A Study of $L$– Smooth Ideals in $(L, \odot)$–Smooth Topological Spaces

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Abstract

In this paper we establish the concept of $L$–smooth ideals and $L$–smooth ideal bases in $(L, \odot)$–smooth topological spaces. Also, we study the images and pre-images of $L$–smooth ideals. Furthermore, we establish the definition of the product of $L$–smooth ideals. Finally we introduce a new definition of $L$–smooth compactness in terms of $L$–smooth ideals.
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1 Introduction

Ideal is one of the most important notions in general topology. A lot of different kinds of ideals have been introduced and studied by many topologists as Kuratowski [13], Hamlett and Jankovice [8]. Throughout a few last year's many types of sets via ideals have been defined and studied by a staff of topologists. As a result of these new sorts of sets, topologists used some of them to construct new forms of topological spaces. This helps us to present several types of functions and investigate some operators which join between the above constructed spaces. Sarkar [19], Abd El-Monsef et al [5] and Mahmoud [14] extended those ideas from general topology to fuzzy topological spaces. Ramadan, Abdel-Sattar and Kim [17] studied the concept of a smooth ideals in $[0,1]$— smooth topological spaces.

ˇSostak [20] introduced the notion of $(L, \land)$— fuzzy topological spaces as a generalization of $L$—topological spaces. Hohle and ˇSostak [9] substitute a complete quasi-monoidal lattice (or GL-monoid) instead of a completely distributive lattice or a unit interval. Also they introduced the concept of $L$—filters for a complete quasi-monoidal lattice $L$ which is the dual of fuzzy ideals. Many authors[15,16,18] studied the structures of fuzzy topology and the structures of fuzzy filters as $[1,3,4,6,7,10,11,12]$. Abdel-Sattar [2] studied the concept of $(L, \odot)$— smooth topological spaces and their properties. In this paper, we introduce the notion of $L$—smooth ideals in $(L, \odot)$— smooth topological spaces and their properties.

2 Preliminaries

Throughout this paper, let $X$ be a nonempty set. $L = (L, \leq, \lor, \odot', 0, 1)$ denotes a completely distributive lattice with order-reversing involution $'$ which has the least and greatest elements, say 0 and 1, respectively. Let $L^X$ be the family of all $L$-fuzzy subsets
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of $X$. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point, $x_t$ for $t \in L$ is an element of $L^X$ such that, for $y \in X$,

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in $X$ is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$. A fuzzy set $\lambda$ is quasi-coincident with $\mu$, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If $\lambda$ is not quasi-coincident with $\mu$, we denote $\lambda \nq \mu$. All the other notations and the other definitions are standard in fuzzy set theory.

**Definition 2.1.** [9,12] A triple $(L, \leq, \odot)$ is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:

(L1) $L = (L, \leq, 1, 0)$ is a complete lattice.

(L2) $(L, \odot)$ is a commutative semigroup.

(L3) $a = a \odot 1$, for each $a \in L$.

(L4) $\odot$ is a distributive over arbitrary joins, i.e. $(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b)$.

**Example 2.2 [11]**

1. Each frame is a stsc-quantale. In particular, the unite interval $(\{0,1\}, \leq, \lor, \land, 0, 1)$ is a stsc-quantale.

2. The unit interval with a left-continuous t-norm $t$, $(\{0,1\}, \leq, t)$, is a stsc-quantale.

3. Every GL-monoid is a stsc-quantale.

4. Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = \max\{0, x+y-1\}$. Then $(\{0,1\}, \leq, \odot)$ is a stsc-quantale.

**Definition 2.3.** [2,9,12] A mapping $T : L^X \rightarrow L$ is called $(L, \odot)$-smooth topology on $X$ if it satisfies the following conditions:

(O1) $T(\underline{0}) = T(\underline{1}) = 1$, where $\underline{0}(x) = 0$ and $\underline{1}(x) = 1$ for all $x \in X$.

(O2) $T(\mu_1 \odot \mu_2) \geq T(\mu_1) \odot T(\mu_2)$, for any $\mu_1, \mu_2 \in L^X$.

(O3) $T(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} T(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset L^X$.

An $(L, \odot)$-smooth topological spaces is called enriched if

(P) $T(\alpha \odot \mu) \geq T(\mu)$, for any $\mu \in L^X$, and $\alpha \in L$.

The pair $(X, T)$ is called $(L, \odot)$-smooth topological spaces (resp. enriched $(L, \odot)$-smooth topological spaces)
Let \((X,\mathcal{T})\) and \((Y,\mathcal{T}')\) be two \((L,\odot)\)-smooth topological spaces and \(f : X \to Y\) be a mapping. Then \(f\) is said to be smooth \textit{continuous} iff \(\mathcal{T}'(\mu) \leq \mathcal{T}(f^{-1}(\mu))\) for each \(\mu \in L^Y\).

**Definition 2.4.** \([9]\) Let \((L,\ast)\) and \((L,\odot)\) be a stsc-quantale. An operation \(\odot\) dominates \(\ast\) if it satisfies: \(\forall x_1, x_2, y_1, y_2 \in L \Rightarrow (x_1 \ast y_1) \odot (x_2 \ast y_2) \geq (x_1 \odot x_2) \ast (y_1 \odot y_2)\)

**Example 2.5.** \([12]\)
(1) For any left-continuous t-norm \(\ast, \wedge\) dominates \(\ast\) because \((x_1 \ast y_1) \wedge (x_2 \ast y_2) \geq (x_1 \wedge x_2) \ast (y_1 \wedge y_2)\).

