Uniqueness of Meromorphic Functions of Differential Polynomials Sharing a Small Function IM

Xinhua Shi

College of Science
Civil Aviation University of China
Tianjin 300300, China

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Abstract

In this paper, we shall study the uniqueness problems of meromorphic functions of differential polynomials sharing a small function and ∞ IM. Our results improve or generalize many previous results on value sharing of meromorphic functions.

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1 Introduction and main results

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let a be a small function with respect to f and g . We say that $f(z)$, $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$, $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $T(r, f)$ the Nevanlinna characteristic function of the meromorphic function f and by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure. $N_k(r, f)$ denotes the

truncated counting function bounded by k . Moreover, $GCD(n_1, n_2, \dots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \dots, n_k$.

For the sake of simplicity, let m be a nonnegative integer, $a_0 \neq 0$, $a_1, \dots, a_{m-1}, a_m \neq 0$ be complex constants. Define

$$P(w) = \begin{cases} a_m w^m + a_{m-1} w^{m-1} + \dots + a_1 w + a_0, & m > 0, \\ a_0, & m = 0. \end{cases} \quad (1.1)$$

In 2011, Zhang and Xu [12] considered uniqueness problem on meromorphic functions sharing a small function CM and proved the following result.

Theorem A *Let f and g be two non-constant meromorphic functions, $a(z) (\neq 0, \infty)$ be a small function with respect to f . Let n, k , and m be three positive integers with $n > 3k + m + 8$ and $P(w)$ be defined as in Theorem G. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a CM, then*

(I) *when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \dots + a_1 w + a_0$, one of the following three cases holds:*

(I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$,*

(I2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \dots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \dots + a_0)$,*

(I3) *$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} = a^2$;*

(II) *when $P(w) \equiv a_0$, one of the following two cases holds:*

(II1) *$f \equiv tg$ for a constant t such that $t^n = 1$,*

(II2) *$a_0^2 [f^n]^{(k)} [g^n]^{(k)} = a^2$.*

In 2014, Li, Qiu and Xuan [3] changed CM sharing to IM sharing and got

Theorem B *Let f and g be two transcendental meromorphic functions, $a(z) (\neq 0, \infty)$ be a small function with respect to f . Let n, k , and m be three positive integers with $n > 4m + 9k + 14$ and $P(w)$ be defined as in Theorem G. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a IM, then*

(I) *when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \dots + a_1 w + a_0$, one of the following three cases holds:*

(I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$,*

(I2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \dots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \dots + a_0)$,*

(I3) *$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} = a^2$;*

(II) *when $P(w) \equiv a_0$, one of the following two cases holds:*

(II1) *$f \equiv tg$ for a constant t such that $t^n = 1$,*

(II2) *$a_0^2 [f^n]^{(k)} [g^n]^{(k)} = a^2$.*

In this paper, we shall further improve Theorem B by reducing the lower

bound of n . We got

Theorem 1.1. *Let f and g be two non-constant meromorphic functions, $a(z) (\neq 0, \infty)$ be a small function with respect to f , which has no common zero or poles with f and g . Let n, k , and m be three positive integers with $n > 4m + 9k + 10$ and $P(w)$ be defined as in Theorem G. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share a IM, f and g share ∞ IM, then*

(I) *when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$, one of the following three cases holds:*

(I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \text{GCD}(n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$,*

(I2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_0)$;*

(II) *when $P(w) \equiv a_0$, one of the following two cases holds:*

(II1) *$f \equiv tg$ for a constant t such that $t^n = 1$,*

(II2) *$a_0^2 [f^n]^{(k)} [g^n]^{(k)} = a^2$.*

Theorem 1.2. *Let f and g be two transcendental meromorphic functions, $p(z)$ be a nonzero polynomial with $\deg(p) = l \leq 5$, which has no common zero with f and g . Let n, k and m be three positive integers with $n > 4m + 9k + 10$. Let $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ be a nonzero polynomial. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share p IM, f and g share ∞ IM. Then one of the following three cases holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \text{GCD}(n + m, \dots, n + m - i, \dots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$;*

(2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n (a_m \omega_1^m + a_{m-1} \omega_1^{m-1} + \cdots + a_0) - \omega_2^n (a_m \omega_2^m + a_{m-1} \omega_2^{m-1} + \cdots + a_0)$;*

(3) *$P(z)$ is reduced to a nonzero monomial, namely, $P(z) = a_i z^i \neq 0$ for some $i \in \{0, 1, \dots, m\}$; if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z) dz$, c_1, c_2 and c are constants such that $a_i^2 (c_1 c_2)^{n+i} [(n+i)c]^2 = -1$,*

if $p(z)$ is a nonzero constant b , then $f = c_3 e^{cz}$, $g = c_4 e^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^k a_i^2 (c_3 c_4)^{n+i} [(n+i)c]^{2k} = b^2$.

