Extensions of Some Inequalities for the Gamma Function

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Abstract

In this paper, we improve the results of Shabani [7] concerning some inequalities for the Gamma function. Our approach makes use of the logarithmic derivative of products of the Gamma function. We also present some $p$-analogues.

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1 Introduction

We begin by recalling some definitions related to the Gamma function.

The classical Euler’s Gamma function, $\Gamma(t)$ is defined as

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1} \, dx, \quad t > 0. \quad (1)$$
The logarithmic derivative of the Gamma function is defined as
\[ \phi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0. \] (2)

The \( p \)-analogue of the Gamma Function, \( \Gamma_p(t) \), is defined as
\[ \Gamma_p(t) = \frac{p! p^t}{t(t+1) \ldots (t+p)} = \frac{p^t}{t(1+\frac{1}{t}) \ldots (1+\frac{p}{t})}, \quad p \in \mathbb{R}, \quad t > 0. \] (3)

(see also [3] and [4]).

The equivalent definition of \( \phi(t) \) in terms of the \( p \)-analogue is given as follows.
\[ \phi_p(t) = \frac{d}{dt} \ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad p \in \mathbb{R}, \quad t > 0. \] (4)

and
\[ \lim_{p \to \infty} \Gamma_p(t) = \Gamma(t), \quad \lim_{p \to \infty} \phi_p(t) = \phi(t) \] (5)

Our aim in this paper is to establish and prove an extension of the generalized result of A. S. Shabani:
\[ \frac{\Gamma(a + b)^c}{\Gamma(a + \beta)^f} \leq \frac{\Gamma(a + bt)^c}{\Gamma(a + \beta t)^f} \leq \frac{\Gamma(a)^c}{\Gamma(\alpha)^f}, \quad t \in [0, 1] \] (6)
where \( a, b, c, \alpha, \beta, f \) are positive real numbers such that \( a + bt > 0, \alpha + \beta t > 0, \)
\( a + bt \leq \alpha + \beta t, \quad 0 < bc \leq \beta f \) and \( \phi(a + bt) > 0 \) or \( \phi(\alpha + \beta t) > 0 \).
The result (6) is a generalisation of some earlier results by Alsina and Tomas [1], Bougoffa [2], Sandor [5] and Shabani [6].

2 Preliminaries

We present the following auxiliary results.

**Lemma 2.1.** Let \( t > 0 \). Then \( \phi(t) \) has the following series representation
\[ \phi(t) = -\gamma + (t - 1) \sum_{k=0}^{\infty} \frac{1}{(1+k)(t+k)} \] (7)
where \( \gamma \) is the Euler-Mascheroni’s constant.
Proof. See [8].

**Lemma 2.2.** Let \( s > 0, \ t > 0 \) with \( s \leq t \), then
\[
\phi(s) \leq \phi(t). \tag{8}
\]

**Proof.** From (7), we have the following,
\[
\phi(s) - \phi(t) = (s - 1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+s)} - (t - 1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+t)}
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{(k+1)} \left( \frac{s-1}{k+s} - \frac{t-1}{k+t} \right)
\]
\[
= \sum_{k=0}^{\infty} \frac{(s-t)}{(k+s)(k+t)} \leq 0
\]
and the proof is complete.

**Lemma 2.3.** Let \( a, b, \alpha, \beta \) be real numbers such that \( a + bt > 0, \alpha + \beta t > 0 \). Then \( a + bt \leq \alpha + \beta t \) implies \( \phi(a + bt) \leq \phi(\alpha + \beta t) \).

**Proof.** The proof follows directly from Lemma 2.2. (See also[6] and the references therein.)

We also have the following lemma from the paper [7].

**Lemma 2.4.** Let \( a, b, \alpha, \beta, r, q \), be real numbers such that \( a + bt > 0, \alpha + \beta t > 0 \) and \( q \beta \geq r b \).

If (i) \( \phi(a + bt) > 0 \) or
(ii) \( \phi(\alpha + \beta t) > 0 \),
then \( r b \phi(a + bt) - q \beta \phi(\alpha + \beta t) \leq 0 \).