(2) Define t-norm as \(x \odot y = \frac{x \cdot y}{x + y - x \cdot y}\) and \(x \ast y = x \cdot y\). Then \(\odot\) dominates \(\ast\).

**Definition 2.6.** \([17]\) If \(X\) is a set, then an ideal on \(X\) is a nonempty \(D^* \subset 2^X\) satisfying the following conditions:
1- \(X \notin D^*\).
2- If \(A, B \in D^* \Rightarrow A \cup B \in D^*\).
3- If \(B \in D^*\) and \(A \subset B \Rightarrow A \in D^*, i.e., D^*\) is a lower set.

**Definition 2.7.** \([17]\) If \(X\) is a set, then a preideal on \(X\) is a nonempty \(D \subset I^X\) satisfying the following conditions:
1- \(\underline{1} \notin D\).
2- If \(\lambda, \mu \in D \Rightarrow \lambda \lor \mu \in D\).
3- If \(\mu \in D\) and \(\lambda \leq \mu \Rightarrow \lambda \in D\).

**Definition 2.8.** \([17]\) A nonempty subset \(D\) is called a preideal base on \(X\) if it satisfies the following conditions:
1- \(\underline{1} \notin D\).
2- If \(\mu_1, \mu_2 \in D \Rightarrow \exists \mu_3 \in D\) such that \(\mu_1 \lor \mu_2 \leq \mu_3\)

### 3 The pre images of L-smooth ideals

**Definition 3.1.**
A mapping \(\mathcal{I} : L^X \to L\) is called \(L\)-smooth ideal on \(X\) if it satisfies the following conditions:
(S1) \(\mathcal{I}(\underline{1}) = 0\).
(S2) \( I(\lambda \lor \mu) \geq I(\lambda) \odot I(\mu) \), for \( \lambda, \mu \in L^X \).
(S3) If \( \lambda \geq \mu \), \( I(\lambda) \leq I(\mu) \).

If \( I_1 \) and \( I_2 \) are \( L \)-smooth ideals on \( X \), we say \( I_1 \) is finer than \( I_2 \) (or \( I_2 \) is coarser than \( I_1 \)), denoted by \( I_2 \leq I_1 \), iff \( I_2(\lambda) \leq I_1(\lambda) \) for all \( \lambda \in L^X \).

**Notation 3.2.** Let \( \beta : L^X \to L \) be a smooth mapping and \( \lambda \in L^X \). We denote \( \langle \beta \rangle(\lambda) = \sup_{\lambda \leq \mu} \beta(\mu) \)

**Theorem 3.3.** A mapping \( \beta : L^X \to L \) is called \( L \)-smooth ideal base on \( X \) if it satisfies the following conditions:

(SB1) \( \beta(1) = 0 \).
(SB2) \( \langle \beta(\lambda \lor \mu) \rangle \geq \beta(\lambda) \odot \beta(\mu) \), for \( \lambda, \mu \in L^X \).

**Remark 3.4.** (1) An \( L \)-smooth ideal is \( L \)-smooth ideal base.
(2) If a mapping \( I : L^X \to L \) is an \( L \)-smooth ideal (base ), for \( r \in L \), \( I^r = \{ \mu \in L^X : I(\mu) > r \} \) is a preideal (base).

**Theorem 3.5.** If a mapping \( \beta : L^X \to L \) is an \( L \)-smooth ideal base, then \( \langle \beta \rangle \) is the coarsest \( L \)-smooth ideal satisfying \( \beta(\lambda) \leq \langle \beta \rangle(\lambda) \), for each \( \lambda \in L^X \).

**Proof:** The conditions (S1) and (S3) are easily checked. Suppose there exist \( \lambda, \mu \in L^X \) and \( t \in L \) such that

\[
\langle \beta \rangle(\lambda) \odot \langle \beta \rangle(\mu) > t > \langle \beta \rangle(\lambda \lor \mu)
\]  

(1)

Since \( \langle \beta \rangle(\lambda) > t \) and \( \langle \beta \rangle(\mu) > t \), there exist \( \lambda_1, \mu_2 \in L^X \) with \( \lambda \leq \lambda_1, \mu \leq \mu_1 \) such that

\[
\langle \beta \rangle(\lambda) \odot \langle \beta \rangle(\mu) \geq \beta(\lambda_1) \odot \beta(\mu_1) > t.
\]

Since \( \beta \) is \( L \)-smooth ideal base, \( \langle \beta \rangle(\lambda_1 \lor \mu_1) \geq \beta(\lambda_1) \odot \beta(\mu_1) > t \).

Since \( \lambda \lor \mu \leq \lambda_1 \lor \mu_2 \Rightarrow \langle \beta \rangle(\lambda \lor \mu) \geq \langle \beta \rangle(\lambda_1 \lor \mu_2) > t \). It is a contradiction for the relation (1).

Thus, for every \( \lambda, \mu \in L^X \), \( \langle \beta \rangle(\lambda) \odot \langle \beta \rangle(\mu) \leq \langle \beta \rangle(\lambda \lor \mu) \). Hence, \( \langle \beta \rangle \) is a smooth ideal.

