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Miyachi's Theorems Associated with a Differential-Difference Operator on the Real Line

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Abstract

We consider a singular differential operator $\triangle$ on the real line .we establish Miyachi's theorems for the generalized Fourier transform on $\mathbb{R}$ tied the Differential-difference operator .

Keywords: Differential-difference operator, generalized Fourier transform, Miyachi's theorems

1- Introduction

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized . That is, it is impossible for a non-zero function and its Fourier transform to be simultaneously small. Here a concept of the smallness had taken different interpretations in different contexts. Hardy[6], Cowling and Price[2],and Miyachi[9],for example, interpreted the smallness as sharp pointwise estimates or integrable decay of functions. Hardy's theorem[6] for the usual Fourier transform $\mathcal{F}$ on $\mathbb{R}$ asserts that f and its Fourier $\widehat{f} = \mathcal{F}(f)$ can not both be very small. More precisely, let a and b be a positive constants and assume that f is a measurable function on $\mathbb{R}$ such that $|f(x)| \leq Ce^{-ax^2}$ a.e . and $\left|\widehat{f}(y)\right| \leq Ce^{-by^2}$for some positive constant C .Then $f = 0$ a.e. if $ab > \frac{1}{4}$, f is a constant multiple of e^{-ax^2} if $ab = \frac{1}{4}$, and there are infinitely many nonzero functions satisfying the assumptions if $ab < \frac{1}{4}$ .Considerable attention has been devoted to discovering generalizations to new contexts for Hardy's theorem. In particular,Cowling and Price[2] have studied an L^p version of Hardy's theorem which states that for $p, q \in [1, \infty]$,at least one of them is finite, if $\left\| e^{ax^2} f \right\|_p < \infty$ and $\left\| e^{by^2} \widehat{f} \right\|_q < \infty$,then $f = 0$ a.e. If $ab \geq \frac{1}{4}$. Another generalization of Hardy's theorem is given by Miyachi[9] where it is proved that , if f is a measurable function on $\mathbb{R}$ such that

$$e^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \log^+ \frac{\left| \widehat{f}(\xi) e^{\frac{1}{4a}\xi^2} \right|}{\lambda} d\xi < \infty$$

for some positive constants a and λ , then f is a constant multiple of e^{-ax^2}. As a generalisation of these Euclidean uncertainty principles for $\mathcal{F}$,recently Mourou[10] have proved Hardy's theorem and Cowling Price's theorem for the Differential-Difference transform F,many authors have established the analogous of Cowling-Price's and Hardy's theorems in other various settings of harmonic analysis

(see for instance $[1, 4, 5, 8, 12, 10]$).The purpose of this paper is a generalisation of Miyachi's theorem for F

The structure of this paper is as follows. In Section 2,we deal with harmonic analysis associated with the Differential-Difference operator.

The third section, is devoted to Miyachi's theorem for the Differential-Difference transform. We establish that for all $p, q \in [1, +\infty]$ and f is a mesurale function on $\mathbb{R}$ such that :

$$e^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \quad \text{and} \quad \int \log^+ \frac{\left| F(f)(y) e^{by^2} \right|}{\lambda} dy < \infty$$

for some constants $a, b, \lambda > 0$, If $ab > \frac{1}{4}$ then $f = 0$ a.e ; If $ab = \frac{1}{4}$ then $f = CE_a$ with $|C| \leq \lambda$ and if $ab < \frac{1}{4}$ the function having form $f = CE_t$ then $F(f)(y) = Ce^{-\frac{1}{4t}y^2}$ and f and $F(f)$ satisfying these conditions for all $t \in \left] a, \frac{1}{4b} \right[$.

Throughout this paper , the letter C indicates a positive constant that is not necessarily the same in each occurrence.

2- Differential-Difference transform

In this paper; we consider the first-order singular differential - difference operator on $\mathbb{R}$

$$\triangle f = \frac{df}{dx} + \frac{A'(x)}{A(x)} \left(\frac{f(x) - f(-x)}{2} \right)$$

where: $A(x) = |x|^{2\alpha+1} B(x)$, $\alpha > -\frac{1}{2}$

B being a positive C^{∞} even function on $\mathbb{R}$ .We suppose in addition that :

(i) A is increasing on $[0, \infty[$;

(ii) There exists a costant $\delta > 0$. such that the function $e^{\delta x} \frac{B'(x)}{B(x)}$ is bounded for large $x \in [0, \infty[$ together with its derivatives

The generalized Fourier transform related to $\triangle$ is defined for a suitable function f on $\mathbb{R}$.by $\mathcal{F} f(\lambda) = \int_{\mathbb{R}} f(x) \Phi_{-i\lambda}(x) A(x) dx$.Where $\Phi_{-i\lambda}(x)$ is the solution of the differential -difference equation $\triangle u = -i\lambda u$, $u(0) = 1$

The intention of this paper is to establish an analogue of Miyachi's theorems for the generalized Fourier transform $\mathcal{F}$.