**Proof.** (i) If \( \phi(a + bt) > 0 \), then by Lemma 2.3, we have \( \phi(a + bt) \leq \phi(\alpha + \beta t) \).
Multiplying both sides of \( q \beta \geq r b \) by \( \phi(\alpha + \beta t) \) yields;
\[
q \beta \phi(\alpha + \beta t) \geq r b \phi(\alpha + \beta t) \geq r b \phi(a + bt)
\]
which implies;
\[
r b \phi(a + bt) - q \beta \phi(\alpha + \beta t) \leq 0.
\]

(ii) From Lemma 2.3, we have \( \phi(a + bt) \leq \phi(\alpha + \beta t) \).
If \( \phi(\alpha + \beta t) > 0 \), then there are two possible values of \( \phi(a + bt) \). That is,
Case 1: \( \phi(a + bt) \leq 0 \) or
Case 2: \( \phi(a + bt) > 0 \).
For Case 1, we have \( r b \phi(a + bt) \leq 0 \) and \( q \beta \phi(\alpha + \beta t) > 0 \).
Hence \( r b \phi(a + bt) - q \beta \phi(\alpha + \beta t) \leq 0 \).
Case 2 is shown in (i).
Lemma 2.5. The function $\phi_p(t)$ as defined in (4) has the following series representation.

$$\phi_p(t) = \ln(p) - \sum_{k=0}^{p} \frac{1}{t+k}, \quad p \in \mathbb{R}, \quad t > 0. \quad (9)$$

Proof. From inequality (3), we have

$$\ln \Gamma_p(t) = t \ln p - \left( \ln t + \ln(1+t) + \ln(1+\frac{t}{2}) + \cdots + \ln(1+\frac{t}{p}) \right)$$

Thus

$$\phi_p(t) = \frac{d}{dt} \ln(\Gamma(t)) = \ln p - \left( \frac{1}{t} + \frac{1}{1+t} + \cdots + \frac{1}{p+t} \right)$$

$$= \ln p - \sum_{k=0}^{p} \frac{1}{k+t}.$$ 

See also [4].

Lemma 2.6. Let $s > 0$, $t > 0$ with $s \leq t$, then

$$\phi_p(s) \leq \phi_p(t). \quad (10)$$

Proof. From (9), we have the following.

$$\phi_p(s) - \phi_p(t) = \left( \ln(p) - \sum_{k=0}^{p} \frac{1}{s+k} \right) - \left( \ln(p) - \sum_{k=0}^{p} \frac{1}{t+k} \right)$$

$$= \sum_{k=0}^{p} \left( \frac{1}{t+k} - \frac{1}{s+k} \right)$$

$$= \sum_{k=0}^{p} \frac{(s-t)}{(t+k)(s+k)} \leq 0$$

and that ends the proof.

The following Lemmas (See [4]) are the $p$-analogues of Lemmas 2.3 and 2.4 with similar proofs.

Lemma 2.7. Let $a, b, \alpha, \beta$ be real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$. Then $a + bt \leq \alpha + \beta t$ implies $\phi_p(a + bt) \leq \phi_p(\alpha + \beta t)$.

Lemma 2.8. Let $a, b, \alpha, \beta, r, q$, be real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$, $a + bt \leq \alpha + \beta t$ and $q\beta \geq rb$.

If (i) $\phi_p(a + bt) > 0$ or

(ii) $\phi_p(\alpha + \beta t) > 0$, then

$$rb\phi_p(a + bt) - q\beta\phi_p(\alpha + \beta t) \leq 0.$$
3 Main Results

We state and prove the results of this paper here.

**Theorem 3.1.** Define a function \( \Lambda \) by

\[
\Lambda(t) = \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}}, \quad t \in [0, \infty)
\]

(11)

where \( a_i, b_i, \alpha_i, \beta_i, r_i, q_i, \) \( i = 1, 2, ..., n \) are real numbers such that \( a_i + b_i t > 0, \alpha_i + \beta_i t > 0, a_i + b_i t \leq \alpha_i + \beta_i t \) and \( q_i \beta_i \geq r_i b_i \). If \( \phi(a_i + b_i t) > 0 \) or \( \phi(\alpha_i + \beta_i t) > 0 \) then \( \Lambda \) is decreasing and for every \( t \in [0, 1] \), the following inequality holds.

\[
\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i)^{q_i}}
\]

(12)

**Proof.** Let \( g(t) = \ln \Lambda(t) \) for every \( t \in [0, \infty) \). Then,

\[
g(t) = \ln \left( \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \right)
\]

\[
= \ln \left( \prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i} \right) - \ln \left( \prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i} \right)
\]

Then,

\[
g'(t) = \sum_{i=1}^{n} r_i b_i \frac{\Gamma'(a_i + b_i t)}{\Gamma(a_i + b_i t)} - \sum_{i=1}^{n} q_i \beta_i \frac{\Gamma'(\alpha_i + \beta_i t)}{\Gamma(\alpha_i + \beta_i t)}
\]

\[
= \sum_{i=1}^{n} r_i b_i \phi(a_i + b_i t) - \sum_{i=1}^{n} q_i \beta_i \phi(\alpha_i + \beta_i t)
\]

\[
= \sum_{i=1}^{n} \left[ r_i b_i \phi(a_i + b_i t) - q_i \beta_i \phi(\alpha_i + \beta_i t) \right] \leq 0. \quad \text{(by Lemma 2.4)}
\]

