Numerical Solution for Hybrid Fuzzy Systems by Milne's Fourth Order Predictor-Corrector Method

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Abstract

In this paper we study numerical methods for hybrid fuzzy differential equations by an application of the Milne's fourth order predictor-corrector method for fuzzy differential equations. We prove a convergence result and give numerical examples to illustrate the theory.

Keywords: Hybrid Systems; Fuzzy Differential Equations; Milne's fourth order predictor-corrector method

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainty and processing vague or subjective information in mathematical models. Its main directions of development and its applications to the very varied real-world problems have been diverse. Fuzzy differential equations are recently gaining more and more attention in the literature. The first and the most popular approach in dealing with FDEs is using the Hukuhara differentiability, or the Seikkala derivative for fuzzy-number-valued functions. Hybrid systems are devoted to modelling, design, and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modelled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems.

In the last few years, many works have been performed by several authors in numerical solutions of fuzzy differential equations [1, 2, 3, 9, 10, 11, 12, 15, 18]. Pederson and Sambandam [20, 21] have investigated the numerical solution of hybrid fuzzy differential equations by using Runge-Kutta and Euler methods. Recently, the numerical solutions of fuzzy differential equations by predictor-corrector method has been studied in [4]. Allahviranloo [5] discussed numerical solution of fuzzy differential equation by Adams-Bashforth Two-step method.

In this paper we develop numerical method for hybrid fuzzy differential equation by an application of the Milne's fourth order predictor-corrector method. In Section 2 we list some basic definitions to fuzzy valued functions. Section 3 reviews hybrid fuzzy differential systems. Section 4 contains the Milne's fourth order predictor-corrector method for hybrid fuzzy differential equations and convergence theorem. Section 5, the MPC four-step algorithm is discussed. Section 6, contains numerical examples to illustrate this method. Finally, conclusion is present in Section 7.

2. Preliminary Notes

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of R^n . If $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then $\alpha(A + B) = \alpha A + \alpha B$, $\alpha(\beta A) = (\alpha\beta)A$, $1A = A$ and if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. Denote by E^n the set of $u : R^n \rightarrow [0, 1]$ such that u satisfies (i) – (iv) mentioned below:

- (i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;
- (ii) u is fuzzy convex, that is, for $x, y \in R$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

- (iii) u is upper semicontinuous;

- (iv) $[u]^0 = \text{closure of } [x \in R : u(x) > 0]$ is compact.

For $0 < r \leq 1$, we define $[u]^r = [x \in R^n : u(x) \geq r]$. Then from (i) to (iv), it follows that r -level sets $[u]^r = [x \in P_k(R^n)]$ for $0 \leq r \leq 1$. An example of a $u \in E^1$ is given by

$$u(x) = \begin{cases} 4x - 3, & \text{if } x \in (0.75, 1], \\ -2x + 3, & \text{if } x \in (1, 1.5), \\ 0, & \text{if } x \notin (0.75, 1.5). \end{cases} \quad (1)$$

The r -level sets of u in (1) are given by

$$[u]^r = [0.75 + 0.25r, 1.5 - 0.5r]. \quad (2)$$

Let I be real interval. A mapping $g : I \rightarrow E$ is called a fuzzy process and its r -level set is denoted by $[y(t)]^r = [\underline{y}^r(t), \overline{y}^r(t)]$, $t \in I$, $r \in (0, 1]$.

Definition 2.1. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad y(0) = y_0, \quad (3)$$

where $f : [a, b] \times R^n \rightarrow R^n$. An m -step method for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation :

$$y_{i+1} = \sum_{j=0}^{m-1} a_{m-j-1} y_{i-j} + h \sum_{j=0}^{m-1} b_j f(t_{i-j}, y_{i-j}) \quad (4)$$

for $i = m-1, m, \dots, N-1$, such that $a = t_0 \leq t_1 \leq \dots \leq t_N = b$, $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ and $a_0, \dots, a_{m-1}, b_0, \dots, b_m$ are constants with the starting values

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_{m-1} = \alpha_{m-1}.$$

When $b_m = 0$, the method is known as explicit, since Equation (4) gives y_{i+1} explicit in terms of previously determined values. When $b_m \neq 0$, the method is known as implicit, since y_{i+1} occurs on both sides of Equation (4) and is specified only implicitly.

A special case of multistep method is . Here, we set

$$y_{i+1} = y_{i-1} + h \sum_{m=0}^q k_m \nabla^m f(t_i, y_i), \quad q = 0, 1, 2, \dots, \quad (5)$$

where the constants

$$k_m = (-1)^m \int_0^1 \begin{pmatrix} -s \\ k \end{pmatrix} ds$$

are independent of $f, t = t_0 + sh$, $\nabla f(t_i, y_i)$ is the first backward difference of the $f(t, y(t))$ at the point of $t = t_i$ and higher backward difference are defined by $\nabla^k f(t, y_i) = \nabla(\nabla^{k-1} f(t, y_i))$. The special case $q = 3$ of Milnes method are as follows:

Milne explicit method:

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \\ y_{i+1} = y_{i-3} + \frac{4h}{3} [2f(t_i, y_i) - f(t_{i-1}, y_{i-1}) + 2f(t_{i-2}, y_{i-2})],$$

where $i = 3, 4, 5, \dots, N-1$.