If \( I \) is \( L \)-smooth ideal satisfying \( \beta(\lambda) \leq I(\lambda) \) for each \( \lambda \in L^X \), we will show that \( \langle \beta \rangle \leq I \).
suppose there exist \( \mu \in L^X \) and \( r \in L \) such that
\[
\langle \beta \rangle(\mu) > r > \mathcal{I}(\mu).
\]
(2) Since \( \langle \beta \rangle(\mu) > r \), there exist \( \mu_1 \) with \( \mu \leq \mu_1 \) such that \( \langle \beta \rangle(\mu_1) \geq \beta(\mu_1) > r \).

On the other hand, since \( \beta(\mu_1) \leq \mathcal{I}(\mu_1) \), we have \( \mathcal{I}(\mu_1) \geq \mathcal{I}(\mu) \geq \beta(\mu_1) > r \).
It is a contradiction for the relation (2). Hence \( \langle \beta \rangle \leq \mathcal{I} \).

**Definition 3.6.** Let \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) be two \( L \)-smooth ideals on \( X \) and \( Y \), respectively, and \( f : X \rightarrow Y \) be a function then

1. \( f \) is said to be \( L \)-smooth ideal map (for short, \( \mathcal{I} \)-map) iff \( \mathcal{I}_2(\mu) \leq \mathcal{I}_1(f^{-1}(\mu)) \) for each \( \mu \in L^Y \).
2. \( f \) is said to be \( L \)-smooth ideal preserving map (for short, \( \mathcal{I} \)-preserving map) iff \( \mathcal{I}_1(\mu) \leq \mathcal{I}_2(f(\lambda)) \) for each \( \lambda \in L^X \).

Naturally, the composition of \( \mathcal{I} \)-maps (resp. \( \mathcal{I} \)-preserving maps) is an \( \mathcal{I} \)-maps (resp. \( \mathcal{I} \)-preserving maps).

**Theorem 3.7.** Let \( \beta_1 \) and \( \beta_2 \) be two \( L \)-smooth ideals bases on \( X \) and \( Y \), respectively, and \( f : X \rightarrow Y \) be a function then.

1. \( f : (X, \langle \beta_1 \rangle) \rightarrow (Y, \langle \beta_2 \rangle) \) is an \( \mathcal{I} \)-map iff \( \beta_2(\mu) \leq \langle \beta_1 \rangle(f^{-1}(\mu)) \) for each \( \mu \in L^Y \).
2. \( f : (X, \langle \beta_1 \rangle) \rightarrow (Y, \langle \beta_2 \rangle) \) is an \( \mathcal{I} \)-preserving map iff \( \beta_1(\lambda) \leq \langle \beta_2 \rangle(f(\lambda)) \) for each \( \lambda \in L^X \).
3. If \( \beta_2(\mu) \leq \beta_1(f^{-1}(\mu)) \) for each \( \mu \in L^Y \), then \( f : (X, \langle \beta_1 \rangle) \rightarrow (Y, \langle \beta_2 \rangle) \) is an \( \mathcal{I} \)-map.
4. If \( \beta_1(\lambda) \leq \beta_2(f(\lambda)) \) for each \( \lambda \in L^X \), then \( f : (X, \langle \beta_1 \rangle) \rightarrow (Y, \langle \beta_2 \rangle) \) is an \( \mathcal{I} \)-preserving map.

**Proof:**

1. \( (\Rightarrow) \) Since \( \beta_2(\mu) \leq \langle \beta_2 \rangle(\mu) \) for each \( \mu \in L^Y \), it is trivial.
   
   \( (\Leftarrow) \) Suppose there exist \( \mu \in L^Y \) and \( r \in L \) such that
   \[
   \langle \beta_2 \rangle(\mu) > r > \langle \beta_1 \rangle(f^{-1}(\mu)).
   \]
   (3) Since \( \langle \beta_2 \rangle(\mu) > r \), there exists \( \mu_1 \) with \( \mu \leq \mu_1 \) such that
   \[
   \langle \beta_2 \rangle(\mu) \geq \beta_2(\mu_1) > r.
   \]
   On the other hand, since \( \beta_2(\mu_1) \leq \langle \beta_1 \rangle(f^{-1}(\mu_1)) \), we have
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\[
\langle \beta_1 \rangle(f^{-1}(\mu)) \geq \langle \beta_1 \rangle(f^{-1}(\mu_1)) \geq \beta_2(\mu_1) > r.
\]

It is a contradiction for the relation (3). Thus, \( \langle \beta_2 \rangle(\mu) \leq \langle \beta_1 \rangle(f^{-1}(\mu_1)) \), for each \( \mu \in L^Y \).

(2), (3) and (4) are similarly proved.

