About Necessary and Sufficient Condition for Strong Stationarity of the Positive Quadratic Form

Gulomov Otabek and Shodiev Sagdullo

Karshi State University, Karshi, Uzbekistan

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Abstract

In the paper the finite criteria of the strong stationarity is given for the positive quadratic forms. Before known criteria requires of infinite number of equalities.

Keywords: positive quadratic form, perfect form, strong stationarity

1 Introduction. Statement of the problem

Consider $N = \frac{n(n+1)}{2}$ - dimensional Euclid space R^N and ordinary orthogonal system of coordinates. The axis and coordinates of points we denote

$$v_{11}, v_{22}, \dots, v_{nn}, v_{12}, v_{13}, \dots, v_{n-1n}.$$

For every point $(a_{ij} = (a_{11}, \dots, a_{nn}, a_{12}, \dots, a_{n-1n}))$ of this space we find corresponding form

$$\sum_{1 \leq i, j \leq n} a_{ij} x_i x_j = a_{11} x_1^2 + \dots + a_{nn} x_n^2 + 2a_{12} x_1 x_2 + \dots + 2a_{n-1n} x_{n-1} x_n.$$

The set of all points of the space R^N which correspond to positive quadratic forms we denote by K^N .

Suppose the positive quadratic form of n variables (p.q.f.)

$$f = f(x) = f(x_1, \dots, x_n) = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j \quad (1)$$

Is given with real coefficients $a_{ij} = a_{ji}$, determinant $\det f = \det(a_{ij}) > 0$, arithmetic

minimum $m = m(f)$, and with representations of minimum m :

$$\pm m_k = \pm(m_{1k}, \dots, m_{nk}), \quad k = 1, \dots, s \quad (2)$$

and $G(f)$ is the group of integer valued automorphisms of the form f .

P.q.f. f of the form (1) is called [1] perfect form (p.f.) if it is determined by its arithmetic minimum $m(f)$ and representations (2), i.e. if the following system has unique solution with respect to coefficients a_{ij} :

$$\sum_{1 \leq i, j \leq n} a_{ij} m_{ik} m_{jk} = m, \quad (k = 1, \dots, s).$$

Let s be complex number; we consider Epstein zeta-function given for $\operatorname{Re} s > 1$ with series

$$Z(f, s) = \sum_{x \neq 0} \{f(x)\}^{-\frac{ns}{2}}.$$

The summation $\sum_{x \neq 0}$ here and further is given by all integer vectors $x = (x_1, \dots, x_n)$ with the exception of $0 = (0, \dots, 0)$, i.e. $x \in \mathbb{Z}^n \setminus \{0\}$.

We fix the real number $s > 1$.

It is said [2], that p.q.f. f of the determinant $\det f$ is s -extremal, if f is the point of local minimum of the function $Z(f, s)$, in the given cone of positivity K^N under the condition $\det f = d_0$.

It is said [3], that p.q.f. f is final extremal, if there exists the number $s_0 \geq 1$, such that for all $s > s_0$ the form f is s -extremal.

P.q.f. $f_0 = \sum a_{ij}^0 x_i x_j$, satisfying the following conditions is called [3] s -stationary:

$$\sum_{x \neq 0} \frac{x_i x_j}{f_0^{\frac{ns}{2}+1}} = \frac{Z(f_0, s)}{n \cdot \det f_0} A_{ij}^0 \quad (1 \leq i \leq j \leq n), \quad (3)$$

where A_{ij}^0 is the algebraic complement of the element a_{ij}^0 of the matrix $a_{ij}^0 = A_0$ of the form f_0 .

P.q.f. f_0 is called [3] strong-stationary, if it is found such unbounded increasing sequence s_t of real numbers $s_t (t = 1, 2, \dots), s_t \rightarrow \infty$ for $t \rightarrow \infty$, the form f_0 is s_t stationary for all t .

The concept of perfect, final extremality and strong stationarity of p.q.f are connected with each other by the following [3].

Delone-Rizhkov Theorem. [3] P.q.f. f is final extremal if and only if when it is perfect and strong stationary.

Let arbitrary finite group $\{g\}$ of integer valued unimodular form $f = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$, which transforms itself with all matrix of the group $\{g\}$, be given.