Milne implicit method :

$$y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \\ y_{i+1} = y_{i-1} + \frac{h}{3} [f(t_{i-1}, y_{i-1}) + 4f(t_i, y_i) + f(t_{i+1}, y_{i+1})]. \\ \text{where } i = 3, 4, 5, \dots, N-1.$$

Definition 2.2. Assosisted with the difference equation

$$\begin{aligned} y_{i+1} &= a_{m-1}y_i + a_{m-2}y_{i-1} + \cdots + a_0y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \cdots, y_{i+1-m}), \\ y_0 &= \alpha_0, \quad y_1 = \alpha_1, \quad y_{m-1} = \alpha_{m-1}, \end{aligned} \quad (6)$$

the characteristic polynomial of the method is defined by

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0.$$

If $|\lambda_i| \leq 1$ for each $i = 1, 2, \cdots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 2.1. A multistep method of the form (4) is stable if and only if it satisfies the roots condition.

Notations used in this paper are as follows:

A tilde is placed over a symbol to denote a fuzzy set so $\tilde{\alpha}_1, \tilde{f}(t), \cdots$

An arbitrary fuzzy number with an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$. $0 \leq r \leq 1$, which satisfy the following requirements is represented.

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Definition 2.3. The supremum metric d_∞ on E is defined by

$$d_\infty(U, V) = \sup \{d_H([U]^r, [V]^r) : r \in I\},$$

and (E, d_∞) is a complete metric space.

Definition 2.4. A mapping $F : T \rightarrow E$ is a Hukuhara differentiable at $t_0 \in T \subseteq R$ if for some $h_0 > 0$ the Hukuhara differences $F(t_0 + \Delta t) \sim_h F(t_0)$, $F(t_0) \sim_h F(t_0 - \Delta t)$, exist in E for all $0 < \Delta t < h_0$ and if there exist an $F'(t_0) \in E$ such that

$$\lim_{\Delta t \rightarrow 0^+} d_\infty \left(\frac{F(t_0 + \Delta t) -_h F(t_0)}{\Delta t, F'(t_0)} \right) = 0$$

and

$$\lim_{\Delta t \rightarrow 0^+} d_\infty \left(\frac{F(t_0) -_h F(t_0 - \Delta t)}{\Delta t, F'(t_0)} \right) = 0$$

the fuzzy set $F'(t_0)$ is called the Hukuhara derivative of F at t_0 .

Recall that $U \sim_h V = W \in E$ are defined on level sets, where $[U]^r \sim_h [V]^r = [W]^r$ for all $r \in I$. By consideration of definition of the metric d_∞ , all the level set mappings $[F(\cdot)]^r$ are Hukuhara differentiable at t_0 with Hukuhara

derivatives $[F'(t_0)]^r$ for each $r \in I$ when $F : T \rightarrow E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Definition 2.5. The fuzzy integral $\int_a^b y(t)dt, 0 \leq a \leq b \leq 1$, is defined by

$$\left[\int_a^b y(t)dt \right]^r = \left[\int_a^b \underline{y}^r(t)dt, \int_a^b \overline{y}^r(t)dt \right],$$

provided the Lebesgue integrals on the right exist.

Definition 2.6. If $F : T \rightarrow E$ is Hukuhara differentialble and its Hukuhara derivative F' is integrable over $[0,1]$, then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s)ds,$$

for all values of t_0, t where $0 \leq t_0 \leq t \leq 1$.

Definition 2.7. A mapping $y : I \rightarrow E$ is called a fuzzy process. We denote

$$[y(t)]^r = [\underline{y}^r(t), \overline{y}^r(t)], \quad t \in I, \quad r \in (0, 1].$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by $[y(t)]^r = [\underline{y}^r(t), \overline{y}^r(t)]$, $t \in I, r \in (0, 1]$ provided the equation defines a fuzzy number $y'(t) \in E$.

Definition 2.8. If $y : I \rightarrow E$ is Seikkala differentiable and its Seikkala derivative y' is integrable over $[0, 1]$, then

$$y(t) = y(t_0) + \int_{t_0}^t y'(s)ds,$$

for all values of t_0, t where $t_0, t \in I$.

3. The Hybrid Fuzzy Differential System

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{cases} \quad (7)$$

where ' denotes Seikkala differentiation, $0 \leq t_0 < t_1 < \dots < t_k < \dots, t_k \rightarrow \infty$, $f \in C[R^+ \times E^1 \times E^1, E^1]$, $\lambda_k \in C[E^1, E^1]$. To be specific the system would look like

$$x'(t) = \begin{cases} x_0(t) = f(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0, & t_0 \leq t \leq t_1, \\ x_1(t) = f(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1, & t_1 \leq t \leq t_2, \\ \dots & \\ x_k(t) = f(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k, & t_k \leq t \leq t_{k+1}, \\ \dots & \end{cases}$$

Assuming that the existence and uniqueness of solution of (7) hold for each $[t_k, t_{k+1}]$, by the solution of (7) we mean the following function:

$$x'(t) = \begin{cases} x_0(t), & t_0 \leq t \leq t_1, \\ x_1(t), & t_1 \leq t \leq t_2, \\ \dots & \\ x_k(t), & t_k \leq t \leq t_{k+1} \\ \dots & \end{cases},$$

We note that the solution of (7) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E^1$ and $k = 0, 1, 2, \dots$

Using a representation of fuzzy numbers studied by Goetschel and Voxman [8] and Wu and Ma [23], we may represent $x \in E^1$ by a pair of functions $(\underline{x}(r), \bar{x}(r))$, $0 \leq r \leq 1$, such that (i) $\underline{x}(r)$ is bounded, left continuous, and nondecreasing, (ii) $\bar{x}(r)$ is bounded, left continuous, and nonincreasing, and (iii) $\underline{x}(r) \leq \bar{x}(r)$, $0 \leq r \leq 1$. For example, $u \in E^1$ given in (1) is represented by

$$(\underline{u}(r), \bar{u}(r)) = (0.75 + 0.25r, 1.5 - 0.5r), \quad 0 \leq r \leq 1,$$

which is similar to $[u]^\alpha$ given by (2).

