A Generalization of Weakly Co-hopfian Modules

Xu Wang

School of Science, Lanzhou University of Technology
287, Langongping Road, Qilihe Area, Lanzhou
Gansu, 730050, P.R. China

Tingting Li

College of Science, Guilin University of Technology
12, Jiangan Road, Qixing Area, Guilin
Guangxi, 451004, China

Copyright © 2014 Xu Wang and Tingting Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

An R -module M is called t -weakly co-hopfian if any monomorphic endomorphism of M is t -essential, which is discussed in [3]; still its property much familiar with weakly co-hopfian class, though, M is different due to its non-zero $Z_2(M)$. Several equivalent conditions are given for a module to be t -weakly co-hopfian.

Mathematics Subject Classification: 16D10

Keywords: t -weakly co-hopfian, weakly co-hopfian

1 Introduction

Rings will have unit elements and modules will be unitary. The terminology not defined here may be found in [1]. Let R be a ring and M_R a right R -module. M is called co-hopfian if any injective endomorphism of M is an isomorphism, and called weakly co-hopfian if any injective endomorphism of M is essential; see [2] for a discussion of such modules. We say an R -module M is t -weakly co-hopfian if any injective endomorphism f of M is t -essential; that is, for any submodules N of M , $f(M) \cap N \leq_{tes} Z_2(M)$ implies that $N \leq Z_2(M)$.

2 Main Results

Lemma 2.1 *Let A, B be submodules of M , $A \leq B$, if A is t -essential in M , then also B .*

Proof: If B is not t -essential in M , then there exist $S \leq M$ satisfies $S \cap B \leq Z_2(M)$, S is not in $Z_2(M)$. Because $A \leq B$, then $S \leq M$ satisfies $S \cap A \leq Z_2(M)$, S is not in $Z_2(M)$, contradicts.

Theorem 2.2 *The followings are equivalent conditions on a right R -module M .*

(1) M_R is t -weakly co-hopfian. (2) For any right R -module N , if there is an R -monomorphism $f: M \oplus N \rightarrow M$, then $f(N) \leq Z_2(M)$; that is, N is Z_2 -torsion.

(2') For any right R -module N , if $f: M \oplus N \rightarrow M$ is a t -essential monomorphism, then $f(N) \leq Z_2(M)$; that is, N is Z_2 -torsion.

(3) $M' = M/Z_2(M)$, then M' is weakly co-hopfian (Dedekind finite).

(4) There exists a fully invariant (t -)essential submodule which is t -weakly co-hopfian.

(5) Injective endomorphisms of M_R map t -essential submodules to t -essential submodules.

(6) The inverse image of any submodule not in $Z_2(M)$ under any injective endomorphism of M is not in $Z_2(M)$.

Proof:

(1) $\Rightarrow$ (2): Let $f: M \oplus N \rightarrow M$ be a monomorphism, $i: M \rightarrow M \oplus N$ is the canonical injection, then fi is an injective endomorphism of M , hence fi is t -essential. If now there exists a nonzero $K \leq f(N)$, $K \cap Z_2(M) = 0$, then we get a nonzero $K' = K \cap fi(M)$, hence $K' \leq f(M \oplus 0) \cap f(0 \oplus N)$, conversely. So $f(N) \leq Z_2(M)$; hence N is Z_2 -torsion.

(2) $\Rightarrow$ (2'): this is trivial.

(2') $\Rightarrow$ (1): Let g be an injective endomorphism of M without t -essential image. Then there exists a submodule K not in $Z_2(M)$ satisfies $g(M) \oplus K \leq_{tes} M$. But this yields a t -essential monomorphism $M \oplus K \rightarrow M$, contradicting (2'). Hence $g(M) \leq_{tes} M$.

(1) $\Rightarrow$ (3): Let N be the injective endomorphism of $Z_2(M)$, then N is nonsingular in M , so N is weakly co-hopfian. For $N \oplus Z_2(M) \leq_e M$ we have $((N \oplus Z_2(M))/Z_2(M)) \leq_e M'$. Then suppose f be an injective endomorphism of M' , $f(M')$ is not essential in M' . So we have a $K \leq M$, $f(M) \oplus ((K + Z_2(M))/Z_2(M)) \leq_e M$, thus $(K \oplus Z_2(M))/Z_2(M) \cap (N \oplus Z_2(M))/Z_2(M) \neq 0$. But $(N \oplus Z_2(M))/Z_2(M)$ is weakly co-hopfian, then $f((N \oplus Z_2(M))/Z_2(M)) \cap f((K \oplus Z_2(M))/Z_2(M)) \neq 0$, which is in $f(M') \cap (K \oplus Z_2(M))/Z_2(M)$, contradicts. Then M' is weakly co-hopfian.