Example 3.8. Let \( L = I = [0, 1] \) and \( X = \{a, b, c, d\} \) be a set. We define two functions \( \beta_1, \beta_2 : I^X \rightarrow I \) as follows:

\[
\beta_1(\lambda) = \begin{cases} 
0.5, & \text{if } \lambda = 0, \\
0.4, & \text{if } \lambda \in \{1(a), 1(b), 1(a,b,c)\}, \\
0, & \text{otherwise}
\end{cases}
\]

\[
\beta_2(\mu) = \begin{cases} 
0.4, & \text{if } \mu = 0, \\
0.3, & \text{if } \mu \in \{1(a), 1(b), 1(a,b)\}, \\
0, & \text{otherwise}
\end{cases}
\]

Since \( \langle \beta_1 \rangle(1(a) \lor 1(b)) = \beta_1(1(a,b,c)) = 0.4 \), \( \beta_1 \) is \( L \)-smooth ideal base. Similarly \( \beta_2 \) is \( L \)-smooth ideal base. We obtain

\[
\langle \beta_1 \rangle(\lambda) = \begin{cases} 
0.5, & \text{if } \lambda = 0, \\
0.4, & \text{if } 0 < \lambda \leq 1(a,b,c), \\
0, & \text{otherwise}
\end{cases}
\]

\[
\langle \beta_2 \rangle(\mu) = \begin{cases} 
0.4, & \text{if } \mu = 0, \\
0.3, & \text{if } 0 < \mu \leq 1(a,b), \\
0, & \text{otherwise}
\end{cases}
\]

Then, \( id_X : (X, \langle \beta_1 \rangle) \rightarrow (X, \langle \beta_2 \rangle) \) be an \( I \)-map and \( id_X : (X, \langle \beta_2 \rangle) \rightarrow (X, \langle \beta_1 \rangle) \) is an \( I \)-preserving map. But \( 0 = \beta_1(1(a,b,c)) < \beta_2(1(a,b)) = 0.3 \).

Hence, the converse of Theorem 3.7 (3-4) need not be true.

Notation 3.9. Let \( I \)- be \( L \)-smooth ideal on X. We denote \( I_0 = \{ \lambda \in L^X : I(\lambda) > 0 \} \).

Theorem 3.10. Let \( f_i : X \rightarrow X_i \) be a function, for each \( i \in \Gamma \). Let \( \{ \beta_i \}_{i \in \Gamma} \) be a family of \( L \)-smooth ideal bases on \( X_i \) satisfying the following condition
(C) If $v_i \in (\beta_i)_0$ for all $i \in \Gamma$, then we have $\bigvee_{i \in K} f_i^{-1}(v_i) \neq 1$ for every finite subset $K$ of $\Gamma$.

We define a function $\bigvee_{i \in \Gamma} f_i^{-1}(\beta_i) : L^X \to L$ as

$$
\bigvee_{i \in \Gamma} f_i^{-1}(\beta_i)(\lambda) = \begin{cases} 
\sup\{\bigwedge_{i \in K} \beta_i(v_i)\} & \text{if } \lambda = \bigvee_{i \in K} f_i^{-1}(v_i), \; v_i \in (\beta_i)_0 \\
0 & \text{if otherwise.}
\end{cases}
$$

where the supremum is taken for every finite subset of $K$ of $\Gamma$ such that $\lambda = \bigvee_{i \in K} f_i^{-1}(\mu_i)$. Let $\beta = \bigvee_{i \in \Gamma} f_i^{-1}(\beta_i)$ be given. Then

1. $\beta$ is $L$--smooth ideal bases on $X$.
2. $\langle \beta \rangle$ is the coarsest $L$--smooth ideal on $X$ for which each function $f_i : (X, \langle \beta \rangle) \to (X_i, \langle \beta_i \rangle)$ is an $\mathcal{I}$--map.
3. A map $f : (Y, \mathcal{I}^*) \to (X, \langle \beta \rangle)$ is an $\mathcal{I}$--map iff for each $i \in \Gamma$, $f_i \circ f : (Y, \mathcal{I}^*) \to (X_i, \langle \mathcal{I}_i \rangle)$ is an $\mathcal{I}$--map.
4. $\langle \bigvee_{i \in \Gamma} f_i^{-1}(\beta_i) \rangle = \langle \bigvee_{i \in \Gamma} f_i^{-1}(\langle \beta_i \rangle) \rangle$.

**Proof:**

1. Let $\beta = \bigvee_{i \in \Gamma} f_i^{-1}(\beta_i)$. We show that $\beta$ is $L$--smooth ideal base. Since $\beta_i$ is a nonzero function, there exists $\mu_i \in (\beta_i)_0$ such that

$$
\beta(f_i^{-1}(\mu_i)) \geq \beta_i(\mu_i) > 0
$$

Thus, $\beta$ is nonzero function.

(SB1) It is trivial that $\beta(1) = 0$.

(SB2) Suppose there exists $\lambda_1, \lambda_2 \in L^X$ and $r \in L$ such that

$$
\langle \beta \rangle(\lambda_1 \lor \lambda_2) < r < \beta(\lambda_1) \circ \beta(\lambda_2), 
$$

Since

$$
\beta(\lambda_1) > r \text{ and } \beta(\lambda_2) > r, \text{ by definition of } \beta, \text{ there exist two finite subsets } K \text{ and } J \text{ of } \Gamma \text{ with } \lambda_1 = \bigvee_{k \in K} f_k^{-1}(v_k) \text{ and } \lambda_2 = \bigvee_{j \in J} f_j^{-1}(\mu_j) \text{ such that}
$$

$$
\beta(\lambda_1) \geq \bigwedge_{k \in K} \beta_k(v_k) > r, \quad \beta(\lambda_2) \geq \bigwedge_{j \in J} \beta_j(\mu_j) > r.
$$