Therefore we may replace (7) by an equivalent system

$$\begin{cases} \underline{x}'(t) = \underline{f}(t, x, \lambda_k(x_k)) \equiv F_k(t, \underline{x}, \bar{x}), & \underline{x}(t_k) = \underline{x}_k, \\ \bar{x}'(t) = \bar{f}(t, x, \lambda_k(x_k)) \equiv G_k(t, \underline{x}, \bar{x}), & \bar{x}(t_k) = \bar{x}_k, \end{cases} \quad (8)$$

which possesses a unique solution $(\underline{x}, \bar{x})$ which is a fuzzy function. That is for each t , the pair $[\underline{x}(t; r), \bar{x}(t; r)]$ is a fuzzy number, where $\underline{x}(t; r), \bar{x}(t; r)$ are respectively the solutions of the parametric form given by

$$\begin{cases} \underline{x}'(t; r) = F_k[t, \underline{x}(t; r), \bar{x}(t; r)], & \underline{x}(t_k; r) = \underline{x}_k(r), \\ \bar{x}'(t; r) = G_k[t, \underline{x}(t; r), \bar{x}(t; r)], & \bar{x}(t_k; r) = \bar{x}_k(r), \end{cases} \quad (9)$$

4. Milne's fourth order predictor-corrector method

In this section for a hybrid fuzzy differential equation (7). We develop the Milne's fourth order predictor-corrector method in [3]. When f and λ_k in (7) can be obtained via, the Zadeh extension principal from $f \in [R^+ \times R \times R, R]$ and $\lambda_k \in [R, R]$. We assume that the existence and uniqueness of solution (7) hold for each $[t_k, t_{k+1}]$.

For a fixed r , to integrate the system in (7) in $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}], \dots$, we replace each interval by a set of $N_k + 1$ discrete equally spaced grid points (including the end points) at which the exact solution $(\underline{x}(t; r), \overline{x}(t; r))$ is approximated by some $(\underline{y}_k(t; r), \overline{y}_k(t; r))$. For the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1} - t_k}{N_k}$, $0 \leq n \leq N_k$.

Let $(\underline{Y}_k(t; r), \overline{Y}_k(t; r)) \equiv (\underline{y}(t, r), \overline{y}(t, r)), (\underline{Y}_k(t; r), \overline{Y}_k(t; r))$ and $(\underline{y}_k(t; r), \overline{y}_k(t; r))$ may be denoted respectively by $(\underline{Y}_k(t; r), \overline{Y}_k(t; r))$ and $(\underline{y}_k(t; r), \overline{y}_k(t; r))$. The Milne's fourth order predictor-corrector method is approximation of $\underline{y}_k(t; r)$, and $\overline{y}_k(t; r)$ which can be written as

$$\begin{aligned}
 y_0(r) &= [\underline{\alpha}_0(r), \overline{\alpha}_0(r)], & y_1(r) &= [\underline{\alpha}_1(r), \overline{\alpha}_1(r)] \\
 y_2(r) &= [\underline{\alpha}_2(r), \overline{\alpha}_2(r)], & y_3(r) &= [\underline{\alpha}_3(r), \overline{\alpha}_3(r)] \\
 \underline{Y}_{n+1}^p &= \underline{Y}_{n-3}(r) + \frac{4h}{3}[2\underline{f}(t_n, y_n(r), \lambda_k(u_k)) - \underline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) \\
 &\quad + 2\underline{f}(t_{n-2}, y_{n-2}(r), \lambda_k(u_k))], \\
 \overline{Y}_{n+1}^p &= \overline{Y}_{n-3}(r) + \frac{4h}{3}[2\overline{f}(t_n, y_n(r), \lambda_k(u_k)) - \overline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) \\
 &\quad + 2\overline{f}(t_{n-2}, y_{n-2}(r), \lambda_k(u_k))], \\
 \underline{Y}_{n+1}^c &= \underline{Y}_{n-1}(r) + \frac{h}{3}[\underline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) + 4\underline{f}(t_n, y_n(r), \lambda_k(u_k)) \\
 &\quad + \underline{f}(t_{n+1}, y_{n+1}(r), \lambda_k(u_k))], \\
 \overline{Y}_{n+1}^c &= \overline{Y}_{n-1}(r) + \frac{h}{3}[\overline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) + 4\overline{f}(t_n, y_n(r), \lambda_k(u_k)) \\
 &\quad + \overline{f}(t_{n+1}, y_{n+1}(r), \lambda_k(u_k))],
 \end{aligned} \tag{10}$$

where

$$\begin{aligned}
 Y_n(r) &= [\underline{y}_n(r), \overline{y}_n(r)] \\
 \underline{f}(t_n, y_n(r), \lambda_k(u_k)) &= \min\{f(t, u, \lambda_k(u_k)) | u \in [\underline{y}_n(r), \overline{y}_n(r)]\}, \\
 \overline{f}(t_n, y_n(r), \lambda_k(u_k)) &= \max\{f(t, u, \lambda_k(u_k)) | u \in [\underline{y}_n(r), \overline{y}_n(r)]\}.
 \end{aligned}$$