2 Preliminary lemmas

Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right), \quad (2.1)$$

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F} \right) - \left(\frac{G'}{G-1} - \frac{G'}{G} \right), \quad (2.2)$$

where $F = [f^n P(f)]^{(k)}/a(z)$ and $G = [g^n P(g)]^{(k)}/a(z)$, both f and g are meromorphic functions, $a(z) (\not\equiv 0, \infty)$ be a small function with respect to both f and g , which has no common zero or poles with f and g .

Lemma 2.1. [6] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \dots, a_n(z) (\not\equiv 0)$ be small functions with respect to f . Then

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [4] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then

$$N_s(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f),$$

$$N_s(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.3. [2, 8, 9] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

By using the similar method to Banerjee [1, Lemma 2.14], we can prove the following Lemma.

Lemma 2.4. Let F, G and H be defined as in (2.1). If F and G share 1 IM and ∞ IM, and $H \not\equiv 0$, then $F \not\equiv G$, and

$$\begin{aligned} T(r, F) &\leq N_2(r, 1/F) + N_2(r, 1/G) + 2\overline{N}(r, 1/F) + \overline{N}(r, 1/G) \\ &+ 6\overline{N}(r, F) + S(r, F) + S(r, G), \end{aligned}$$

the same inequality holding for $T(r, G)$.

Lemma 2.5. [10] Let F, G and V be defined as in (2.2). If F and G share ∞ IM, and $V \equiv 0$, then $F \equiv G$.

Lemma 2.6. [11] If F and G share 1 IM, then

$$\overline{N}_L(r, \frac{1}{F-1}) \leq \overline{N}(r, 1/F) + \overline{N}(r, F) + S(r, F) + S(r, G),$$

Lemma 2.7. Let f, g be two non-constant meromorphic functions, F, G and V be defined as in (2.2), $P(w)$ is defined as in (1.1), $n > 0, k > 0$ and $m \geq 0$ are three integers. If $V \not\equiv 0$, F and G share 1 and ∞ IM, then

$$\begin{aligned} (n+m+k-3)\overline{N}(r, f) &= (n+m+k-3)\overline{N}(r, g) \\ &\leq 2\overline{N}(r, 1/F) + 2\overline{N}(r, 1/G) + S(r, f) + S(r, g). \end{aligned} \quad (2.3)$$

Proof. Since $V \not\equiv 0$, f and g share ∞ IM, suppose that z_0 is a pole of f with multiplicity p , a pole of g with multiplicity q , then z_0 is a pole of F with multiplicity $(n+m)p+k$, a pole of G with multiplicity $(n+m)q+k$, thus z_0 is a zero of $\frac{F'}{F-1} - \frac{F'}{F} = \frac{F'}{F(F-1)}$ with multiplicity $(n+m)p+k-1 (\geq n+m+k-1)$, and z_0 is a zero of $\frac{G'}{G-1} - \frac{G'}{G} = \frac{G'}{G(G-1)}$ with multiplicity $(n+m)q+k-1 (\geq n+m+k-1)$, hence z_0 is a zero of V with multiplicity at least $n+m+k-1$. So

$$(n+m+k-1)\overline{N}(r, f) = (n+m+k-1)\overline{N}(r, g) \leq N(r, \frac{1}{V}). \quad (2.4)$$

By the logarithmic derivative lemma, we have $m(r, V) = S(r, f) + S(r, g)$. Note that F and G share 1 IM, by Lemma 2.6, so we have

$$\begin{aligned} N(r, \frac{1}{V}) &\leq T(r, V) = m(r, V) + N(r, V) \\ &\leq \overline{N}(r, 1/F) + \overline{N}(r, 1/G) + \overline{N}_L(r, \frac{1}{F-1}) \\ &\quad + \overline{N}_L(r, \frac{1}{G-1}) + S(r, f) + S(r, g) \\ &\leq 2\overline{N}(r, 1/F) + 2\overline{N}(r, 1/G) + 2\overline{N}(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (2.5)$$

From (2.4) and (2.5) we get (2.3). This proves Lemma 2.6.

By the same reason as in Lemma 5 of [5], we obtain the following lemma.

Lemma 2.8. *Let f and g be two non-constant meromorphic functions. Let $P(w)$ be defined as in (1.1), and $n > 0, k > 0$, and $m \geq 0$ be three integers with $n > 2k + m + 1$. If $[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}$, then $f^n P(f) = g^n P(g)$.*

By using the similar method to Lemma 2.9 of [12], we have

Lemma 2.9. *Let f and g be non-constant meromorphic functions, n, k be two positive integers with $n > k + 2$, and let $P(w)$ be defined as in (1.1), $a(z) (\not\equiv 0, \infty)$ be a small function with respect to f , which has no common zero or poles with f and g . If $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} = a^2$, f and g share ∞ IM, then $P(w)$ is reduced to a nonzero monomial.*

3 Proof of Theorem 1.1

Let F, G, H and V be defined in (2.1) and (2.2). $F_1 = f^n P(f)$, $G_1 = g^n P(g)$, then F and G share 1 and ∞ IM.

Suppose that $H \not\equiv 0$, then $F \not\equiv G$, and $V \not\equiv 0$.