Since the quadratic form $f = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$, in transformation with matrix $g = \{g_{ik}\}$ passes to the quadratic form with coefficients

$$a'_{kl} = \sum_{1 \leq i, j \leq n} a_{ij} g_{ik} g_{jl},$$

then if g transforms the form f itself (g is the integer valued automorphism of the f), we get the following equality

$$\sum_{1 \leq i, j \leq n} a_{ij} g_{ik} g_{jl} = a_{kl} \quad (1 \leq k \leq l \leq n).$$

Here we obtain $\frac{n(n+1)}{2} = N$ linear homogeneous equations with respect to coefficients

a_{ij} of the form f . If the form f transforms to itself by any transformation of the group $\{g\}$ of order $|\{g\}|$, then we get $|\{g\}| \frac{n(n+1)}{2}$ linear homogeneous equations with respect to coefficients a_{ij} . By Kronicer-Kapelli theorem, the part of these coefficients, m , remain arbitrarily, and other coefficients are expressed in them by Kramer's formula. Therefore in the space of coefficients of quadratic forms it is obtained linear manifold. This manifold is m -dimensional (where $1 \leq m \leq N$) and passes through vertex O of the cone K^N .

But, according to Mashke's theorem [4], for any finite group of integer valued unimodular matrix there exists positive quadratic form, which transforms to itself by all matrixes of this group. Therefore the manifold which considered above is called [4] Brave manifold, corresponding to the given group $\{g\}$, and is denoted by $B_{\{g\}}$.

Thus, Brave manifold $B_{\{g\}}$ is the collection of all that points of the cone K^N , every of which transforms to itself by that finite subgroup $\{g\}$ of the equivalence group $\{G\}$, i.e. of the group all integer valued unimodular matrixes $(n \cdot n)$.

In the case, when $B_{\{g\}}$ consists of one point, it is said that Brave manifold $B_{\{g\}}$ is point set. This means, that the group $\{g\}$ is the group of iteger valued automorphisms only one (unique) p.q.f. $f_0 = \sum_{1 \leq i, j \leq n} a_{ij}^0 x_i x_j$. Further as $\{g\}$ we consider $G(f_0)$ group of integer valued automorphisms of the group f_0 .

One point Brave manifold $B_{\{g\}}$ means that

$$G(f_0) \langle f(x_1, \dots, x_n) \rangle = F_0(a_{11}, \dots, a_{n-1n}) f_0(x_1, \dots, x_n). \quad (4)$$

In particular,

$$G(f_0) \langle x_i x_j \rangle = c_{ij}^0 f_0(x_1, \dots, x_n) (1 \leq i \leq j \leq n), \quad (5)$$

where $G(f_0) \langle f(x_1, \dots, x_n) \rangle = \sum_{g \in G(f_0)} g \langle f(x_1, \dots, x_n) \rangle$,

$g\langle(x_1, \dots, x_n)\rangle$ is action of the automorphism g to real quadratic form of the form (1), $F_0(a_{11}, \dots, a_{n-1n})$ is linear function of the coefficients a_{ij} of the form $f = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j$, c_{ij}^0 is constant depending only on the group $G(f_0)$.

It should be noted that equalities (4) and (5) are equivalent. Indeed, assuming in $f = \sum_{1 \leq i, j \leq n} a_{ij} x_i x_j a_{ii} = 1$, $a_{ij} = \frac{1}{2}$ ($1 \leq i \leq j \leq n$), using (4), we get (5). And inversely, multiplying both parts of (5) by a_{ii} and $2a_{ij}$, and summing over $1 \leq i \leq j \leq n$, we obtain (4).

In the work [5] S.S.Rizhkov posted the problem of finding finite necessary and sufficient conditions of strong stationarity, because existing such conditions require checking of infinite number equalities [5].

Here we solve this problem in some sense.

2 The main result

Theorem. Assume Brave manifold $B_{G\{g\}}$ with respect to integer valued automorphisms of p.q.f. $f_0 = \sum_{1 \leq i, j \leq n} a_{ij}^0 x_i x_j$ is point set. Then in order to f_0 becomes strong stationary it is necessary and sufficient fulfilling of the following conditions

$$\frac{n \cdot \det f_0}{|G(f_0)|} c_{ij}^0 = A_{ij}^0 (1 \leq i \leq j \leq n). \quad (6)$$