Now we define

$$\begin{aligned}
F^p[t, y(r)] &= [2\underline{f}(t_n, y_n(r), \lambda_k(u_k)) - \underline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) \\
&\quad + 2\underline{f}(t_{n-2}, y_{n-2}(r), \lambda_k(u_k))], \\
G^p[t, y(r)] &= [2\overline{f}(t_n, y_n(r), \lambda_k(u_k)) - \overline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) \\
&\quad + 2\overline{f}(t_{n-2}, y_{n-2}(r), \lambda_k(u_k))], \\
F^c[t, y(r)] &= [\underline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) + 4\underline{f}(t_n, y_n(r), \lambda_k(u_k)) \\
&\quad + \underline{f}(t_{n+1}, y_{n+1}(r), \lambda_k(u_k))], \\
G^c[t, y(r)] &= [\overline{f}(t_{n-1}, y_{n-1}(r), \lambda_k(u_k)) + 4\overline{f}(t_n, y_n(r), \lambda_k(u_k)) \\
&\quad + \overline{f}(t_{n+1}, y_{n+1}(r), \lambda_k(u_k))].
\end{aligned} \tag{11}$$

The exact and approximate solutions at t_n , $0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [\underline{Y}(t; r), \overline{Y}(t; r)]$ and $[y(t_n)]_r = [\underline{y}(t; r), \overline{y}(t; r)]$ respectively. By (10), (11) we have

$$\left. \begin{aligned}
\underline{Y}^c(t_{n+1}; r) &\approx \underline{Y}^c(t_n; r) + \frac{h}{3} F[t_n; \underline{Y}^c(t_n; r), \overline{Y}^c(t_n; r)] \\
\overline{Y}^c(t_{n+1}; r) &\approx \overline{Y}^c(t_n; r) + \frac{h}{3} G[t_n; \underline{Y}^c(t_n; r), \overline{Y}^c(t_n; r)]
\end{aligned} \right\} \tag{12}$$

we define

$$\left. \begin{aligned}
\underline{y}^c(t_{n+1}; r) &= \underline{y}^c(t_n; r) + \frac{h}{3} F[t_n; \underline{y}^c(t_n; r), \overline{y}^c(t_n; r)] \\
\overline{y}^c(t_{n+1}; r) &= \overline{y}^c(t_n; r) + \frac{h}{3} G[t_n; \underline{y}^c(t_n; r), \overline{y}^c(t_n; r)]
\end{aligned} \right\} \tag{13}$$

However, (12) will use $\underline{y}_{0,0} = \underline{x}_0(r)$, $\overline{y}_{0,0} = \overline{x}_0(r)$, and $\underline{y}_{k,0}(r) = \underline{y}_{k-1}(r)N_{k-1}$, $\overline{y}_{k,0}(r) = \overline{y}_{k-1}(r)N_{k-1}$. If $k \geq 1$, then (12) represents an approximation of $\underline{y}_k(t; r), \overline{y}_k(t; r)$ for each of intervals $t_0 \leq t \leq t_1, t_1 \leq t \leq t_2, \dots, t_k \leq t \leq t_{k+1}, \dots$

Lemma 4.1. Suppose $i \in Z^+$, $\epsilon_i > 0$, $r \in [0, 1]$, and $h_i < 1$ are fixed. Let $\{Z_{i,n}(r)\}_{n=0}^{N_i}$ be the Milne's fourth order predictor-corrector approximation with $N = N_i$ to the fuzzy IVP:

$$\left\{ \begin{aligned} x'(t) &= f(t, x(t), \lambda_i(x_i)), \quad t \in [t_i, t_{i+1}], \\ x(t_i) &= x_i. \end{aligned} \right. \tag{14}$$

If $\{y_{i,n}(r)\}_{n=0}^{N_i}$ denotes the result of (10) from some $y_{i,0}(r)$, then there exist a $\delta_i > 0$ such that $|\underline{z}_{i,0}(r) - \underline{y}_{i,0}(r)| < \delta_i$, $|\overline{z}_{i,0}(r) - \overline{y}_{i,0}(r)| < \delta_i$ implies $|\underline{z}_{i,0}(r) - \underline{y}_{i,0}(r)| < \epsilon_i$, $|\overline{z}_{i,0}(r) - \overline{y}_{i,0}(r)| < \epsilon_i$

proof: Proof is similar to lemma 3.1 [19]

Theorem 4.1. Consider the systems (9) and (11). For a fixed $k \in Z^+$ and $r \in [0, 1]$,

$$\lim_{h_0, \dots, h_k \rightarrow 0} \underline{y}_{k, N_k}(r) = \underline{x}(t_{k+1}; r), \quad (15)$$

$$\lim_{h_0, \dots, h_k \rightarrow 0} \overline{y}_{k, N_k}(r) = \overline{x}(t_{k+1}; r). \quad (16)$$

Proof. See [19].

5. Algorithm

The following algorithm is based on Milne's fourth order predictor-corrector method.

To approximate the solution of following fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t), \lambda_k(u_k)), & t_0 \leq t \leq T, \\ \underline{y}^\alpha = \alpha_0, & \underline{y}^\alpha = \alpha_1 & \underline{y}^\alpha = \alpha_2, & \underline{y}^\alpha = \alpha_3, \\ \overline{y}^\alpha = \alpha_4, & \overline{y}^\alpha = \alpha_5 & \overline{y}^\alpha = \alpha_6, & \overline{y}^\alpha = \alpha_7, \end{cases},$$

positive integer N is chosen

Step 1. Let $h = \frac{(T-t_0)}{N}$,

$$\begin{aligned} y'(t) &= f(t, y(t), \lambda_k(u_k)), & t_0 \leq t \leq T, \\ \underline{w}^\alpha &= \alpha_0, & \underline{w}^\alpha = \alpha_1 & \underline{w}^\alpha = \alpha_2, & \underline{w}^\alpha = \alpha_3, \\ \underline{w}^\alpha &= \alpha_4, & \overline{w}^\alpha = \alpha_5 & \overline{w}^\alpha = \alpha_6, & \overline{w}^\alpha = \alpha_7, \end{aligned}$$

Step 2. For $n = 0, 1, 2$, do Steps 3-4.