Preliminaries:

In this section we recall some facts about harmonic analysis related to the differential-difference operator $\triangle$

We cite here ,as briefly as possible ,only those properties actually required for the discussion .For more details we refer to[7]

If $\lambda \in \mathbb{C}$, it is know that the differential-difference equation $\triangle u = \lambda u$, $u(0) = 1$, admits a unique C^{∞} solution on $\mathbb{R}$, denoted Φ_{λ} given by:

$$\Phi_{\lambda}(x) = \begin{cases} \varphi_{i\lambda}(x) + \frac{1}{\lambda} \frac{d}{dx} \varphi_{i\lambda}(x) & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0 \end{cases}$$

$\Phi_{\lambda}(x)$ is entire in λ.

Notation:

For a Borel positive measure μ on $\mathbb{R}$, and $1 \leq p \leq \infty$, we write $L^p(\mathbb{R}, \mu)$ for the Lebesgue Space equiped with the norm $\|\|_{p,\mu}$ defined by

$\|f\|_{p,\mu} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$, if $p < \infty$, and $\|f\|_{\infty,\mu} = ess\sup_{x \in \mathbb{R}} |f(x)|$

When $u = w(x) dx$, with w a nonnegative function on $\mathbb{R}$, we replace the μ in the norms by w.

The generalized Fourier transform of a function $f \in L^1(\mathbb{R}, A(x)\,dx)$ is defined by:

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x) \times \Phi_{-i\lambda}(x)\, A(x)\, dx \quad , \quad \lambda \in \mathbb{R}$$

a Plancherel type for the transform $\mathcal{F}$ is as follows.

Theorem 1 *1. There is an even positive tempered measure σ (and only one) on $\mathbb{R}$ .such that for all .$f \in L^1 \cap L^2(\mathbb{R}, A(x)\,dx)$*

$$\int_{\mathbb{R}} |f(x)|^2 A(x)\, dx = \int_{\mathbb{R}} |\mathcal{F}f(\lambda)|^2\, d\sigma(\lambda)$$

2. The generalized Fourier transform $\mathcal{F}$ extends uniquely to a unitary isomorphism from :$L^2(\mathbb{R}, A(x)\,dx)$ onto $L^2(\mathbb{R}, \sigma)$ .The inverse transform is given by :

$$\mathcal{F}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda) \times \Phi_{i\lambda}(x)\, d\sigma(\lambda)$$

where the integral converges in $L^2(\mathbb{R}, A(x)\,dx)$.

The measure σ is called . the spectral measure associated with the differential-difference operator $\triangle$.under our assumptions on the fuction A ,it is known (see[7, 11]) that the spectral measure σ takes the form

$$d\sigma(\lambda) = \frac{d\lambda}{|c(|\lambda|)|^2} \quad , \quad \lambda \in \mathbb{R}$$

where $c(s)$ is a cotinuous function on $]0, \infty[$ such that :

$$c(s)^{-1} \backsim k_1 s^{\alpha+\frac{1}{2}} \quad \text{as} \quad s \longrightarrow \infty$$

$$c(s)^{-1} \backsim k_2 s^{\alpha+\frac{1}{2}} \quad \text{as} \quad s \longrightarrow 0$$

for some $k_1, k_2 \in \mathbb{C}$.