(3) $\Rightarrow$ (1): Suppose $g: M \rightarrow M$ be injective and $g(M)$ is not t -essential in M . Then we have an submodule K of M , $g(M) \oplus K \leq_e M$ and K is

not in $Z_2(M)$. But $(g(M) + Z_2(M))/Z_2(M) \oplus (K + Z_2(M))/Z_2(M) \leq_e M'$, contradicts the Dedekind finiteness of M' . So M_R is t -weakly co-hopfian.

(1) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Assume that $K \leq_{tes} M$, K is fully invariant and t -weakly co-hopfian. g is an injective endomorphism of M , then $g|_K$ is an injective endomorphism of K . So $g(K) \leq_{tes} K$, and $K \leq_{tes} M$, then we have $g(K) \leq_{tes} M \Rightarrow g(M) \leq_{tes} M$.

(1) $\Rightarrow$ (5): Suppose A is a t -essential submodule of M , f is an injective endomorphism of M . If there is a submodule N in M which is not in $Z_2(M)$, satisfies $f(A) \cap N \leq Z_2(M)$. Then we have $A \cap f^{-1}(N) \leq Z_2(M)$, but $f^{-1}(N)$ is not in $Z_2(M)$, contradicting that A is t -essential. So $f(A)$ is t -essential too.

(5) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (6): Suppose M_R submodule N is not in $Z_2(M)$, f is an injective endomorphism of M , and $f(M) \cap N$ is not in $Z_2(M)$. Let n be an element of N not in $Z_2(M)$, and there exists an m in M , such that $n = f(m)$. If m is in $Z_2(M)$, we have $f(m)$ in $Z_2(M)$, contradicts. So m is not in $Z_2(M)$.

(6) $\Rightarrow$ (1): Assume an injective endomorphism f of M satisfies $f(M) \cap N \leq Z_2(M)$ where N is not in $Z_2(M)$, then we get $f^{-1}(f(M) \cap N) \leq f^{-1}(Z_2(M)) \leq Z_2(M) \Rightarrow f^{-1}(N) \leq Z_2(M)$, contradicting (6).

Corollary 2.3 *If M_R is t -weakly co-hopfian and f is an injective endomorphism of M , then $N \leq_{tes} M$ if and only if $f(N) \leq_{tes} M$ if and only if $f^{-1}(N) \leq_{tes} M$.*

Corollary 2.4 *The following hold.*

- (1) *A direct summand of a t -weakly co-hopfian module is t -weakly co-hopfian.*
- (2) *A module is t -weakly co-hopfian whenever the injective envelope of $M/Z_2(M)$ is Dedekind finite.*

Corollary 2.5 *Let M be quasi-injective.*

(1) *If M is t -weakly co-hopfian so are any submodule and any finite direct sum of copies of M .*

(2) *Suppose N is a fully invariant essential submodule of M . Then N is t -weakly co-hopfian if and only if so is M . Moreover, M is t -weakly co-hopfian if and only if so is $E(M)$.*

Proof:

(1) *For M is t -weakly co-hopfian then we have $M' = M/Z_2(M)$ is weakly co-hopfian. As in [2], M^n is weakly co-hopfian, so $M^n = \bigoplus_n M/Z_2(M) = \bigoplus_n M / \bigoplus_n Z_2(M) = \bigoplus_n M/Z_2(\bigoplus_n M)$. Then, due to theorem 1.1 (3), t -weakly co-hopfian property keeps the direct summand. Similarly, to prove that a submodule A of M is t -weakly co-hopfian, let $Z_2(M) = Z_2(A) + K$, then we notice that $A' = A/Z_2(A) \cong (A + K)/Z_2(M) \leq M/Z_2(M) = M'$, which means that A' is Dedekind finite. Thus A is t -weakly co-hopfian.*

(2) The first section is obvious from theorem 1.1 and (1) above. For the second, we just notice that M is a fully invariant t -essential submodule of $E(M)$ and then apply the first assertion.

References

- [1] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*(2nd), Springer-Verlag, New York, 1992.
- [2] A. Haghany, M. R. Vedadi, Modules whose Injective Endomorphisms are Essential, *Journal of Algebra*, **1** (2001), 243, 765-779.
- [3] A. Haghany, Sh. Asgari, T-Extending Modules and T-Baer Modules, *communication in Algebra*, **9** (2011),39 , 1605-1623.
- [4] A. Haghany, Sh. Asgari, Y. Tolooei, T-Semisimple Modules and T-Semisimple Rings, *Communication in Algebra*, **9** (2013),41, 1882-1902.

Received: January 5, 2014