Compute starting values using fourth order Runge - Kutta method

Step 3. Let

$$\underline{k}_1(t_{k,n}; y_{k,n}(r)) = \min\{h_k \cdot f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]\},$$

$$\overline{k}_1(t_{k,n}; y_{k,n}(r)) = \max\{h \cdot f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]\},$$

$$\underline{k}_2(t_{k,n}; y_{k,n}(r)) = \min\{h \cdot f(t_{k,n} + h/2, u, \lambda_k(u_k)) | u \in [\underline{z}_1(t_{k,n}, y_{k,n}(r)), \overline{z}_1(t_{k,n}, y_{k,n}(r))]\},$$

$$\overline{k}_2(t_{k,n}; y_{k,n}(r)) = \max\{h \cdot f(t_{k,n} + h/2, u, \lambda_k(u_k)) | u \in [\underline{z}_1(t_{k,n}, y_{k,n}(r)), \overline{z}_1(t_{k,n}, y_{k,n}(r))]\},$$

$$\begin{aligned}\underline{k}_3(t_{k,n}, y_{k,n}(r)) &= \min\{h.f(t_{k,n} + h/2, u, \lambda_k(u_k)) | u \in [\underline{z}_2(t_{k,n}, y_{k,n}(r)), \bar{z}_2(t_{k,n}, y_{k,n}(r))]\}, \\ \bar{k}_3(t_{k,n}, y_{k,n}(r)) &= \max\{h.f(t_{k,n} + h/2, u, \lambda_k(u_k)) | u \in [\underline{z}_2(t_{k,n}, y_{k,n}(r)), \bar{z}_2(t_{k,n}, y_{k,n}(r))]\}, \\ \underline{k}_4(t_{k,n}, y_{k,n}(r)) &= \min\{h.f(t_{k,n} + h/2, u, \lambda_k(u_k)) | u \in [\underline{z}_3(t_{k,n}, y_{k,n}(r)), \bar{z}_3(t_{k,n}, y_{k,n}(r))]\}, \\ \bar{k}_4(t_{k,n}, y_{k,n}(r)) &= \max\{h.f(t_{k,n} + h, u, \lambda_k(u_k)) | u \in [\underline{z}_3(t_{k,n}, y_{k,n}(r)), \bar{z}_3(t_{k,n}, y_{k,n}(r))]\},\end{aligned}$$

where

$$\begin{aligned}\underline{z}_1(t_{k,n}, y_{k,n}(r)) &= \underline{y}_{k,n}(r) + \frac{1}{2}\underline{k}_1(t_{k,n}, y_{k,n}(r)), \\ \bar{z}_1(t_{k,n}, y_{k,n}(r)) &= \bar{y}_{k,n}(r) + \frac{1}{2}\bar{k}_1(t_{k,n}, y_{k,n}(r)), \\ \underline{z}_2(t_{k,n}, y_{k,n}(r)) &= \underline{y}_{k,n}(r) + \frac{1}{2}\underline{k}_2(t_{k,n}, y_{k,n}(r)), \\ \bar{z}_2(t_{k,n}, y_{k,n}(r)) &= \bar{y}_{k,n}(r) + \frac{1}{2}\bar{k}_2(t_{k,n}, y_{k,n}(r)), \\ \underline{z}_3(t_{k,n}, y_{k,n}(r)) &= \underline{y}_{k,n}(r) + \underline{k}_3(t_{k,n}, y_{k,n}(r)), \\ \bar{z}_3(t_{k,n}, y_{k,n}(r)) &= \bar{y}_{k,n}(r) + \bar{k}_3(t_{k,n}, y_{k,n}(r)).\end{aligned}$$

step 4. Let

$$\begin{aligned}\underline{y}(t_{k,n+1}, r) &= \underline{y}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}(\underline{k}_1(t_{k,n}, y_{k,n}(r)) + 2\underline{k}_2(t_{k,n}, y_{k,n}(r)) \\ &\quad + 2\underline{k}_3(t_{k,n}, y_{k,n}(r)) + \underline{k}_4(t_{k,n}, y_{k,n}(r))), \\ \bar{y}(t_{k,n+1}, r) &= \bar{y}(t_{k,n}, y_{k,n}(r)) + \frac{1}{6}(\bar{k}_1(t_{k,n}, y_{k,n}(r)) + 2\bar{k}_2(t_{k,n}, y_{k,n}(r)) \\ &\quad + 2\bar{k}_3(t_{k,n}, y_{k,n}(r)) + \bar{k}_4(t_{k,n}, y_{k,n}(r))).\end{aligned}$$

