Binding Number of Corona and Join of Graphs¹

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Abstract

The binding number of a graph G is defined as $bind(G) = \min \frac{|N(S)|}{|S|}$, $S \in F(G)$ where $F(G) = \{S \subseteq V(G) : S \neq \emptyset \text{ and } N(S) \neq V(G)\}$. This paper provides some results on the binding numbers of corona and join of graphs and characterize them in terms of independent binding set.

Mathematics Subject Classification: 05C69

Keywords: binding number, corona, join of graphs, independent binding set

1 Introduction

In 1973, Wodall [8] introduced the concept of a binding number of a graph. He published the first known results on the binding numbers of some graphs including the binding numbers of paths, cycles, and complete graphs. While in 1985, Ronghua [7] proved Woodall's conjecture that if $bind(G) \geq \frac{3}{2}$ then G is pancyclic. Looking back in 1981, Kane and Mohanty [6] theorized that if $bind(G) \geq \frac{3}{2}$ and $|V(G)| \geq 5$ then G contains a 4-cycle or a 5-cycle. While Goddard and Swart [5] published results on the binding numbers of cartesian and lexicographic product of graphs.

¹This research is funded by the Commission on Higher Education (CHED).

In this paper, we determine the binding numbers of corona and join of graphs and provide the necessary and sufficient conditions for these graphs to have the independent binding sets.

Throughout this paper, the graph we consider is simple, that is, no loop or multiple edges. We use the terminologies of Chartrand and Oellerman[3], and Harary [4].

2 Preliminaries

Definition 2.1 A *graph* G is a finite nonempty set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices of G called the *edges*. The vertex set of G is denoted by $V(G)$, while the edge set is denoted by $E(G)$. If G has no loops, that is, no edge joining a vertex to itself, and no multiple edges, that is, no two edges join the same pair of vertices, then the graph G is *simple*.

Definition 2.2 The *cardinality* of $V(G)$ denoted by $|V(G)|$ is called the *order* of G and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G .

Definition 2.3 A graph G is *connected* if every pair of distinct vertices is joined by a path, otherwise it is *disconnected*.

Definition 2.4 The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set defined by $N(v) = \{u \in V(G) : uv \in E(G)\}$. This is more commonly called the open neighborhood of a vertex. The neighborhood of a set $X \subseteq V(G)$, denoted by $N(X) = \bigcup_{x \in X} N(x)$.

Definition 2.5 A set S of vertices in a graph G is *independent* if no two vertices of S are adjacent in G . A set containing independent vertices is called an *independent set*. An independent set S of vertices in G is called a *maximal independent set* if S is not a proper subset of any other independent set of vertices of G .

Definition 2.6 The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G of order n and n copies H_i of H , and then joining the i th vertex of G to every vertex of H_i .

Definition 2.7 The **join** $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \dot{\cup} V(H)$ and

$$E(G + H) = E(G) \dot{\cup} E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}.$$

Definition 2.8 [8] Let G be a graph with vertex set $V(G)$ and define $F(G) = \{S \subseteq V(G) : S \neq \emptyset \text{ and } N(S) \neq V(G)\}$. The binding number of G is given by

$$\text{bind}(G) = \min_{S \in F(G)} \left\{ \frac{|N(S)|}{|S|} \right\}.$$

The binding set of G is any set $S \in F(G)$ such that $\text{bind}(G) = \frac{|N(S)|}{|S|}$.

Proposition 2.9 [8] For any graph G on n vertices and with minimum degree $\delta(G)$, $\text{bind}(G) \leq \frac{n-1}{n-\delta(G)}$.

Proposition 2.10 [6] $\text{bind}(\bigcup_{i=1}^r G_i) = \min\{\text{bind}(G_1), \dots, \text{bind}(G_r), 1\}$.

Proposition 2.11 [8] (a) $\text{bind}(K_n) = n - 1$ ($n \geq 1$).

The binding set of K_n is a singleton.

(b) If $n \geq 3$, then

$$\text{bind}(C_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ \frac{n-1}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

The binding set of C_n is $S = \{a_i : i \text{ is odd}\}$ or $S = \{a_i : i \text{ is even}\}$ if n is even and $S_i = V(C_{2k+1}) - N(x_i)$ if n is odd. Moreover, $|S| = \frac{n}{2}$ if n is even and $|S| = n - 2$ if n is odd.