Put $m \in K \cup J$ such that

$$
\rho_m = \begin{cases} 
v_m \in L^X & \text{if } m \in K - (K \land J), \\
\mu_m \in L^X & \text{if } m \in J - (K \land J), \\
(v_m \lor \mu_m) \in L^X & \text{if } m \in (K \land J).
\end{cases}
$$
For each \( m \in K \cup J \), since \( \beta_m(v_m) > r \) and \( \beta_m(\mu_m) > r \), we have

\[
\langle \beta_m \rangle(v_m \vee \mu_m) \geq \beta_m(v_m) \odot \beta_m(\mu_m) > r.
\]

From the definition of \( \langle \beta_m \rangle \), there exists \( \omega_m \in L^X \) with \( \omega_m \geq v_m \vee \mu_m \) such that

\[
\langle \beta_m \rangle(v_m \vee \mu_m) \geq \beta_m(\omega_m) > r. \quad (5)
\]

Since

\[
\lambda_1 \vee \lambda_2 = (\bigvee_{k \in K} f_k^{-1}(\nu_k)) \vee (\bigvee_{j \in J} f_j^{-1}(\mu_j)) = \bigvee_{m \in (K \cup J)} f_m^{-1}(\rho_m) \leq (\bigvee_{m \in (K \cup J)} f_m^{-1}(\rho_m)) \vee (\bigvee_{m \in (K \cap J)} f_m^{-1}(\omega_m)),
\]

there exists a finite index \( K \cup J \) such that

\[
\langle \beta \rangle(\lambda_1 \vee \lambda_2) \geq (\bigvee_{m \in (K \cup J) - (K \cap J)} \beta_m(\rho_m)) \odot (\bigvee_{m \in (K \cap J)} \beta_m(\omega_m)) \\
\geq (\bigvee_{m \in (K \cup J) - (K \cap J)} \beta_m(\rho_m)) \odot r \quad \text{(by (5))}
\]

It is a contradiction for the relation (4). Hence, \( \langle \beta \rangle(\lambda_1 \vee \lambda_2) \geq \beta(\lambda_1) \odot \beta(\lambda_2) \), \( \forall \lambda_1, \lambda_2 \in L^X \).

(2) From Theorem 3.7 (3), we only show that \( \beta(f^{-1}_i(\lambda_i)) \geq \beta_i(\lambda_i) \) for each \( i \in \Gamma \) from the following:

If \( \beta_i(\lambda_i) = 0 \), it is trivial.

If \( \beta_i(\lambda_i) > 0 \), for a one family \( \{ \lambda_i \in (\beta_i)_0 \} \), we have \( \beta(f^{-1}_i(\lambda_i)) \geq \beta_i(\lambda_i) \).

Let \( \mathcal{I}(f^{-1}_i(\lambda_i)) \geq \langle \beta_i \rangle(\lambda_i) \) for each \( i \in \Gamma \). Suppose there exist \( \lambda \in L^X \) and \( r \in L \) such that

\[
\langle \beta \rangle(\lambda) > r > \mathcal{I}(\lambda). \quad (6)
\]

Since \( \langle \beta \rangle(\lambda) > r \), by definition of \( \langle \beta \rangle \), there exist a finite subset \( K \) of \( \Gamma \) with \( \lambda \leq \bigvee_{k \in K} f_k^{-1}(\nu_k) \) such that

\[
\langle \beta \rangle(\lambda) \geq \bigwedge_{k \in K} \beta_k(\nu_k) > r.
\]

On the other hand, since \( \mathcal{I}(f^{-1}_k(\nu_k)) \geq \langle \beta_k \rangle(\nu_k) \) for all \( k \in K \), we have

\[
\mathcal{I}(\lambda) \geq \mathcal{I}(\bigvee_{k \in K} \beta_k(\nu_k)) \geq \bigwedge_{k \in K} \mathcal{I}(f^{-1}_k(\nu_k)) \geq \bigwedge_{k \in K} \langle \beta_k \rangle(\nu_k) \geq \bigwedge_{k \in K} \beta_k(\nu_k) \quad \text{(by S3)}
\]
> r.
It is a contradiction for the relation (6). Hence \( \mathcal{I} \geq \langle \beta \rangle \).

(3) Necessity of the composition condition is clear since the composition of \( \mathcal{I} \)-maps is an \( \mathcal{I} \)-map.

Conversely, suppose \( f : (Y, \mathcal{I}^*) \to (X, \langle \beta \rangle) \) is not an \( \mathcal{I} \)-map. There exist \( \mu \in \mathcal{L}^X \) and \( r \in \mathcal{L} \) such that
\[
\langle \beta \rangle(\mu) \geq \bigwedge_{k \in K} \beta_k(\nu_k) > r > \mathcal{I}^*(f^{-1}(\mu)) .
\]
From the other hand, since for each \( i \in \Gamma \), \( f_i : (Y, \mathcal{I}^*) \to (X_i, \langle \beta_i \rangle) \) is an \( \mathcal{I} \)-map, \( \langle \beta \rangle(\mu) \geq \bigwedge_{k \in K} \beta_k(\nu_k) > r \).
and \( \langle \beta_i \rangle(\nu_i) \leq \mathcal{I}^*(f^{-1}(f_i^1(\nu_i))) \).