Step 5. For $i=3,4,\dots,N-1$. Let

$$\left\{ \begin{array}{l} \underline{y}_{i+1}^p = \underline{y}_{i-3} + \frac{4h}{3}[2\underline{f}(t_i, y_i, \lambda_k(u_k)) - \underline{f}(t_{i-1}, y_{i-1}, \lambda_k(u_k)) + 2\underline{f}(t_{i-2}, y_{i-2}, \lambda_k(u_k)) \\ \quad - 9\underline{f}(t_{i-3}, y_{i-3}, \lambda_k(u_k))], \\ \bar{y}_{i+1}^p = \bar{y}_{i-3} + \frac{4h}{3}[2\bar{f}(t_i, y_i, \lambda_k(u_k)) - \bar{f}(t_{i-1}, y_{i-1}, \lambda_k(u_k)) + 2\bar{f}(t_{i-2}, y_{i-2}, \lambda_k(u_k)) \\ \quad - 9\bar{f}(t_{i-3}, y_{i-3}, \lambda_k(u_k))]. \end{array} \right.$$

step 6. Let $t_{i+2} = t_0 + (i + 2)h$

Step 7. Let

$$\begin{cases} \underline{y}_{i+1}^c = \underline{y}_{i-1} + \frac{h}{3}[\underline{f}(t_{i-1}, y_{i-1}, \lambda_k(u_k)) + 4\underline{f}(t_i, y_i, \lambda_k(u_k)) + \underline{f}(t_{i+1}, y_{i+1}, \lambda_k(u_k))], \\ \overline{y}_{i+1}^c = \overline{y}_{i-1} + \frac{h}{3}[\overline{f}(t_{i-1}, y_{i-1}, \lambda_k(u_k)) + 4\overline{f}(t_i, y_i, \lambda_k(u_k)) + \overline{f}(t_{i+1}, y_{i+1}, \lambda_k(u_k))] \end{cases}$$

Step 8. For $i = i + 1$

Step 9. If $i \leq N - 2$ go to step 3.

Step 10.

Algorithm will be completed and $(\underline{w}^\alpha(T), \overline{w}^\alpha(T))$ approximates real value of $(\underline{Y}^\alpha(T), \overline{Y}^\alpha(T))$.

6. Numerical Examples

Before illustrating the numerical solution of hybrid fuzzy IVP, first we recall the Example 3 of [18] studying the fuzzy IVP;

$$x'(t) = x(t), \quad x(0; r) = [0.75 + 0.25r, 1.125 - 0.125r], \quad 0 \leq r \leq 1. \quad (17)$$

By Milne's predictor-corrector method with $N=10$. Since the exact solution of (17) is $x(t; r) = [(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t]$, $0 \leq r \leq 1$, we see that $x(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e]$, $0 \leq r \leq 1$.

Example 6.1. Next consider the following hybrid fuzzy IVP,

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k x(t_k), & t \in [t_k, t_{k+1}], \quad t_k = k, \quad k = 0, 1, 2, 3, \dots, \\ x(0; r) = [(0.75 + 0.25r), (1.125 - 0.125r)], & 0 \leq r \leq 1, \end{cases} \quad (18)$$

where

$$m(t) = \begin{cases} 2(t \bmod 1) & \text{if } t \bmod 1 \leq 0.5, \\ 2(1 - t \bmod 1) & \text{if } t \bmod 1 > 0.5, \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \in \{1, 2, \dots\}. \end{cases}$$

The hybrid fuzzy IVP (18) is equivalent to the following systems of fuzzy IVPs:

$$\begin{cases} x'_0(t) = x_0(t), & t \in [0, 1], \\ x(0; r) = [(0.75 + 0.25r), (1.125 - 0.125r)], & 0 \leq r \leq 1, \\ x'_i(t) = x_i(t) + m(t)x_{i-1}(t), & t \in [t_i, t_{i+1}], \quad x_i(t) = x_{i-1}(t_i), \quad i = 1, 2, \dots, \end{cases}$$

In (14) $x(t) + m(t)\lambda_k(x(t_k))$ is continuous function of t, x and $\lambda_k(x(t_k))$. Therefore by Example 6.1 of Kaleva [14], for each $k = 0, 1, 2, \dots$, the fuzzy IVP

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_k}, \end{cases} \quad (19)$$

has a unique solution on $[t_k, t_{k+1}]$.

Therefore $x(1; r) = [(0.75 + 0.25r)e, (1.125 - 0.125r)e], x(1; r)(3\sqrt{e} - 3)$. Then $x(1.5; 1)$ is approximately 5.290221726 and $y_1(1.5; r)$ is approximately 5.290221962. For $t \in [1.5; 2]$ the exact solution satisfies

$$x(t; r) = x(1; r)(2t - 2 + e^{t-1.5}(3\sqrt{e} - 4)).$$

Therefore

$$x(2.0; r) = x(1; r)(2 + 3e - 4\sqrt{e}).$$

Then $x(2.0; r)$ is approximately 9.676975672 and $y(2.0; 1)$ is approximately 9.6769741822.

The exact and approximate solution by using Milne's predictor-corrector method. With $N = 10$ the following results are obtained. The exact and approximate solution by fourth order Runge-Kutta method and Milne's Predictor Corrector method are compared at $t = 2$ see Table 1 and Figure 1, Error in fourth order Runge-Kutta method and Milne's predictor-corrector method see Table 2.

Table 1.

r	RK-Fourth Order		Predictor Corrector		Exact	
	$y_1(t_i; r)$	$y_2(t_i; r)$	$y_1(t_i; r)$	$y_2(t_i; r)$	$Y_1(t_i; r)$	$Y_2(t_i; r)$
0.1	7.499642836	10.76561633	7.499654991	10.76563378	7.499656146	10.76563660
0.2	7.741566798	10.64465435	7.741579346	10.64467160	7.741580538	10.64467324
0.3	7.983490761	10.52369237	7.983503700	10.52370942	7.983504929	10.52371104
0.4	8.225414723	10.40273039	8.225428055	10.40274725	8.225429321	10.40274885
0.5	8.467338686	10.28176840	8.467352409	10.28178507	8.467353713	10.28178665
0.6	8.709262648	10.16080642	8.709276764	10.16082289	8.709278105	10.16082446
0.7	8.951186611	10.03984444	8.951201119	10.03986071	8.951202497	10.03986226
0.8	9.193110573	9.918882460	9.193125473	9.918898537	9.193126888	9.918900064
0.9	9.435034536	9.797920479	9.435049828	9.797936359	9.435051280	9.797937868
1.0	9.676958498	9.676958498	9.676974182	9.676974182	9.676975672	9.676975672