Proof. Necessity. Suppose p.q.f. f_0 is strong stationary. Therefore, it is stationary and taken place the equality (3) (see [3]). Applying the group $G(f_0)$ to both parts of equality (3) and using (5), consequently we get

$$\sum_{g \in G(f_0)} g \left\langle \sum_{x \neq 0} \frac{x_i x_j}{\{f(x)\}^{\frac{ns}{2}+1}} \right\rangle = \sum_{g \in G(f_0)} g \left\langle \frac{Z(f_0, s)}{n \cdot \det f_0} A_{ij}^0 \right\rangle, \quad (7)$$

$$\sum_{x \neq 0} \frac{\sum_{g \in G(f_0)} g \langle x_i x_j \rangle}{\{f(x)\}^{\frac{ns}{2}+1}} = \frac{Z(f_0, s)}{n \cdot \det f_0} A_{ij}^0 |G(f_0)|, \quad (8)$$

$$Z(f_0, s) c_{ij}^0 = \frac{Z(f_0, s)}{n \cdot f_0} A_{ij}^0 |G(f_0)|, \quad (9)$$

$$c_{ij}^0 = \frac{|G(f_0)|}{n \cdot \det f_0} A_{ij}^0 (1 \leq i \leq j \leq n). \quad (10)$$

Necessity of the Theorem is proved.

Sufficiency. Assume the equalities (5) and (6) are valid. Then, doing all things in

inverse order of the proof of necessity of the theorem, starting with (10) and finishing (7), we get (3).

Really, from (10) using (5) we have

$$\frac{\sum_{g \in G(f_0)} g \langle x_i x_j \rangle}{f_0(x)} = \frac{|G(f_0)|}{n \cdot \det f_0} A_{ij}^0 (1 \leq i \leq j \leq n). \quad (11)$$

Multiplying both parts of (11) by $\{f_0(x)\}^{\frac{ns}{2}}$, and summing over $x \neq 0$, we obtain

$$\sum_{x \neq 0} \frac{\sum_{g \in G(f)} g \langle x_i x_j \rangle}{\{f_0(x)\}^{\frac{ns}{2}+1}} = Z(f_0, s) \frac{|G(f_0)|}{n \cdot \det f_0} A_{ij}^0.$$

Hence

$$\sum_{g \in G(f_0)} g \left\langle \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}} \right\rangle = Z(f_0, s) \frac{|G(f_0)|}{n \cdot \det f_0} A_{ij}^0.$$

The last equality is equivalent to (3), or the series in the left and the right parts of the equality

$$g \left\langle \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}} \right\rangle = \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}}$$

differ from each other at most permutation of the terms. Therefore

$$\begin{aligned} \sum_{g \in G(f_0)} g \left\langle \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}} \right\rangle &= \sum_{g \in G(f_0)} \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}} = \\ &= \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}} \cdot \sum_{g \in G(f_0)} 1 = |G(f_0)| \sum_{x \neq 0} \frac{x_i x_j}{\{f_0(x)\}^{\frac{ns}{2}+1}}. \end{aligned}$$

Hence, f_0 is s stationary. The conditions (2) and (6) do not depend on parameter s . Therefore f_0 is s stationary for any $s > 1$ i.e. f_0 is strong stationary. Sufficiency of the Theorem is proved.

Example. Assuming in (10) $f_0 = \varphi_1^4, \varphi_1^5, \varphi_2^6, \varphi_2^7$, it is easy to write the coefficients c_{ij}^0 which in the work [6] were calculated by Computer.

References

- [1] G.F.Voronoy, On some properties of positive perfect and quadratic forms, Selected works. V.II. Izdatelstvo AN SSSR. (1952). 171-238.

- [2] R.A.Rankin, A minimum problem for the Epshtein zeta-funktion, Proc. Glasgow Math. Assoc. V.1 (1953), 149-158.
- [3] B.N.Delone, S.S.Rizhkov, To the theory of extremums of multidimensinal ζ - function, Doklady SSSR. V.173.N 5 (1967) 991-994.
- [4] B.N.Delone, R.V.Galuilin, M.N.Shtogrii, Theory of Brave and its generalization to n-dimensional lattice, O.Brave Selected works. "Nauka", Leningrad (1974), 384-389.
- [5] S.S.Rizhkov, Main extremal problems of the geometry of positive quadratic forms, Doctoral dissertation. Moscow (1970) 171p.
- [6] K.M.Endibaev, Local extremality of Epstein zeta-function for the forms corresponding to the densest lattice packing of equal balls, Candidate thesis. Leningrad, (1985) 91p.

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