(c) If $n \geq 1$, then

$$\text{bind}(P_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ \frac{n-1}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

The binding set of P_n is $S = \{x_1, x_2, \dots, x_{n-2}, x_n\}$ or $S = \{x_1, x_3, x_4, \dots, x_n\}$ if n is even and $S = \{x_1, x_3, x_5, \dots, x_{2k-1}, x_{2k+1}\}$ if n is odd. Furthermore, $|S| = n - 1$ if n is even and $|S| = \frac{n+1}{2}$ if n is odd.

3 Binding Number of Corona of Graphs

Theorem 3.1 Given H a connected or trivial graph and K a connected graph with $|V(H)| = m$ and $|V(K)| = n$. If $G = H \circ K$ then $\text{bind}(G) = \frac{(m-1)(n+1)+|N(X)|}{(m-1)(n+1)+|X|}$, where $X \in F(K^v)$ such that X is of maximum order.

Proof: Let's prove when H and K are connected and the case when H is trivial follows. Suppose that $G = H \circ K$. Define $A = \{V(G) \setminus V(K^v) + v : v \in V(H)\}$. Then $A \in F(G)$ and $N(A) = \{V(G) \setminus V(K^v) : v \in V(H)\}$. Since H and K are connected, for every $A \in F(G)$, $|N(A)| > |A|$. From the definition of A , we obtain $|A| = n(m-1) + (m-1) = (m-1)(n+1)$ and $|N(A)| =$

$(m-1)(n+1)+1$. Note that $\min\{\frac{|N(A)|}{|A|}\} = \min\{\frac{(m-1)(n+1)+1}{(m-1)(n+1)}\} > \frac{n-1}{n-\delta(G)}$ which violates Proposition 2.9. Thus, A does not give the binding number. So now, let's consider the set $F(K^v) = \{X \subseteq V(K^v) : X \neq \emptyset \text{ and } N(X) \neq V(K^v)\}$ where $v \in V(H)$ and let $T = A \cup X$ where $X \in F(K^v)$ is of maximum order. Then $T \in F(G)$ and $N(T) = \{V(G) \setminus V(K^v) : v \in V(H)\} \cup N(X)$. It follows that $|T| = (m-1)(n+1) + |X|$ and $|N(T)| = (m-1)(n+1) + |N(X)|$. Let $S \in F(G)$ such that $S = A$ or $S = \{V(G) \setminus V(K^v) : v \in V(H)\} \cup X$ where $X \in F(K^v)$ is either of maximum order or not. Then $|S| \leq |T|$ and $\frac{|N(S)|}{|S|} \geq \frac{|N(T)|}{|T|}$. Thus, for every $S \in F(G)$, $\frac{|N(S)|}{|S|} \geq \frac{(m-1)(n+1)+|N(X)|}{(m-1)(n+1)+|X|}$. Hence, $\text{bind}(G) \geq \frac{(m-1)(n+1)+|N(X)|}{(m-1)(n+1)+|X|}$. By definition, $\text{bind}(G) = \min \frac{|N(S)|}{|S|}$, so we have $\text{bind}(G) \leq \frac{(m-1)(n+1)+|N(X)|}{(m-1)(n+1)+|X|}$. Combining these inequalities, we obtain $\text{bind}(G) = \frac{(m-1)(n+1)+|N(X)|}{(m-1)(n+1)+|X|}$. Thus, if H is trivial then $m = 1$ and $\text{bind}(G) = \frac{0+|N(X)|}{0+|X|} = \frac{|N(X)|}{|X|}$. $\square$

Theorem 3.2 Let H be a disconnected graph and K a connected graph. Then $\text{bind}(H \circ K) = \text{bind}(\bigcup_{i=1}^r H_i \circ K)$
 $= \min\{\text{bind}(H_1 \circ K), \text{bind}(H_2 \circ K), \dots, \text{bind}(H_r \circ K), 1\}$,
 where H_i are the subgraphs of H for $1 \leq i \leq r$.