Table 2.

r	Rk 4 th -Order		Predictor Corrector	
	$y_1(t_i; r)$	$y_2(t_i; r)$	$y_1(t_i; r)$	$y_2(t_i; r)$
0.1	1.331×10^{-5}	2.027×10^{-5}	1.155×10^{-6}	2.82×10^{-6}
0.2	1.374×10^{-5}	1.889×10^{-5}	1.192×10^{-6}	1.64×10^{-6}
0.3	1.417×10^{-5}	1.867×10^{-5}	1.229×10^{-6}	1.62×10^{-6}
0.4	1.459×10^{-5}	1.846×10^{-5}	1.266×10^{-6}	1.60×10^{-6}
0.5	1.373×10^{-5}	0.981×10^{-5}	1.304×10^{-6}	1.58×10^{-6}
0.6	1.411×10^{-5}	1.804×10^{-5}	1.341×10^{-6}	1.57×10^{-6}
0.7	1.588×10^{-5}	1.782×10^{-5}	1.378×10^{-6}	1.55×10^{-6}
0.8	1.631×10^{-5}	1.818×10^{-5}	1.415×10^{-6}	1.52×10^{-6}
0.9	1.674×10^{-5}	1.739×10^{-5}	1.452×10^{-6}	1.52×10^{-6}
1.0	1.717×10^{-5}	1.717×10^{-5}	1.490×10^{-6}	1.49×10^{-6}

Figure 1: $h=0.1$

Example 6.2. Next consider the following hybrid fuzzy IVP,

$$\begin{cases} x'_0(t) = x_0(t) + m(t)\lambda_k x(t_k), & t \in [0, 1], \\ x(0; r) = [(0.75 + 0.25r), (1.125 - 0.125r)], & 0 \leq r \leq 1, \\ x'_i(t) = x_i(t) + m(t)x_{i-1}(t), t \in [t_i, t_{i+1}], x_i(t) = x_{i-1}(t_i), i = 1, 2, \dots, \end{cases}$$

where

$$m(t) = |\sin(\pi t)|, \quad k = 0, 1, 2, \dots,$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \in \{1, 2, \dots\}. \end{cases}$$

Then $x(t) + m(t)\lambda_k(x(t_k))$ is continuous function of t, x , and $\lambda_k(x(t_k))$. Therefore by Example of 6.1 of Kaleva [14], for each $k=0,1,2,\dots$, the fuzzy IVP

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_k}, \end{cases} \quad (20)$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (18) we will apply the Milne's predictor-corrector method for hybrid fuzzy differential equation with $N = 10$ to obtain $y_{1,2}(r)$ approximating $x(2.0; r)$.

The exact solution of (20) satisfies

$$\begin{aligned} \underline{x}(t; r) &= \underline{x}(1; r) \frac{\pi \cos(\pi t) + \sin(\pi t)}{\pi^2} + \frac{e^t}{e} \underline{x}(1; r) \left(1 + \frac{\pi}{\pi^2 + 1}\right), \\ \overline{x}(t; r) &= \overline{x}(1; r) \frac{\pi \cos(\pi t) + \sin(\pi t)}{\pi^2} + \frac{e^t}{e} \overline{x}(1; r) \left(1 + \frac{\pi}{\pi^2 + 1}\right). \end{aligned}$$

Therefore

$$\begin{aligned} x(1; r) &= [(0.75 + 0.25r)e, (1.125 - 0.125r)e], \\ x(2; r) &= \left(\frac{\pi}{\pi^2 + 1} + e \left(1 + \frac{\pi}{\pi^2 + 1}\right)\right) x(1; r). \end{aligned}$$

Then $x(2; 1)$ is approximately 10.31033432 where as $y_1(2.0; 1)$ is approximately 10.31033194 see in Fig.2.

Table 3.

r	RK-Fourth Order		Predictor Corrector		Exact	
	$y_1(t_i; r)$	$y_2(t_i; r)$	$y_1(t_i; r)$	$y_2(t_i; r)$	$y_1(t_i; r)$	$y_2(t_i; r)$
0.1	7.990499916	11.47023375	7.990507256	11.47024429	7.990509096	11.47024693
0.2	8.248257977	11.34135472	8.248265554	11.34136514	8.248267456	11.34136775
0.3	8.506016039	11.21247569	8.506023853	11.21248599	8.506025814	11.21248857
0.4	8.763774101	11.08359666	8.763782152	11.08360684	8.763784172	11.08360939
0.5	9.021532163	10.95471763	9.021540450	10.95472769	9.021542530	10.95473022
0.6	9.279290225	10.82583860	9.279298749	10.82584854	9.279300888	10.82585104
0.7	9.537048286	10.69695956	9.537057047	10.69696939	9.537059246	10.69697186
0.8	9.794806348	10.56808058	9.794815346	10.56809024	9.794817604	10.56809268
0.9	10.05256441	10.43920150	10.05257364	10.43921109	10.05257596	10.43921350
1.0	10.31032247	10.31032247	10.31033194	10.31033194	10.31033432	10.31033432

Table 4.