It follows \( \mathcal{I}^*(f^{-1}(f_i^1(\nu_i))) \geq \beta_k(\nu_k) \) for all \( k \in K \). Since \( f^{-1}(\mu) \leq \bigvee_{k \in K} f^{-1}(f_k^1(\nu_k)) \), we have
\[
\mathcal{I}^*(f^{-1}(\mu)) \geq \bigwedge_{k \in K} \mathcal{I}^*(f^{-1}(f_k^1(\nu_k))) 
\geq \bigwedge_{k \in K} \langle \beta_k \rangle(\nu_k)
\geq \bigwedge_{k \in K} \beta_k(\nu_k)
> r.
\]
It is a contradiction for the relation (7). Hence, \( f \) is an \( \mathcal{I} \)-map.

(4) Let \( \mathcal{I} = (\bigvee_{i \in \Gamma} f_i^{-1}(\langle \beta_i \rangle)) \). Since \( \beta_i(\lambda_i) \leq \beta(f_i^{-1}(\lambda_i)) \), by Theorem 3.7(3), \( f_i : (X, \langle \beta \rangle) \to (X_i, \langle \beta_i \rangle) \) is an \( \mathcal{I} \)-map.

From (3), the identity map \( id_X : (X, \langle \beta \rangle) \to (X, \mathcal{I}) \) is an \( \mathcal{I} \)-map. Thus, \( \mathcal{I} \leq \langle \beta \rangle \).

From the definition of \( \mathcal{I} \), \( \langle \beta_i \rangle(\lambda_i) \leq \mathcal{I}(f_i^{-1}(\lambda_i)) \), that is, \( f_i : (X, \mathcal{I}) \to (X_i, \langle \beta_i \rangle) \) is an \( \mathcal{I} \)-map. From (2), \( \langle \beta \rangle \leq \mathcal{I} \).

**Theorem 3.11.** Let \( \{ \mathcal{I}_i \}_{i \in \Gamma} \) be a family of \( L \)-smooth ideals on \( X \) satisfying the following condition:

(C) If \( \lambda_i \in (\mathcal{I}_i)_0 \) for all \( i \in \Gamma \), then we have \( \bigvee_{i \in K} \lambda_i \neq 1 \), for every finite subset \( K \) of \( \Gamma \).

We define the function \( \bigvee_{i \in \Gamma} \mathcal{I}_i : \mathcal{L}_X \to \mathcal{L} \) as
\[
\bigvee_{i \in \Gamma} \mathcal{I}_i(\mu) = \begin{cases} 
\sup \{ \bigwedge_{i \in K} \mathcal{I}_i(\mu_i) \} & \text{if } \mu = \bigvee_{i \in K} \mu_i , \mu_i \in (\mathcal{I}_i)_0 \\
0 & \text{if otherwise.}
\end{cases}
\]
where the supremum is taken for every finite subset $K$ of $\Gamma$ such that $\mu = \bigvee_{i \in K} \mu_i$. Then $\mathcal{I}$ is the coarsest $L$-smooth ideal finer than $i \in \Gamma$.

**Proof:**

From Theorem 3.10, put $(X_i, \beta_i) = (X, \mathcal{I}_i)$ and $g_i = id_X$ where $id_X$ is an identity map for each $i \in \Gamma$. Let $\mathcal{I} = \bigvee_{i \in \Gamma} \mathcal{I}_i$ be given. We only show that $\mathcal{I} = \langle \mathcal{I} \rangle$. It is trivially show that $\mathcal{I} \leq \langle \mathcal{I} \rangle$.

Suppose that $\mathcal{I} \geq \langle \mathcal{I} \rangle$. (8) There exist $\nu \in L^X$ and $r \in L$ such that $\mathcal{I}(\nu) < r < \langle \mathcal{I} \rangle(\nu)$.

Since $\langle \mathcal{I} \rangle(\nu) \geq r$, there exists $\mu \in L^X$ with $\nu \leq \mu$ such that $\langle \mathcal{I} \rangle(\nu) \geq \mathcal{I}(\mu)$.

By definition of $\mathcal{I}$, there exists a finite index set $K$ with $\mu = \bigvee_{k \in K} \mu_k$ such that

$$\mathcal{I}(\mu) \geq \bigwedge_{k \in K} \mathcal{I}_i(\mu_k) > r.$$

On the other hand, since $\nu = \nu \wedge \mu = \bigwedge_{k \in K} (\nu \wedge \mu_k)$, we have

$$\mathcal{I}(\nu) \geq \bigwedge_{k \in K} \mathcal{I}_k(\nu \wedge \mu_k) \geq \bigwedge_{k \in K} \mathcal{I}_k(\mu_k) > r.$$

It is a contradiction for the relation (8). Hence $\mathcal{I} \geq \langle \mathcal{I} \rangle$.

**Example 3.12.** Let $L = I = [0,1]$ and $X = \{a, b, c, d\}$ be a set. We define a function $\beta : I^X \to I$ as follows:

$$\beta(\lambda) = \begin{cases} 0.5, & \text{if } \lambda = \emptyset, \\ 0.4, & \text{if } \lambda \in \{1(a), 1(b), 1(a,b,c)\}, \\ 0, & \text{otherwise}. \end{cases}$$

(1) Let $A = \{a, b, c\}$ be a set and $i_A : A \to X$ be an inclusion map. Since $i_A^{-1}(1(a,b,c)) = 1(a,b,c)$, we can not define $i^{-1}(\beta)$ from the condition (C) of Theorem 3.10.