Proof: Let H_i be the subgraphs of H where H_i is either connected or trivial graph for $1 \leq i \leq r$. Then $H \circ K = H_1 \circ K \cup H_2 \circ K \cup \dots \cup H_r \circ K$
 $= \bigcup_{i=1}^r H_i \circ K$. By Proposition 2.10, $\text{bind}(H \circ K) = \text{bind}(\bigcup_{i=1}^r H_i \circ K)$
 $= \min\{\text{bind}(H_1 \circ K), \text{bind}(H_2 \circ K), \dots, \text{bind}(H_r \circ K), 1\}$. $\square$

Theorem 3.3 Let $G = H \circ K$ with H a connected graph of order m and K a disconnected graph of order n with no trivial subgraph. Then $\text{bind}(G) = \frac{n(m+1)+|N(Y)|}{n(m+1)+|Y|-1}$, where $Y \in F(M_t^v)$ is largest and M_t is the smallest among the disconnected subgraphs of the i th copy of K .

Proof: Let $M_1, M_2, \dots, M_s$ be the disconnected subgraphs of K and M_t be the smallest subgraph. Define $A = \{V(G) \setminus V(M_t^v)\}$ where $v \in V(H)$. Since M_r is nontrivial for each $1 \leq r \leq s$, we can use parallel argument in the proof of Result 3.1. Define $F(V(M_t^v)) = \{Y \subseteq V(M_t^v) : Y \neq \emptyset \text{ and } Y \neq V(M_t^v)\}$. Let $T = A \cup Y$ where $Y \in F(V(M_t^v))$ is of maximum order. Then $T \in F(G)$ and $N(T) = N(A) \cup N(Y)$. It follows that $|T| = m(n-1) + (n-1) + m + |Y| = n(m+1) + |Y| - 1$ and $|N(T)| = n(m+1) + |N(Y)|$. Let $S \in F(G)$ such that $S = A$ or $S = A \cup Y$ where $Y \in F(V(K_t^v))$ is not necessarily of maximum order. Then by the same argument in the proof of Result 3.1, $|T| \leq |S|$ and $\frac{|N(S)|}{|S|} \geq \frac{|N(T)|}{|T|}$. Thus, T gives the binding number of G and therefore, $\text{bind}(G) = \frac{n(m+1)+|N(Y)|}{n(m+1)+|Y|-1}$. $\square$

Remark 3.4 Let $G = H \circ K$ with H a connected graph of order m and K a disconnected graph of order n . If K has a trivial subgraph then $\text{bind}(G) = 1$.

Theorem 3.5 Let H and K be graphs of order m and n respectively. If $G = H \circ K$ has independent binding set S then $\text{bind}(G) = \frac{1}{|S^*|}$ where S^* is the largest subset of S in the i th copy of K .

Proof: Suppose that $G = H \circ K$ has independent binding set. Let S be a binding set of G . Then S consists of nonadjacent vertices in G . Hence, $S \subseteq V(K^v)$ where $v \in V(H)$. It follows that $N(S) \subseteq V(H)$. Let S^* be a subset of S in the i th copy of K of maximum order. Since there are m times K^v , we have $|S| = m|S^*|$ where $1 \leq |S^*| \leq n$ and $|N(S)| = m$. Therefore, $\text{bind}(G) = \frac{|N(S)|}{|S|} = \frac{m}{m|S^*|} = \frac{1}{|S^*|}$. $\square$

Theorem 3.6 Let G be a corona of graphs. Then G has independent binding set if and only if G has a subgraph tree.

Proof: Let G be a corona of graphs. Suppose that G has independent binding set. Let S be a binding set of G . Then S is independent. It follows that the elements of S are the nonadjacent vertices of G . Thus, G has a subgraph tree.