r	Rk 4 th -Order		Predictor Corrector	
	$y_1(t_i; r)$	$y_2(t_i; r)$	$y_1(t_i; r)$	$y_2(t_i; r)$
0.1	9.140×10^{-6}	1.054×10^{-5}	1.840×10^{-6}	2.64×10^{-6}
0.2	9.479×10^{-6}	1.303×10^{-5}	1.902×10^{-6}	2.61×10^{-6}
0.3	9.775×10^{-6}	1.288×10^{-5}	1.961×10^{-6}	2.58×10^{-6}
0.4	1.007×10^{-5}	1.102×10^{-5}	2.020×10^{-6}	2.55×10^{-6}
0.5	1.037×10^{-5}	1.259×10^{-5}	2.080×10^{-6}	2.53×10^{-6}
0.6	8.524×10^{-6}	1.244×10^{-5}	2.139×10^{-6}	2.50×10^{-6}
0.7	1.096×10^{-5}	1.230×10^{-5}	2.199×10^{-6}	2.47×10^{-6}
0.8	1.136×10^{-5}	1.210×10^{-5}	2.258×10^{-6}	2.44×10^{-6}
0.9	1.155×10^{-5}	1.200×10^{-5}	2.320×10^{-6}	2.41×10^{-6}
1.0	1.185×10^{-5}	1.185×10^{-5}	2.380×10^{-6}	2.38×10^{-6}

Figure 2: $h=0.1$

7. Conclusion

In this paper, we have applied iterative solution of Milne's predictor-corrector fourth order method for finding the numerical solution of hybrid fuzzy differential equations. Comparison of solution of Example (6.1) and (6.2) shows that our proposed method gives better solution than fourth order Runge-Kutta method.

References

- [1] S.Abbasbandy, T.Allah Viranloo, Numerical solution of fuzzy differential equation by Talor method, *Journal of Computational Methods in Applied mathematics*, 2 (2002) 113-124.
- [2] S.Abbasbandy, T.Allah Viranloo, Numerical solution of fuzzy differential equation, *Mathematical and Computational Applications*, 7 (2002) 41-52.
- [3] S.Abbasbandy, T.Allah Viranloo, Numerical solution of fuzzy differential equation by Runge-Kutta method, *Nonlinear Studies*, 11 (2004) 117-129.
- [4] T.Allah Viranloo, N.Ahmady, E.Ahmady, Numerical solution of fuzzy differential equation by predictor-corrector method, *Information Sciences* 177 (2007) 1633-1647
- [5] T.Allah Viranloo, Numerical solution of fuzzy differential equation by Adams-Bashforth Two-Step Method, *Journal of Applied Mathematics Islamic Azad University Lahijan* (2004), 36-47
- [6] B.Bede, Note on Numerical solutions of fuzzy differential equations by predictor corrector method, *Information science* 178 (2008) 1917-1922.
- [7] J.J.Bukley, T.Feuring, Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 110 (2000) 43-54.
- [8] R.Goetschel, Woxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems*, 18 (1986) 31-43.
- [9] T. Jayakumar, K. Kanagarajan, Numerical Solution for Hybrid Fuzzy System by Improved Euler Method, *International Journal of Applied Mathematical Science*, 38 (2012) 1847-1862.
- [10] T. Jayakumar, K. Kanagarajan, S. Indrakumar, Numerical Solution of N^{th} -Order Fuzzy Differential Equation by Runge-Kutta Nystrom Method, *International Journal of Mathematical Engineering and Science*, 1(5) 2012 1-13.
- [11] T. Jayakumar, D. Maheshkumar, K. Kanagarajan, Numerical Solution of Fuzzy Differential Equations by Runge-Kutta Method of Order Five, *International Journal of Applied Mathematical Science*, 6 (2012) 2989-3002.
- [12] T. Jayakumar, K. Kanagarajan, Numerical Solution for Hybrid Fuzzy System by Runge-Kutta Method of Order Five, *International Journal of Applied Mathematical Science*, 6 (2012) 3591-3606.

- [13] A.Khastan, K.Ivaz, Numerical Solution for Fuzzy Differential Equations by Nystrom method, *Chaos, Solitons and Fractals* 41 (2009) 859-868.
- [14] O.Kaleva, Fuzzy Differential Equations, *Fuzzy Sets and Systems*, 24 (1987) 301-317.
- [15] K. Kanagarajan, M.Sambath Numerical Solution for Fuzzy Differential Equations by Third Order Runge-Kutta Method, *International Journal of Applied Mathematics and Computation*, 2(4) (2010) 1-10.
- [16] V.Lakshmikantham, X.Z.Liu, Impulsive Hybrid systems and stability theory, *International Journal of Nonlinear Differential Equations*, 5 (1999) 9-17.
- [17] V.Lakshmikantham, R.N.Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, United Kingdom, 2003.
- [18] M.Ma, M.Friedman, A.Kandel, Numerical solutions of fuzzy differential equations, *Fuzzy Sets and Systems*, 105 (1999) 133-138.
- [19] S.Pederson, M.Samandham, Numerical solution to Hybrid fuzzy systems, *Mathematical and Computer Modelling*, 45(2007) 1133-1144.
- [20] S.Pederson, M.Samandham, The Runge-Kutta method for Hybrid Fuzzy Differential Equations, *Nonlinear Analysis Hybrid Systems*, 2 (2008) 626-634.
- [21] M.Samandham, Perturbed Lyapunov-like functions and Hybrid Fuzzy Differential Equations, *International Journal of Hybrid Systems*, 2 (2002) 23-34.
- [22] S.Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems*, 24 (1987) 319-330.
- [23] C.X.Wu, M.Ma, Embedding problem of fuzzy number space Part I, *Fuzzy Sets and Systems*, 44 (1991) 33-38.

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