That implies \( g \) is decreasing on \( t \in [0, \infty) \). Hence, \( \Lambda \) is decreasing for every \( t \in [0, \infty) \). Then for every \( t \in [0, 1] \) we have,

\[
\Lambda(1) \leq \Lambda(t) \leq \Lambda(0) \quad \text{yielding},
\]

\[
\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i)^{q_i}}.
\]
Corollary 3.2. If \( t \in (1, \infty) \), then the following inequality holds.

\[
\frac{\prod_{i=1}^{n} \Gamma(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma(\alpha_i + \beta_i)^{q_i}}
\]

Proof. If \( t \in (1, \infty) \), then we have \( \Lambda(t) \leq \Lambda(1) \) yielding the result.

In the following, we present the \( p \)-analogues of Theorem 3.1 and Corollary 3.2.

Theorem 3.3. Define a function \( \Omega \) by

\[
\Omega(t) = \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}}, \quad t \in [0, \infty), \quad p \in \mathbb{N}
\]

where \( a_i, b_i, \alpha_i, \beta_i, r_i, q_i, \) \( i = 1, 2, \ldots, n \) are real numbers such that \( a_i + b_i t > 0, \alpha_i + \beta_i t > 0, a_i + b_i t \leq \alpha_i + \beta_i t \) and \( q_i \beta_i \geq r_i b_i \). If \( \phi_p(a_i + b_i t) > 0 \) or \( \phi_p(\alpha_i + \beta_i t) > 0 \) then \( \Omega \) is decreasing and for every \( t \in [0, 1] \), the following inequality holds.

\[
\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i)^{q_i}}
\]

Proof. Let \( h(t) = \ln \Omega(t) \) for every \( t \in [0, \infty) \). Then by a similar argument as in in the proof of Theorem 3.1 we arrive at,

\[
h'(t) = \sum_{i=1}^{n} \left[ r_i b_i \phi_p(a_i + b_i t) - q_i \beta_i \phi_p(\alpha_i + \beta_i t) \right] \leq 0. \quad \text{(by Lemma 2.8)}
\]

That implies \( h \) is decreasing on \( t \in [0, \infty) \). Hence, \( \Omega \) is decreasing for every \( t \in [0, \infty) \). Then for every \( t \in [0, 1] \) we have,

\[
\Omega(1) \leq \Omega(t) \leq \Omega(0) \quad \text{yielding},
\]

\[
\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i)^{q_i}}
\]

Corollary 3.4. If \( t \in (1, \infty) \), then the following inequality holds.

\[
\frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i t)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i t)^{q_i}} \leq \frac{\prod_{i=1}^{n} \Gamma_p(a_i + b_i)^{r_i}}{\prod_{i=1}^{n} \Gamma_p(\alpha_i + \beta_i)^{q_i}}, \quad p \in \mathbb{N}
\]

Proof. If \( t \in (1, \infty) \), then we have \( \Omega(t) \leq \Omega(1) \) giving the result.
4 Concluding Remarks

We dedicate this section to some remarks concerning inequalities (12) and (14).

Remark 4.1. In inequality (12), put $i = 1$, $a_1 = b_1 = \alpha_1 = q_1 = 1$ and $\beta_1 = r_1 = n$, then we obtain
\[
\frac{1}{n!} \leq \frac{\Gamma(1 + t)^n}{\Gamma(1 + nt)} \leq 1, \quad t \in [0, 1], \quad n \in \mathbb{R}
\]
as in [1].

Remark 4.2. In inequality (12), put $i = 1$, $a_1 = \alpha_1 = q_1 = 1$ and $\beta_1 = r_1 = a$, then we obtain
\[
\frac{1}{\Gamma(1 + a)} \leq \frac{\Gamma(1 + t)^a}{\Gamma(1 + at)} \leq 1, \quad t \in [0, 1], \quad a \geq 1
\]
as in [5].

Remark 4.3. In inequality (12), put $i = 1$, $a_1 = a$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $q_1 = q$ and $r_1 = r$, then
\[
\frac{\Gamma(a)^r}{\Gamma(a)^q} \leq \frac{\Gamma(a + bt)^r}{\Gamma(a + \beta t)^q} \leq \frac{\Gamma(a + b)^r}{\Gamma(a + \beta)^q}, \quad t \in [0, 1]
\]
where $a \geq b > 0$, $r, q$ are positive real numbers such that $rb \geq q\beta > 0$ and $\phi(a + \beta t) > 0$.

Remark 4.4. Using (5) together with Theorem 3.3 and Corollary 3.4, the entire results of Theorem 3.1 and Corollary 3.2 are respectively recovered.

References


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