Conversely, suppose that $G = H \circ K$ has a subgraph tree. Define $X = \{v \in V(G) : \deg(v) = 1\}$. Then X consists of the nonadjacent vertices of G . It follows that $X \subseteq V(T_n)$ where T_n is a tree and $N(X) \subseteq V(H)$. Clearly, $X \in F(G)$ and $|X| > |N(X)|$. Let $S \in F(G)$. If $S \subseteq V(H)$ then $|S| < |N(S)|$. Since $\frac{|N(X)|}{|X|} < \frac{|N(S)|}{|S|}$ the possible binding set is a subset of X . Let $S \subseteq X$. If $|S| = 1$ then $|N(S)| = 1$. If $S = X$ then $|S| = |V(T_n)|$ and $|N(S)| = |H|$. Consequently, X gives the minimum value of $\frac{|N(S)|}{|S|}$. Thus, X is the independent binding set. Therefore, G has independent binding set. $\square$

4 Binding Number of Join of Graphs

Theorem 4.1 Let G and H be graphs. Then

$$\begin{aligned} \text{bind}(G + H) &= \min\{E_1, E_2\}, \text{ where} \\ E_1 &= \min\left\{\frac{|N_G(S)| + |V(H)|}{|S|}\right\} = \frac{|N_G(S^*)| + |V(H)|}{|S^*|}, S \in F(G) \\ &\quad \text{and} \\ E_2 &= \min\left\{\frac{|N_H(T)| + |V(G)|}{|T|}\right\} = \frac{|N_H(T^*)| + |V(G)|}{|T^*|}, T \in F(H) \end{aligned}$$

where S^* and T^* are the maximum binding sets of G and H respectively.

Proof: Let G and H be graphs and let $S, T \in F(G + H)$. We have to show that $S \in F(G)$ or $S \in F(H)$ and similar argument will be applied to T .

Claim. $F(G) \cup F(H) = F(G + H)$.

Clearly, $F(G) \cup F(H) \subseteq F(G+H)$. We are left to show that $F(G+H) \subseteq F(G) \cup F(H)$. Let $X \in F(G+H)$. Then $N(X) \neq V(G+H)$. Since every vertex of G is adjacent to all vertices of H and conversely, $N(X) = N_G(X) \cup V(H)$ or $N(X) = N_H(X) \cup V(G)$. This implies that $X \in F(G)$ or $X \in F(H)$. Thus, $X \in F(G) \cup F(H)$. Hence, $F(G) \cup F(H) = F(G+H)$.

Thus, if $S \in F(G+H)$ then by the claim, $S \in F(G)$ or $S \in F(H)$. We just assume that $S \in F(G)$. Then $N(S) = N_G(S) \cup V(H)$. It follows that $|N(S)| = |N_G(S)| + |V(H)|$. Since S is arbitrary, the minimum of $\frac{|N(S)|}{|S|}$ can be obtained for some choice of S , say S^* . Hence, $\min\{\frac{|N_G(S)|+|V(H)|}{|S|}\} = \frac{|N_G(S^*)|+|V(H)|}{|S^*|}$. Set $E_1 = \frac{|N_G(S^*)|+|V(H)|}{|S^*|}$. Furthermore, if $T \in F(G+H)$ then $T \in F(G)$ or $T \in F(H)$. Let $T \in F(H)$. Then $N(T) = N_H(T) \cup V(G)$ and $|N(T)| = |N_H(T)| + |V(G)|$. With the same argument above we have, $\min\{\frac{|N_H(T)|+|V(G)|}{|T|}\} = \frac{|N_H(T^*)|+|V(G)|}{|T^*|}$. Set $E_2 = \frac{|N_H(T^*)|+|V(G)|}{|T^*|}$. Therefore, $\text{bind}(G+H) = \min\{E_1, E_2\}$. $\square$

Corollary 4.2 Let G be a graph. Then

$$\text{bind}(G + K_n) = \text{bind}(G) + \frac{n}{|S|}, S \in F(G)$$

where S is a maximum binding set of G .