The generalized Gaussian kernel E_a, $a > 0$,is defined by:

$$E_a(x) = \int_{\mathbb{R}} e^{-\frac{\lambda^2}{4a}} \varphi_{i\lambda}(x)\, d\sigma(\lambda), \quad x \in \mathbb{R}.$$

From[3] we know that E_a is a positive even function on $\mathbb{R}$,and belongs to the Schwartz space . Moreover,there are two positive constants c_1 and c_2 depending on a such that:

$$c_1 e_{-a} \leq E_a \leq c_2 e_{-a} \qquad (2.1)$$

where $e_{-a}(x) = e^{-ax^2}$, $x \in \mathbb{R}$

3-Miyachi's theorems

The proofs of this paper depends on the folowing lemmas

Lemma 1 [10]*Let* $q \in [1, \infty]$,$a > 0$ *and* $\lambda = \xi + i\eta$ *with* $\xi, \eta \in \mathbb{R}$. *Then there exists a positive constant* C *such that*

$$\|e_{-a}\Phi_{-i\lambda}\|_{q,A} \leq C\left(1 + |\eta|\right)^{\frac{2\alpha+2}{q}} e^{\frac{\eta^2}{4a}}$$

$e_{-a}(x) = e^{-ax^2}$

Lemma 2 *If* $f(z)$ *is an entiere function of* $z \in \mathbb{C}$ *,if there exist constants* $A, B > 0$ *and a positive integer* m *such that*

$$|f(z)| \leq A(1 + |z|)^m \exp\left(B(\Re ez)^2\right)$$

and if

1.

$$\int_{-\infty}^{+\infty} \log^+ \frac{|f(t)|}{(1+t)^m} dt < \infty$$

then $f(z)$ *is a polynomial in* z *of degree at most* m

Corollary 1 *If in Lemma2 ,the assumption* (1) *is replaced with*

1.

$$\int_{-\infty}^{+\infty} \log^+ |f(t)|\, dt < \infty$$

then $f(z)$ *is a constant.*

Lemma 3 *Let* p, q *in* $[1, \infty]$ *and* f *a measurable function on* $\mathbb{R}$ *such that:*

$$e^{ax^2} f \in L^1(\mathbb{R}, A) + L^\infty(\mathbb{R}, A)$$

for some $a > 0$.*then for all* $z \in \mathbb{C}$,*the integral*

$$F(f)(\lambda) = \int_{\mathbb{R}} f(x)\, \Phi_{-i\lambda}(x)\, A(x)\, dx$$

is well defined .$F(f)(z)$ *is entire and there exists* $C > 0$*such that for all* ζ, η *in* $\mathbb{R}$

$$|F(f)(\zeta + i\eta)| \leq C\left(1 + |\eta|\right)^{2\alpha+2} e^{\frac{\eta^2}{4a}}$$

Proof:

From analyticity theorem under the integral sign, we deduce that the function defined on $\mathbb{C}$ by [13] is well defined and entire on $\mathbb{C}$

$$\begin{aligned}
|Ff(\lambda)| &= \left|\int_{\mathbb{R}} f(x)\,\Phi_{-i\lambda}(x)\,A(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}} e^{ax^2} f(x) \times e^{-ax^2}\Phi_{-i\lambda}(x)\,A(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}} (f_1(x)+f_2(x)) \times e^{-ax^2}\Phi_{-i\lambda}(x)\,A(x)\,dx\right| \quad \left(e^a f = f_1+f_2 \in L^1(\mathbb{R},A)+L^\infty(\mathbb{R},A)\right) \\
&= \left|\int_{\mathbb{R}} f_1(x) \times e^{-ax^2}\Phi_{-i\lambda}(x)\,A(x)\,dx + \int_{\mathbb{R}} f_2(x) \times e^{-ax^2}\Phi_{-i\lambda}(x)\,A(x)\,dx\right| \\
&\leq \|f_1\|_{1,A} \times \left\|e^{-a}\Phi_{-i\lambda}\right\|_{\infty,A} + \|f_2\|_{\infty,A} \times \left\|e^{-a}\Phi_{-i\lambda}\right\|_{1,A} \quad \text{by Holder's inequality} \\
&\leq (C_1+C_2) \times C\,(1+|Im\lambda|)^{2\alpha+2}\,e^{\frac{(Im\lambda)^2}{4a}} \quad \text{by Lemma 1} \\
&\leq C'\,(1+|Im\lambda|)^{2\alpha+2}\,e^{\frac{(Im\lambda)^2}{4a}}
\end{aligned}$$

Theorem 2 *f is a mesurable function on $\mathbb{R}$ such that:*

1.

$$e^{ax^2} f \in L^1(\mathbb{R},A) + L^\infty(\mathbb{R},A) \qquad (3.1)$$

2.

