an open problem for taxicab equation (A) above in the sense that numerical results are awaited in which paramatization of chains having less than six terms can be attempted. Numerical solutions are available for \((5^4-4)\) but they are not equal to twice a fifth power as required in this paper. Beyond \(k=5\), we have Euler’s conjecture to contend with, but with computers becoming more powerful, maybe in the future we will have numerical solutions which might help us paramatize them.

**References**


6) Duncan Moore-Taxicab webpage, 
   http://homepage.nt/world.com/duncan_moore/taxicab
7) Tito Piezas- Online collection of algebraic identities,
http://sites.google.com/site/tpiezas


10) Jaroslaw Wroblewski, tables of Numerical solutions for 3-1-3 & 4-1-4 equations.
Website, www.math.uni.wroc.pl/~jwr/eslp

Received: January 9, 2014