Proof: Let G be a graph and S a binding set of $G + K_n$. If G is trivial then we can take $S = G$ and the conclusion follows. If G is complete then either $S \in F(G)$ or $S \in F(K_n)$. Suppose that G is nontrivial and noncomplete. Then $S \in F(G)$ and by Result 4.1, $\text{bind}(G + K_n) = \frac{|N_G(S)|+|V(K_n)|}{|S|} = \frac{|N(S)|}{|S|} + \frac{|V(K_n)|}{|S|} = \text{bind}(G) + \frac{n}{|S|}$ where S is a maximum binding set of G . $\square$

Corollary 4.3 For $n \geq 3$,

$$\text{bind}(W_n) = \begin{cases} \frac{n+2}{n}, & \text{if } n \text{ is even} \\ \frac{n}{n-2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let W_n be a wheel for $n \geq 3$. Since $W_n = C_n + K_1$, by Result 4.2, $\text{bind}(W_n) = \text{bind}(C_n + K_1) = \text{bind}(C_n) + \frac{1}{|S|}$ where S is a binding set of C_n . By Proposition 2.11 (b), $\text{bind}(C_n) = 1$ if n is even and $|S| = \frac{n}{2}$. Thus, $\text{bind}(W_n) = 1 + \frac{1}{n/2} = 1 + \frac{2}{n} = \frac{n+2}{n}$. On the other hand, if n is odd then $|S| = n-2$ and $\text{bind}(C_n + K_1) = \frac{n-1}{n-2} + \frac{1}{n-2}$. Thus, $\text{bind}(W_n) = \frac{n-1+1}{n-2} = \frac{n}{n-2}$. $\square$

Corollary 4.4 For $n \geq 3$,

$$\text{bind}(F_n) = \begin{cases} \frac{n}{n-1}, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let F_n be a fan with $n \geq 3$. Since $F_n = P_n + K_1$, by Result 4.2, $bind(F_n) = bind(P_n + K_1) = bind(P_n) + \frac{1}{|S|}$ where S is a binding set of P_n . By Proposition 2.11 (c), $bind(P_n) = 1$ if n is even and $|S| = n - 1$. Thus, $bind(F_n) = 1 + \frac{1}{n-1} = \frac{n-1+1}{n-1} = \frac{n}{n-1}$. Likewise, $bind(P_n) = \frac{n-1}{n+1}$ if n is odd and $|S| = \frac{n+1}{2}$. Hence, $bind(F_n) = \frac{n-1}{n+1} + \frac{1}{(n+1)/2} = \frac{n-1}{n+1} + \frac{2}{n+1} = 1$. $\square$

Theorem 4.5 If G is a graph with independent binding set then $G + K_n$ has independent binding set.

Proof: If G is trivial or complete graph then the conclusion follows. Suppose that G is nontrivial and noncomplete graph. Let S be a binding set of G . We have to show that S is a binding set of $G + K_n$. Suppose that S^* is a binding set of $G + K_n$. Using parallel argument in the proof of Theorem 4.1, $S^* \in F(G)$ or $S^* \in F(K_n)$. If $S^* \in F(K_n)$ then S^* must be a singleton by Proposition 2.11. Since S^* need not be a singleton, $S^* \in F(G)$. Consequently, $S = S^*$. Therefore, $G + K_n$ has independent binding set. $\square$

Theorem 4.6 Let G and H be graphs. If S is a binding set of $G + H$ then S is a binding set of either G or H .

Proof: Let G and H be graphs. Suppose that S is a binding set of $G + H$. Then by Result 4.2, $S \in F(G)$ or $S \in F(H)$. Note that the minimum of $\frac{|N(T)|}{|T|}$ for any $T \in F(G)$ or $T \in F(H)$ remains if we compute $bind(G \setminus H) = bind(G)$ or $bind(H \setminus G) = bind(H)$. Taking $S = T$, then S is a binding set of either G or H . $\square$

Theorem 4.7 Let G and H be graphs. Then $G + H$ has independent binding set if and only if either G or H has independent binding set.

Proof: Let G and H be graphs. Suppose that $G + H$ has independent binding set. Let S be a binding set of $G + H$. By Result 4.6, S is a binding set of either G or H . But S is independent, it follows that G or H has independent binding set.

Conversely, suppose that either G or H has independent binding set. If G has independent binding set then by Result 4.2, $bind(G + H) = \frac{|N_H(S)| + |V(H)|}{|S|}$ which shows that S is a binding set of $G + H$. If H has independent binding set S then we have $bind(G + H) = \frac{|N_G(S)| + |V(G)|}{|S|}$. This implies that S is also a binding set of $G + H$. In either case $G + H$ has independent binding set. $\square$

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Received: Decmber 20, 2013