$$\int_{-\infty}^{+\infty} \log^+ \left|\frac{Ff(\lambda)\,e^{b\lambda^2}}{C}\right| d\lambda < \infty \qquad (3.2)$$

for some constants $a, b, \lambda > 0$ and $1 \leq p, q \leq +\infty$

(i) If $ab > \frac{1}{4}$ then $f = 0$ almost everywhere

(ii) If $ab = \frac{1}{4}$ then $f = CE_a$ with $|C| \leq \lambda$

(iii) If $ab < \frac{1}{4}$ then the function having forme $f = CE_t$ then $F(f) = Ce^{-\frac{1}{4t}y^2}$ and f and $F(f)$ satisfy (3.1) and (3.2) for all $t \in \left]a, \frac{1}{4b}\right[$

Proof:

Let

$$h(\lambda) = Ff(\lambda) \times e^{\frac{\lambda^2}{4a}}$$

$$\begin{aligned} |h(\lambda)| &= |Ff(\lambda)| \times \left| e^{\frac{\lambda^2}{4a}} \right| \quad \text{with } \lambda = \xi + i\eta \\ &= |Ff(\lambda)| \times \left| e^{\frac{\xi^2}{4a}} \right| \times \left| e^{-\frac{\eta^2}{4a}} \right| \\ &\leq Ce^{\frac{\eta^2}{4a}} \times (1+|\eta|)^{2\alpha+2} \times e^{\frac{\xi^2}{4a}} \times e^{-\frac{\eta^2}{4a}} \\ &\leq C \times (1+\eta)^{2\alpha+2} \times e^{\frac{\xi^2}{4a}} \end{aligned}$$

We will divide the proof into three cases.
(i) $ab > \frac{1}{4}$ (i) $ab > \frac{1}{4}$.

$$|h(\lambda)| \leq C(1+|\lambda|)^{2\alpha+2} \times e^{\frac{(Re\lambda)^2}{4a}} \qquad (3.3)$$

we note that

$$\begin{aligned} \int_{\mathbb{R}} \log^+ |h(y)|\, dy &= \int_{\mathbb{R}} \log^+ \left| e^{\frac{y^2}{4a}} F(f)(y) \right| dy \\ &= \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} F(f)(y) \right|}{\lambda} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy \\ &\leq \int_{\mathbb{R}} \log^+ \frac{\left| e^{by^2} F(f)(y) \right|}{\lambda} dy + \int_{\mathbb{R}} \lambda e^{\left(\frac{1}{4a}-b\right)y^2} dy \end{aligned}$$

because $\log^+(cd) \leq \log^+(c) + d$ for all $c, d > 0$,since $ab > \frac{1}{4}$, (3.2) implies that

$$\int_{\mathbb{R}} \log^+ |h(y)|\, dy < +\infty \qquad (3.4)$$

then it follows from (3.3) and (3.4) that h satisfies the assumptions in Lemma 2,and thus, h is a constant and

$$F(f)(y) = Ce^{-\left(\frac{1}{4a}\right)y^2}$$

Since $ab > \frac{1}{4}$, (3.2) holds whenever $C = 0$ and the injectivity of F implies that $f = 0$ almost everywhere.

(ii) $ab = \frac{1}{4}$,As in the previous case, it follows that $F(f)(y) = Ce^{-\frac{y^2}{4a}}$ then (3.2) holds whenever $|C| \leq \lambda$.

then $f(x) = \int_{\mathbb{R}} F(f)(\lambda) \times \Phi_{i\lambda}(x)\, d\sigma(\lambda) = \int_{\mathbb{R}} Ce^{-\frac{\lambda^2}{4a}} \Phi_{i\lambda}(x)\, d\sigma(\lambda) = CE_a(x)$

$(iii)\; ab < \frac{1}{4}$,if $f = CE_t$, then $F(f)(y) = Ce^{-\frac{1}{4t}y^2}$ and for all $t \in \left]a, \frac{1}{4b}\right[$

$e^{ax^2} f = e^{ax^2} CE_t \leq Ce^{(a-t)x^2} \in L^1(\mathbb{R}) \subset L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ by (2.1) and because $a - t < 0$

$\int_{\mathbb{R}} \log^+ \frac{\left|F(f)(y)e^{by^2}\right|}{\lambda} dy = \int \log^+ \frac{\left|Ce^{\left(b-\frac{1}{4t}\right)y^2}\right|}{\lambda} dy < \infty$ because $b - \frac{1}{4t} < 0$

then f and $F(f)$ satisfy (3.1) and (3.2) for all $t \in \left]a, \frac{1}{4b}\right[$

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