Case 1. $m > 0$. By Lemma 2.4 we have

$$\begin{aligned} T(r, F) &\leq N_2(r, 1/F) + N_2(r, 1/G) + 2\overline{N}(r, 1/F) + \overline{N}(r, 1/G) \\ &\quad + 6\overline{N}(r, F) + S(r, F) + S(r, G), \end{aligned} \quad (3.1)$$

By Lemma 2.2 with $s = 2$, we obtain

$$T(r, F_1) \leq T(r, F) - N_2(r, 1/F) + N_{k+2}(r, 1/F_1) + S(r, F), \quad (3.2)$$

and

$$N_2(r, 1/G) \leq N_{k+2}(r, 1/G_1) + k\overline{N}(r, G) + S(r, G). \quad (3.3)$$

Combining (3.1)–(3.3) gives

$$\begin{aligned} T(r, F_1) &\leq N_{k+2}(r, 1/F_1) + N_{k+2}(r, 1/G_1) + (4k+6)\overline{N}(r, f) \\ &\quad + 2N_{k+1}(r, 1/F_1) + N_{k+1}(r, 1/G_1) + S(r, f) + S(r, g) \\ &\leq (3k+4)\overline{N}(r, 1/f) + 3N(r, 1/P(f)) + (2k+3)\overline{N}(r, 1/g) \\ &\quad + 2N(r, 1/P(g)) + (4k+6)\overline{N}(r, f) + S(r, f) + S(r, g). \end{aligned}$$

It follows from Lemma 2.1 and the above inequality that

$$\begin{aligned} (n+m)T(r, f) &\leq (3k+4)\overline{N}(r, 1/f) + 3N(r, 1/P(f)) + (2k+3)\overline{N}(r, 1/g) \\ &\quad + 2N(r, 1/P(g)) + (4k+6)\overline{N}(r, f) + S(r, f) + S(r, g). \end{aligned} \quad (3.4)$$

Similarly we have

$$\begin{aligned} (n+m)T(r, g) &\leq (3k+4)\overline{N}(r, 1/g) + 3N(r, 1/P(g)) + (2k+3)\overline{N}(r, 1/f) \\ &\quad + 2N(r, 1/P(f)) + (4k+6)\overline{N}(r, f) + S(r, f) + S(r, g). \end{aligned} \quad (3.5)$$

From (3.4) and (3.5) we deduce that

$$\begin{aligned} (n-5k-4m-7)(T(r, f) + T(r, g)) &\leq (4k+6)\overline{N}(r, f) \\ &\quad + S(r, f) + S(r, g). \end{aligned} \quad (3.6)$$

Note that $V \neq 0$ and we get (2.3). By Lemma 2.2 with $s = 1$, we obtain

$$\begin{aligned} \overline{N}(r, 1/F) &\leq N_{k+1}(r, 1/F_1) + k\overline{N}(r, f) + S(r, f) \\ &\leq (k+1)\overline{N}(r, 1/f) + N(r, 1/P(f)) \\ &\quad + k\overline{N}(r, f) + S(r, f). \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \overline{N}(r, 1/G) &\leq N_{k+1}(r, 1/G_1) + k\overline{N}(r, g) + S(r, g) \\ &\leq (k+1)\overline{N}(r, 1/g) + N(r, 1/P(g)) \\ &\quad + k\overline{N}(r, g) + S(r, g). \end{aligned} \quad (3.8)$$

From (2.3), (3.7) and (3.8) we get

$$(n + m - 3k - 3)\overline{N}(r, f) \leq 2(k + m + 1)(T(r, f) + T(r, g)) + S(r, f) + S(r, g). \quad (3.9)$$

Combining (3.6)–(3.9) gives

$$[(n - 5k - 4m - 7)(n + m - 3k - 3) - 4(4k + 6)(k + m + 1)](T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \quad (3.10)$$

which is a contradiction since $n > 4m + 9k + 10$.

Thus $H \equiv 0$. Similar to the proof of [7, Lemma 3], we obtain

- (i) $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = a^2$, or
- (ii) $[f^n P(f)]^{(k)} \equiv [g^n P(g)]^{(k)}$.

By Lemma 2.9, the case of (i) is impossible. By Lemma 2.8, we get $f^n P(f) \equiv g^n P(g)$ from (ii).

Case 2. $m = 0$. Similar to the proof of Case 1, we get

$$[(n - 5k - 7)(n - 3k - 3) - 4(4k + 6)(k + 1)](T(r, f) + T(r, g)) \leq S(r, f) + S(r, g), \quad (3.11)$$

which is a contradiction since $n > 9k + 10$.

Thus $H \equiv 0$. and we have

- (iii) $a_0^2[f^n]^{(k)}[g^n]^{(k)} = a^2$, or
- (iv) $[f^n]^{(k)} \equiv [g^n]^{(k)}$.

For (iv), By Lemma 2.8, we get $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

By the similar method to the proof of Theorem 1.1, we only need to consider

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} = p^2. \quad (4.1)$$

Similar to the proof of Theorem 1.3 of [5], we obtain the results.

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