(2) Let $B = \{a, b, d\}$ be a set and $i_B : B \to X$ be an inclusion map. We can obtain $i_B^{-1}(\beta) : I^B \to I$ as follows:

$$i_B^{-1}(\beta)(\mu) = \begin{cases} 0.5, & \text{if } \mu = \emptyset, \\ 0.4, & \text{if } \mu \in \{1(a), 1(b), 1(a,b)\}, \\ 0, & \text{otherwise}. \end{cases}$$

From Theorem 3.10, we have $\langle i_B^{-1}(\beta) \rangle = \langle i_B^{-1}(\langle \beta \rangle) \rangle$ as follows:

$$\langle i_B^{-1}(\beta) \rangle(\mu) = \begin{cases} 0.5, & \text{if } \mu = \emptyset, \\ 0.4, & \text{if } 0 < \mu \leq 1(a, b), \\ 0, & \text{otherwise}. \end{cases}$$
4 The product of L-smooth ideals spaces, the images of L-smooth ideals and \( L \)-smooth ideal compact spaces

**Definition 4.1.** Let \( \{ I_i \}_{i \in \Gamma} \) be a family of \( L \)-smooth ideals on \( X_i \), \( X = \prod_{i \in \Gamma} X_i \) a product set and the function \( \Pi_i : X \rightarrow X_i \) a projection map, for each \( i \in \Gamma \).

The structure \( \{ \bigvee_{i \in \Gamma} \Pi_i^{-1}(I_i) \} \) is called a product \( L \)-smooth ideals on \( X \).

**Theorem 4.2.** Let \( \{ \beta_i \}_{i \in \Gamma} \) be a family of \( L \)-smooth ideals bases on \( X_i \). Let \( X = \prod_{i \in \Gamma} X_i \) be a product set and \( \Pi_i : X \rightarrow X_i \) a projection map, for each \( i \in \Gamma \). We define a function \( \bigvee_{i \in \Gamma} \Pi_i^{-1}(\beta_i) : L^X \rightarrow L \) as:

\[
\bigvee_{i \in \Gamma} \Pi_i^{-1}(\beta_i)(\mu) = \begin{cases} 
\sup \{ \bigwedge_{\mu_i \in (\beta_i)_0} (\beta_i)(\mu_i) \} & \text{if } \mu = \bigvee_{i \in \Gamma} \Pi_i^{-1}(\beta_i), \
0 & \text{if otherwise.}
\end{cases}
\]

Where the supremum is taken for every finite subset \( K \) of \( \Gamma \) such that \( \mu = \bigvee_{i \in K} \Pi_i^{-1}(\beta_i) \). Let \( \beta = \bigvee_{i \in \Gamma} \Pi_i^{-1}(\beta_i) \) be given.

Then:

1. \( \langle \beta \rangle \) is the coarsest \( L \)-smooth ideal on \( X \) for which each projection map \( \Pi_i : (X, \langle \beta \rangle) \rightarrow (X_i, \langle \beta_i \rangle) \) is an \( \mathcal{I} \)-map.
2. A map \( f : (Y, \mathcal{I}) \rightarrow (X, \langle \beta \rangle) \) is an \( \mathcal{I} \)-map iff for each \( i \in \Gamma \), \( \Pi_i \circ f : (Y, \mathcal{I}) \rightarrow (X_i, \langle \beta_i \rangle) \) is an \( \mathcal{I} \)-map.

**Example 4.3.** Let \( L = I = [0, 1] \) and \( X = \{ a, b \}, Y = \{ x, y \} \) be two sets. We define an \( L \)-smooth ideals \( I_1 : L^X \rightarrow L \) and \( I_2 : L^X \rightarrow L \) as follows:

\[
I_1(\nu) = \begin{cases} 
0.6, & \text{if } \nu = 0, \\
0.4, & \text{if } \nu = a_t, t \in (0, 1] \\
0, & \text{otherwise.}
\end{cases}
\]

\[
I_2(\nu) = \begin{cases} 
0.7, & \text{if } \nu = 0, \\
0.3, & \text{if } \nu = x_s, s \in (0, 1] \\
0, & \text{otherwise.}
\end{cases}
\]
Let $\Pi_1 : X \times Y \to X$ and $\Pi_2 : X \times Y \to Y$ be two projection maps. We can obtain the product $L$-smooth ideal

$$\mathcal{I} = \Pi_1^{-1}(\mathcal{I}_1) \lor \Pi_2^{-1}(\mathcal{I}_2)$$

as follows:

$$\mathcal{I}(\lambda) = \begin{cases} 
0.7, & \text{if } \lambda = 0, \\
0.4, & \text{if } 0 < \lambda \leq 1(a_x)(a_y) \\
0.3, & \text{if } 1(a_x)(a_y) \neq \lambda \leq 1(a_x)(a_y)(b_x), \\
0, & \text{otherwise.} 
\end{cases}$$

**Theorem 4.4.** Let $f_i : X_i \to X$ be a function for each $i \in \Gamma$. Let $\{\mathcal{I}_i\}_{i \in \Gamma}$ be a family of $L$–smooth ideals on $X$ satisfying the following condition:

(C) If $\mu_i \in (\mathcal{I}_i)_0$ for all $i \in \Gamma$, then we have $\bigvee_{i \in K} f_i(\mu_i) \neq 1$ for every finite subset $K$ of $\Gamma$. We define a function $\bigvee_{i \in \Gamma} f_i(\mathcal{I}_i) : L^X \to L$ as

$$\bigvee_{i \in \Gamma} f_i(\mathcal{I}_i)(\nu) = \begin{cases} 
\sup\{\bigwedge_{i \in K} \mathcal{I}_i(\mu_i)\} & \text{if } \nu = \bigvee_{i \in K} f_i(\mu_i), \mu_i \in (\mathcal{I}_i)_0, \\
0, & \text{otherwise.} 
\end{cases}$$

where the supremum is taken for every finite subset $K$ of $\Gamma$ such that $\nu = \bigvee_{i \in \Gamma} f_i(\mu_i)$. Put $\mathcal{I} = \bigvee_{i \in \Gamma} f_i(\mathcal{I}_i)$. Then:

(1) $\mathcal{I}$ is the coarsest smooth ideal on $X$ for which each function $f_i : X_i \to X$ is an $\mathcal{I}$-preserving map.

(2) A map $f : (X,\mathcal{I}) \to (Y,\mathcal{H})$ is an $\mathcal{I}$-preserving map iff for each $i \in \Gamma$, $f \circ f_i : (X_i,\mathcal{I}_i) \to (Y,\mathcal{H})$ is an $\mathcal{I}_i$-preserving map.

**Corollary 4.5.** Let $f_i : X_i \to X$ be a function for each $i \in \Gamma$. Let $\{\beta_i\}_{i \in \Gamma}$ be a family of $L$–smooth ideals bases on $X$ satisfying the following condition:

(C) If $\mu_i \in (\beta_i)_0$ for all $i \in \Gamma$, then we have $\bigvee_{i \in K} f_i(\mu_i) \neq 1$ for every finite subset $K$ of $\Gamma$. We define a function $\bigvee_{i \in \Gamma} f_i(\beta_i) : L^X \to L$ as

$$\bigvee_{i \in \Gamma} f_i(\beta_i)(\nu) = \begin{cases} 
\sup\{\bigwedge_{i \in K} \beta_i(\mu_i)\} & \text{if } \nu = \bigvee_{i \in K} f_i(\mu_i), \mu_i \notin (\beta_i)_0, \\
0, & \text{otherwise.} 
\end{cases}$$

where the supremum is taken for every finite subset $K$ of $\Gamma$ such that $\nu = \bigvee_{i \in \Gamma} f_i(\mu_i)$. Put $\beta = \bigvee_{i \in \Gamma} f_i(\beta_i)$. Then:

(1) $\bigvee_{i \in \Gamma} f_i(\beta_i)$ is the $L$–smooth ideal base on $X$ for which each function $\beta_i(\lambda_i) \leq \bigvee_{i \in \Gamma} f_i(\beta_i)(\lambda_i)$ for each $\lambda_i \in L^{X_i}$.

(2) $\langle \beta \rangle = \bigvee_{i \in \Gamma} f_i(\langle \beta_i \rangle)$. 

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A map \( f : (X, \langle \beta \rangle) \to (Y, \mathcal{H}) \) is an \( \mathcal{I} \)-preserving map iff for each \( i \in \Gamma \), \( f \circ f_i : (X_i, \langle \beta_i \rangle) \to (Y, \mathcal{H}) \) is an \( \mathcal{I} \)-preserving map.

**Definition 4.6.** Let \((X, \tau)\) be an \((L, \odot)\)-smooth topological space with \(L\)-smooth ideal \( \mathcal{I} \). A fuzzy set \( \lambda \) is called \( L \)-smooth \( \mathcal{I} \)-compact iff every open cover \((\mu_i)_{i \in \Gamma}\) of \( \mu \) has a finite sub-cover \((\mu_{i_o})_{i_o \in \Gamma}\) such that \( \forall i_o \in \Gamma, \exists x \in X, (\lambda - \bigvee_{i_o \in \Gamma} \mu_{i_o})(x) \leq \nu(x), \forall \nu \in \mathcal{I} \).

An \((L, \odot)\)-smooth topological space with \(L\)-smooth ideal \( \mathcal{I} \) is \( \mathcal{I} \)-compact as a subset is a smooth \( \mathcal{I} \)-compact.

**Theorem 4.7.** An \((L, \odot)\)-smooth topological space \((X, \tau)\) with \(L\)-smooth ideal \( \mathcal{I}_1 \) is a smooth \( \mathcal{I}_1 \)-compact and \( \mathcal{I}_2 \) is an \( L \)-smooth ideal on \( X \) such that \( \mathcal{I}_1 \leq \mathcal{I}_2 \). Then \((X, \tau)\) is a smooth \( \mathcal{I}_2 \)-compact space.

**Notation:** In other paper we study the smooth \( \mathcal{I} \)-compact spaces and separation axioms with respect to \( L \)-smooth ideals.

5 Conclusion

A smoothing by using fuzzy logic gives rather good results. In particular smoothing of ideals, The images and pre images of smooth ideals. The product of smooth ideals seem to be a good examples and corresponding concepts trace back to the (classic) fuzzy ideal structures. We feeling that we can be build a new mathematical object (\((L, \odot)\)-smooth structures) where \( L = (L, \leq, \lor, \odot', 0, 1) \) denotes a completely distributive lattice with order-reversing involution ‘ which has the least and greatest elements, say 0 and 1, respectively. This approach could be a subject of further studies.

**References**


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